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ON THE ACCURACY OF REGULARIZED SOLUTIONS TO QUADRATIC MINIMIZATION PROBLEMS ON A HALF-SPACE, IN CASE OF A NORMALLY SOLVABLE OPERATOR*

I. KRNIĆ, O. OBRADOVIĆ,

Department of Mathematics, University of Montenegro
Podgorica, Serbia and Montenegro

M.M. POTAPOV

Faculty of Computational Mathematics and Cybernetics,
Moscow State University, Moscow, Russia

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Abstract: A new stable quadratic minimization method on a half-space is presented. In case of normally solvable operators this method outperforms approximate solutions having the same optimal order accuracy as earlier methods for unconstrained problems.

Keywords: Quadratic functional, linear constraints, optimization, regularization.

1. STATE OF THE PROBLEM. STRUCTURE OF THE OPTIMAL SOLUTION.

In this paper we consider a quadratic minimization problem with a linear constraint:

$$J(u) = \frac{1}{2} \| Au - f \|_F^2 \to \inf, \quad \langle c, u \rangle_H \le r \tag{1}$$

Here H and F are Hilbert spaces, $f \in F$, $c \in H$, $(c \neq 0)$ and $r \in R$. Operator $A \in L\,(H \to F)$ is a linear bounded operator which is normally solvable, i.e.

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$$R(A) = \overline{R(A)}\,, \tag{2}$$

where $R(A) = \{v \in F \mid \exists u \in H,\, v = Au\}$ is the range of the operator A.

We also assume that, instead of exact data A, f, and c we are given their approximations $A_\mu \in L\,(H \to F),\, f_\delta \in F,\, c_\sigma \in H,$ and corresponding error levels $\mu, \delta, \sigma > 0$:

$$\| A - A_\mu \| \le \mu, \quad \| f - f_\delta \| \le \delta, \quad \| c - c_\sigma \| \le \sigma\,. \tag{3}$$

The same problem (1) with perturbed data (3), but without a normal solvability assumption (2) was considered earlier in [1] and it was shown that the accuracy order of the output approximations generated by the algorithm from [1] was bounded by the value $(\mu + \delta + \sigma)^{\frac{1}{2}}$. On the other hand, for unconstrained minimization problems

$$\frac{1}{2} \| Au - f \|^2 \to \inf, \quad u \in H, \tag{4}$$

it is well known [2, 3, 4] how to bring this accuracy to the order-optimal level $\mu + \delta$ under the same conditions (2), (3).

In this paper we propose a solution to the problem (1) with order-optimal accuracy under the assumptions (2) and (3). In this method we combine the basic minimization algorithm for unconstrained problem (4) from [6] with special structural features of the optimal normal solution to the problem (1). These features were also used in [1], but only to prove convergence of approximate solutions. Here, we include these structural representations directly into the numerical procedure. Let us specify the character of those representations.

If operator A is normally solvable, then the problem (1) has a solution for arbitrary $f \in F$, $c \in H$, $(c \ne 0)$ and $r \in R$ [2, 4, 7] and the unconstrained minimization problem (4) has a solution for each $f \in F$. Let u_* and w be normal solutions to the problems (1) and (4) respectively. By the optimality criteria there exists Lagrange multiplier $\lambda_* \ge 0$ so that

$$\begin{aligned} &A^*(Au_* - f) + \lambda_* c = 0, \\ &\langle c, u_* \rangle \le r, \quad \lambda_*(\langle c, u_* \rangle - r) = 0, \end{aligned} \tag{5}$$

where $A^*Aw = A^*f$. From (5) it is easy to get [1] the following representations of the element u_*:

1) if $c \in R(A^*A)$, i.e. $c = A^*Ah$ for some $f \in R(A^*A)$, then

$$u_* = w - \lambda_* h, \tag{6}$$

where $\lambda_* = 0$ if $\langle c, w \rangle \le r$ and $\lambda_* = \dfrac{\langle c, w \rangle - r}{\langle c, h \rangle}$ if $\langle c, w \rangle > r$;

2) if $c \notin R(A^* A)$, i.e. $c^0 = (I-P)c \neq 0$ (here P is the orthogonal projection operator from H onto the range $R(A^* A)$ of the operator $A^* A$ and $I-P$ is the orthoprojector from H onto the null-space $N(A)$ of the operator A), then in (5) $\lambda_* = 0$ and

$$u_* = w - \gamma_* c^0, \tag{7}$$

where $\gamma_* = 0$ if $\langle c,w \rangle \leq r$ and $\gamma_* = \dfrac{\langle c,w \rangle - r}{\langle c,h \rangle}$ if $\langle c,w \rangle > r$;

