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BI-INDUCED SUBGRAPHS AND STABILITY NUMBER*

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Abstract: We define a 2-parametric hierarchy $\mathcal{CLAP}(m,n)$ of bi-hereditary classes of graphs, and show that a maximum stable set can be found in polynomial time within each class $\mathcal{CLAP}(m,n)$. The classes can be recognized in polynomial time.

Keywords: Stability number, hereditary class, bi-hereditary class, forbidden induced subgraphs, forbidden bi-induced subgraphs.

1. INTRODUCTION

A set $S \subseteq V(G)$ in a graph *G* is *stable* (or *independent)* if *S* does not contain adjacent vertices. A stable set of a graph *G* is called *maximal* if it is not contained in another stable set of *G*. A stable set of a graph *G* is called *maximum* if *G* does not have a stable set containing more vertices. The cardinality of a maximum stable set in *G* is the *stability number* of *G*, and it is denoted by $\alpha(G)$.

Decision Problem 1 (Stable Set).
Instance: *A graph G and an integer k.*
Question: *Is there a stable set in G with at least k vertices?*

This problem is known to be NP-complete (Karp [7], see also Garey and Johnson [3]). A class $\mathcal{P}$ of graphs is α-*polynomial* if there exists a polynomial-time algorithm to solve Stable Set Problem within $\mathcal{P}$. We shall define a hierarchy $\mathcal{CLAP}(m,n)$ of α-polynomial graph classes. The hierarchy covers all graphs.

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Note that it is easy to find the stability number of graphs in any class without large connected induced bipartite subgraphs. In other words, the class $\mathcal{CONNBIP}(N)$-*free* graphs is α-polynomial, where $\mathcal{CONNBIP}(N)$ is the set of all connected bipartite graphs of order *N*. Lozin and Rautenbach [8] used this fact to produce α-polynomial subclasses of $\mathcal{CONNBIP}(N)$-*free* graphs defined by a path and a star as forbidden subgraphs. Specifically, given *m* and *n,* there exists an integer *N* such that each $(P_n, K_{1,m})$-free triangle-free graph is a $\mathcal{CONNBIP}(N)$-*free* graph.

In our hierarchy we also forbid a path, but we do not forbid a star. Instead, we use Hall's theorem to specify a particular family of connected bipartite graphs, thus obtaining a more general result.

2. BI-INDUCED SUBGRAPHS

The neighborhood of a vertex *x* in a graph *G* is denoted by $N(x) = N_G(x)$. For a subset *X* of $V(G)$, we denote $N(X) = \bigcup_{x \in X} N_G(x)$.

Definition 1. *A bipartite graph F is called a bi-induced subgraph of a graph G if*

(BI1): *F is a subgraph of G [not necessarily induced], and*

(BI2): *there exists a bipartition $A \cup B$ of V(F) such that both A and B are stable sets in G.*

In other words, a bi-induced subgraph *F* of a graph *G* is obtained from a bipartite induced sub graph *F′* of *G* by deleting some edges [possibly, none]. As usual, we distinguish bi-induced subgraphs up to isomorphism.

A class $\mathcal{P}$ is *bi-hereditary* if it is closed under taking bi-induced subgraphs. That is, $F \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and *F* is a bi-induced subgraph of *G*. Clearly, a class is bi-hereditary if and only if it can be characterized in terms of *forbidden bi-induced subgraphs*. Also, a bi-hereditary class with finitely many minimal forbidden bi-induced subgraphs can be recognized in polynomial time.

We define a 2-parametric series $\mathcal{CLAP}(m,n)$ of bi-hereditary classes of graphs. As usual, P_n denotes the *n*-vertex path. An *m-claw* is a complete bipartite graph of the form $K_{1,m}$. If we subdivide every edge of an *m*-claw by a vertex, we obtain a bipartite graph of order 2*m* + 1 called a *subdivided m-claw*, $SK_{1,m}$ (see Figure 1).