2. BASIC APPROXIMATION ALGORITHM

For stable approximation of elements w, h, c^0 and coefficients λ_*, γ_* from representations (6), (7) we use well-known regularization methods, generated by functions $g_\alpha(t)$ with the following characteristics [6, p.157]:

$$\sup_{0\leq t\leq a} |g_\alpha(t)| \leq \frac{k}{\alpha}, \quad k = \text{const.} > 0, \tag{8}$$

$$\sup_{0\leq t\leq a} t^p(1 - t\cdot g_\alpha(t)) \leq c_p \cdot \alpha^p, \quad c_p = \text{const.} > 0, \ 0 \leq p \leq p_0, \tag{9}$$

$$0 \leq 1 - t\cdot g_\alpha(t) \leq 1, \quad 0 \leq t \leq a, \tag{10}$$

where $\alpha > 0$ is an arbitrary small real number and $a = \sup\limits_{0\leq\mu\leq\mu_0} \| A_\mu^* A_\mu \|$. The value p_0 from (9) is usually called the qualification of the regularization method, generated by the family of functions $g_\alpha(t)$. In this paper we consider only the case of normally solvable operators A and therefore we use $p_0 \geq 1$ (as in [6, ch.6]). In particular, the classical Tikhonov regularization method generated by function $g_\alpha(t) = \dfrac{1}{t+\alpha}$ with qualification $p_0 = 1$ is fully permitted.

We do not assume that the minimal value of the functional in the global optimization problem (4) is equal to zero. Moreover, we do not know beforehand whether the orthogonal projection c^0 of vector c onto the null-space $N(A)$ is equal to zero or not. That is why we also use (as in [6, ch.6]) along with functions $g_\alpha(t)$ their modifications $\overline{g}_\alpha(t) = t\cdot(g_\alpha(t))^2$. Functions $\overline{g}_\alpha(t)$ possess the same basic properties (8) - (10) as functions $g_\alpha(t)$ [6, p.158]:

$$\sup_{0\leq t\leq a} |\overline{g}_\alpha(t)| \leq \frac{\overline{k}}{\alpha}, \quad \overline{k} = \text{const.} > 0, \tag{11}$$

$$\sup_{0\leq t\leq a} t^p(1 - t\cdot \overline{g}_\alpha(t)) \leq \overline{c}_p \cdot \alpha^p, \quad \overline{c}_p = \text{const.} > 0, \ 0 \leq p \leq p_0, \tag{12}$$

$$0 \le 1 - t \cdot \bar{g}_\alpha(t) \le 1, \tag{13}$$

where $\bar{k} = k$ and $\bar{c}_p = 2c_p$.

As in for [6] we approximate the normal solution w to the global minimization problem (4) by the elements

$$\bar{w}_\alpha = \bar{g}_\alpha(A_\mu^* A_\mu) A_\mu^* f_\delta. \tag{14}$$

For vector h from (6) and projection $c^0 = (I-P)c$ from (7) we choose the approximations

$$\bar{h}_\alpha = \bar{g}_\alpha(A_\mu^* A_\mu) c_\sigma \tag{15}$$

and

$$c_\alpha^0 = K_{\alpha\mu} c_\sigma, \text{ where } K_{\alpha\mu} = I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu) \tag{16}$$

If operator A is normally solvable, and if $\alpha = \mu$ ([6, ch.6]) then the approximate solutions (14) to the unconstrained problem (4) reach order-optimal accuracy. We also set $\alpha = \mu$ in all three approximations (14), (15), (16). Accuracy of the approximations (14)-(16) is estimated in the following lemmas.

Lemma 1. *Let the conditions (2)-(3) and (11)-(13) be satisfied and approximations* $\bar{w}_\alpha$ *of the normal solution w to the problem (4) be determined by (14). Then for* $\alpha = \mu$

$$\| \bar{w}_\alpha - w \| \le M(\mu + \delta) \quad as \quad \mu,\ \delta \to 0. \tag{17}$$

The proof of Lemma 1 with more detailed error estimation than in (17) can be found in [6, ch.6, Th. 2.1].

Lemma 2. *Let* $c \in H,\ c \ne 0$ *be an arbitrary vector with orthogonal projections* $c^* = Pc$ *and* $c^0 = (I-P)c$ *onto the subspaces* $R(A^*A)$ *and* $N(A)$ *respectively. Let conditions (2), (3) be satisfied and let the approximations* $c_\alpha^* = g_\alpha(A_\mu^* A_\mu) A_\mu^* A_\mu c_\sigma$ *and* c_α^0 *from (16) generated by the functions* $g_\alpha(t)$ *with properties (8)-(10) be chosen. Then for* $\alpha = \mu$ *and* $\mu,\ \sigma \to 0$

$$\| c_\alpha^* - c^* \| \le M(\mu + \sigma), \tag{18}$$

$$\| c_\alpha^0 - c^0 \| \le M(\mu + \sigma). \tag{19}$$

The proof of Lemma 2 is similar to the proofs of Theorem 2.1 from [6, ch. 6] and Lemma 1. We make use of the properties (3) along with the fact that the operators

$g_\alpha(A_\mu^* A_\mu)A_\mu^* A_\mu$ and $K_{\alpha\mu} = I - g_\alpha(A_\mu^* A_\mu)A_\mu^* A_\mu$ are uniformly bounded, as is obvious from. The condition of normal solvability (2) becomes important for the estimation of the elements $K_{\alpha\mu}c^*$ and $g_\alpha(A_\mu^* A_\mu)A_\mu^*(A_\mu - A)c^0$. Namely, under condition (2) the element c^* is source wise presented, i.e. $c^* = A^* Ah,\ h \in R(A^* A)$ and the family of operators $g_\alpha(A_\mu^* A_\mu)A_\mu^*$ is uniformly bounded. Therefore, the accuracy order of error estimations can be improved to the level of (18) and (19).

Lemma 3. *Let* $c \in R(A^* A)$, *i.e.* $c = A^* Ah,\ h \in R(A^* A)$ *and the conditions (2)-(3) be satisfied. Let the approximations* $\overline{h}_\alpha,\ c_\alpha^0$ *be determined by (15), (16), where generating functions* $g_\alpha(t),\ \overline{g}_\alpha(t)$ *satisfy the conditions (8)-(13). Then if* $\alpha = \mu$ *and* $\mu,\ \sigma \to 0$

$$\| \overline{h}_\alpha - h \| \le M(\mu + \sigma), \tag{20}$$

$$\| c_\alpha^0 \| \le M(\mu + \sigma). \tag{21}$$

The uniform bound ness of the operator family $\overline{g}_\alpha(A_\mu^* A_\mu)$ follows from (2) and plays an important role in the proof of (20) which is similar to the proof of the Theorem 2.1 from [6, ch. 6]. Under the conditions of Lemma 3 the orthogonal projection $c^0 = 0$ and therefore (21) follows directly from (19).

3. THE ALGORHITM. ACCURACY ESTIMATIONS OF THE APPROXIMATE SOLUTIONS

Input of the algorithm is perturbed data $A_\mu,\ f_\delta,\ c_\sigma$ with the corresponding error levels $\mu,\ \delta,\ \sigma > 0$. The output of the algorithm is the approximation $\tilde{u}_*$ to the desired normal solution u_* of the initial problem (1). Algorithm is the sequence of the following steps.

Step 1. Take $\alpha = \mu$. Calculate $\overline{w}_\alpha$ as in (14) and verify the condition

$$\langle c, \overline{w}_\alpha \rangle \le r + (\mu + \delta + \sigma). \tag{22}$$

If (22) is true, set $\tilde{u}_* = \overline{w}_\alpha$. Otherwise proceed to Step 2.

Step 2. Calculate c_α^0 as in (16) and verify that

$$\| c_\alpha^0 \| \le \sqrt{\mu + \sigma}. \tag{23}$$

If (23) is true, set $c^0 = 0,\ c = A^* Ah,$ calculate $\overline{h}_\alpha$ as in (15) and choose approximation according to (6):

$$\tilde{u}_* = \overline{w}_\alpha - \overline{\lambda}_\alpha \cdot \overline{h}_\alpha, \text{ where } \overline{\lambda}_\alpha = \frac{\langle c_\sigma, \overline{w}_\alpha \rangle - r}{\langle c_\sigma, \overline{h}_\alpha \rangle} > 0. \tag{24}$$

Otherwise proceed to Step 3.

Step 3. By virtue of (21) set $c^0 \neq 0$ and choose approximation according to (7):

$$\tilde{u}_* = \overline{w}_\alpha - \gamma_\alpha \cdot c_\alpha^0, \text{ where } \gamma_\alpha = \frac{\langle c_\sigma, \overline{w}_\alpha \rangle - r}{\langle c_\sigma, c_\alpha^0 \rangle} > 0. \tag{25}$$

In the following Theorem convergence of the described method is proved and the accuracy order of approximations provided by this algorithm is estimated.