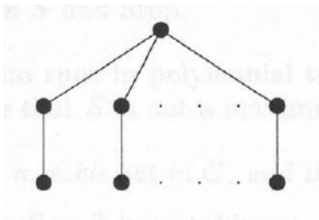

Figure 1: Subdivided *m*-claw $SK_{1,m}$

Definition 2. *Given integers* $m \geq 1$ *and* $n \geq 1$*, the class* $\mathcal{CLAP}(m,n)$ *consists of all graphs that do not contain*

- $SK_{1,m}$ *as bi-induced subgraphs, and*
- P_n *as induced subgraphs.*

Clearly,

$$\mathcal{CLAP}(m,n) \subset \mathcal{CLAP}(m+1,n),$$

$$\mathcal{CLAP}(m,n) \subset \mathcal{CLAP}(m,n+1)$$

for all $m \geq 1$ and $n \geq 1$, and

$$\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\mathcal{CLAP}(m,n)$$

contains all graphs. Note that membership in each $\mathcal{CLAP}(m,n)$ can be checked in polynomial time, since there is one minimal forbidden induced subgraph and there is one minimal forbidden bi-induced subgraph for this class.

3. STABILITY IN $\mathcal{CLAP}(m,n)$

Here is our main result.

Theorem 1. *For all integers* $m \geq 1$ *and* $n \geq 1$*, the class* $\mathcal{CLAP}(m,n)$ *is* α *-polynomial.*

Proof: We define

$$N = N(m,n) = \left\lceil 0.5 + 0.5(m+2)\sum_{d=1}^{n-2}(m+1)^{d-1} \right\rceil \tag{1}$$

if $n \geq 3$, and $N = 1$ if $n \leq 2$. Now we apply the following algorithm to an arbitrary graph $G \in \mathcal{CLAP}(m,n)$.

Algorithm 1.

Step 0. *Set* $S = \emptyset$.

Step 1. For every stable set $T \subseteq V(G) \setminus S$ with $|T| \leq N$, define $S' = (S \setminus N(T)) \bigcup T$. If $|S'| > |S|$, set $S = S'$.

Step 2. *Return S and* Stop.

The algorithm runs in polynomial time, since *N* is a constant. It produces a set $S \subseteq V(G)$. Suppose that *S* is not a maximum stable set.

Claim 1. *S is a stable set in G, and there exists a stable set* $T \subseteq V(G) \setminus S$ *with* $|T| > N$.

Proof: Initially, $S = \emptyset$ is a stable set. Also, the set $S' = (S \setminus N(T)) \bigcup T$ [on Step 2] is stable. Thus S is a stable set in G.

Since S is not a maximum stable set, there exists a stable set I in G with $|I| > |S|$. We denote $T = I \setminus S$. Since $|S| < |I|$ we have $|S \setminus I| < |T|$, and therefore

$$|N(T) \bigcap S| \leq |S \setminus I| < |T|.$$

Step 1 of the algorithm implies that $|T| > N$. ♦

According to Claim 1, there exists a set $T \subseteq V(G) \setminus S$ such that

(Tl): $|T| > N$, and

(T2): $|S'| > |S|$, where $S' = (S \setminus N(T)) \bigcup T$.

We assume that T has the minimum cardinality among all sets that satisfy (Tl) and (T2). Let H be a bipartite graph induced by $T \bigcup U$, where $U = S \setminus I$.

Claim 2. *(i) For every vertex $u \in T$, there exists a matching M in H - u that covers U, and*
(ii) $|T| = |U| + 1$.

Proof: (i) Each proper subset T' of T does not satisfy (T2) [with T' instead of T]. Indeed, if $|T'| \leq N$, then it follows from Step 1 of the algorithm. If $|T| > N$ then it follows from minimality of T.

Let $u \in T$. Each subset of $T' = T \setminus \{u\}$ does not have property (T2). In other words, for every $X \subseteq T'$, we have $|N(X)| \leq |X|$ in $H - u$. By Hall's theorem (Hall [5], see also Hall [4]), there exists a matching M in $H - u$ that covers T'. In particular, $|T'| \leq |U|$. The condition (T2) for T implies that $|T| > |U|$. Therefore $|T'| = |U|$, and M must cover U as well.

(ii) The statement follows directly from (i). ♦

As usual, $\Delta(G)$ is the maximum vertex degree in G.

Claim 3. $\Delta(H) \leq m + 2$.

Proof: Suppose that there exists a vertex $u \in V(H)$ of degree $m + 2$. First let $u \in T$. Let u is adjacent to pairwise distinct vertices $v_1, v_2, ..., v_m \in U$. By Claim 2(i), there exists a matching M in $H - u$ that covers U. We consider the edges of M that are incident to $v_1, v_2, ..., v_m$. Clearly, $H - u$ contains $SK_{1,m}$ as a hi-induced subgraph.

Now let $u \in U$. Let u is adjacent to pairwise distinct vertices $u_1, u_2, ..., u_{m+2} \in T$. We apply Claim 2(i) to the graph $H' = H - u_{m+2}$: there exists a matching M in H' that covers U. At most one edge of M is incident to the vertex u. We see that H' contains $SK_{1,m}$ as a hi-induced subgraph.

It remains to note that a hi-induced subgraph in an induced subgraph of G is also a hi-induced subgraph of G. ♦

Note that Claim 2 implies connectedness of *H*. Indeed, if *H* is not connected then there is a component *K* in *H* such that one part is larger than the other, and therefore deleting a vertex $u \in T \setminus V(K)$ produces a graph without perfect matching.

Claim 4. *H contains* P_n *as an induced subgraph.*
Proof: According to (Tl), $|T| \geq N+1$. By Claim 2(ii), $|U| = |T|-1 \geq N$. Thus,

$$|V(H)| \geq 2N+1 . \tag{2}$$

If $n \geq 2$ then $N=1$ and $2N+1=3$, and the result follows.
Suppose that $n \geq 3$. Using (2) and (1), we obtain

$$|V(H)| \geq 2N+1 \geq 2+(m+2)\sum_{d=1}^{n-2}(m+1)^{d-1} \tag{3}$$

Then (3) and Claim 3 imply

$$|V(H)| \geq 2+\Delta\sum_{d=1}^{n-2}(\Delta-1)^{d-1} . \tag{4}$$

Let $u \in V(H)$. There are at most $\Delta(\Delta-1)^{d-1}$ vertices at distance $d \geq 1$ from *u*. Since *H* is a connected graph, (4) implies that there exists a vertex *v* at distance *n* – 1 from *u*. A shortest (u,v)-path is an induced P_n.

Claim 4 produces a contradiction to the condition that $G \in \mathcal{CLAP}(m,n)$. This contradiction shows that *S* is a maximum stable set in G. ♦

Theorem 1 implies the following results on α-polynomial classes: $(P_5, K_{1,n})$-free graphs (Mosca [10]), a subclass of $(P_5, K_{1,4})$-free graphs (Branstädt and Hammer [2]), $(P_5, P, K_{2,3})$-free graphs (Mahadev [9], see Figure 2), and $(P_2 \cup P_3, K_{1,n})$-free graphs (Alekseev [1]).

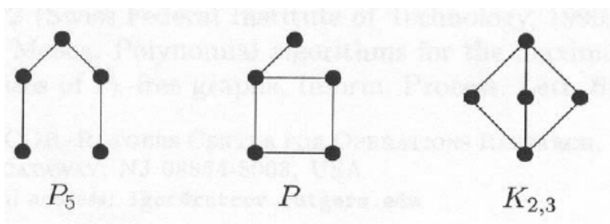

Figure 2: P_5, *P* and $K_{2,3}$

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