Theorem. *Suppose that in the initial problem (1) operator A is normally solvable approximate data satisfy the condition (3), and functions* $g_\alpha(t)$, $\overline{g}_\alpha(t)$ *satisfy the conditions (8)-(13) with qualification degree* $p_0 \geq 1$. *If* $\alpha = \mu$ *and* $\mu,\ \delta,\ \sigma \to 0$ *then approximations* $\tilde{u}_*$ *determined by the algorithm converge to the normal solution* u_* *and*

$$\| \tilde{u}_* - u_* \| \leq M(\mu + \delta + \sigma). \tag{26}$$

Proof:

1) If, for all sufficiently small $\mu,\ \delta,\ \sigma > 0$, algorithm stops at Step 1, then in both initial problems (1) and (4) their normal solutions $\tilde{u}_*$ and w coincide. In this case the desired estimation (26) after choosing approximations $\tilde{u}_* = \overline{w}_\alpha$ is obtained from Lemma 1:

$$\| \tilde{u}_* - u_* \| = \| \overline{w}_\alpha - w \| \leq M(\mu + \delta). \tag{27}$$

2) If, for all sufficiently small $\mu,\ \delta,\ \sigma > 0$, the algorithm stops at Step 2, then by virtue of relation (19) we have $c^0 = 0,\ c = c^* = A^*Ah \neq 0,\ h \in R(A^*A)$. In that case the exact solution u_* is represented in the form (6) and the algorithm provides the output approximations of the form (24). We estimate the difference $\tilde{u}_* - u_*$ using the triangle inequality:

$$\| \tilde{u}_* - u_* \| \leq \| \overline{w}_\alpha - w \| + \lambda_* \| h - \overline{h}_\alpha \| + | \lambda_* - \overline{\lambda}_\alpha | \| \overline{h}_\alpha \|. \tag{28}$$

From Lemma 1 and Lemma 3 it follows that the first and the second addenda from the right hand side of (28) are values of the order $\mu + \delta + \sigma$. The estimation (20) implies uniform bound ness of the norms $\| \overline{h}_\alpha \|$. Let us estimate the difference between the Lagrange multipliers $\lambda_* - \overline{\lambda}_\alpha$. We pass to Step 2 only if the condition (22) is not satisfied. It means that $\langle c, w \rangle \geq r$ and the exact value of $\lambda_* \geq 0$ can be represented by

$$\lambda_* = \frac{\langle c, w \rangle - r}{\langle c, h \rangle}, \quad \text{where} \quad \langle c, h \rangle = \| Ah \|^2 > 0.$$

Here we have

$$\lambda_* - \bar{\lambda}_\alpha = \frac{\langle c, w\rangle - r}{\langle c, h\rangle} - \frac{\langle c_\sigma, \bar{w}_\alpha\rangle - r}{\langle c_\sigma, \bar{h}_\alpha\rangle}. \tag{29}$$

On the basis of (3) and Lemma 1 the difference between numerators of the fractions from (29) is the value of the order $\mu+\delta+\sigma$. Similarly it follows from Lemma 3 that the difference between denominators of the same fractions is the value of the order $\mu+\sigma$, therefore

$$|\lambda_* - \bar{\lambda}_\alpha| \le M(\mu+\delta+\sigma).$$

Now from (28) we get the required estimation (26).

3) If, for all sufficiently small $\mu, \delta, \sigma > 0$, the algorithm stops at Step 3, then by virtue of relation (21) we have $c^0 \ne 0$, representation (7) of the exact normal solution u_*, and output approximations $\tilde{u}_*$ of the form (25). In this case we use, as in (28) the estimation

$$\|\tilde{u}_* - u_*\| \le \|\bar{w}_\alpha - w\| + \gamma_* \|c^0 - c_\alpha^0\| + |\gamma_* - \gamma_\alpha| \|c_\alpha^0\|. \tag{30}$$

It follows from Lemma 1 and Lemma 2 that the first two addenda from the right hand side (30) are values of the order $\mu+\delta+\sigma$ and norms $\|c_\alpha^0\|$ are uniformly bounded. We pass to Step 3 only if the condition (22) is not true, therefore $\langle c, w\rangle \ge r$. In this case we can represent the exact value of $\gamma_* \ge 0$ in the form of $\gamma_* = \dfrac{\langle c, w\rangle - r}{\langle c, c^0\rangle}$ and write the difference $\gamma_* - \gamma_\alpha$ as follows:

$$\gamma_* - \gamma_\alpha = \frac{\langle c, w\rangle - r}{\langle c, c^0\rangle} - \frac{\langle c_\sigma, \bar{w}_\alpha\rangle - r}{\langle c_\sigma, c_\alpha^0\rangle}. \tag{31}$$

The difference between numerators in (31) as well as in (29) is the value of the order $\mu+\delta+\sigma$. It follows from Lemma 2 that the difference between the denominators of these fractions is the value of the order $\mu+\sigma$ and also $\langle c, c^0\rangle = \|c^0\|^2 > 0$ since $c^0 \ne 0$. Thus we have

$$|\gamma_* - \gamma_\alpha| \le M(\mu+\delta+\sigma)$$

and then (30) implies (26). This completes the proof.

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