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STOCHASTIC MODELING OF THE NUMBER OF TREES AND THE NUMBER OF FELLED TREES IN SELECTION STANDS*

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Abstract: This paper solves the problem of forecasting the number of trees in the selection stands predicted for harvesting in a future period, so that the present resource of the number of trees is sustained. This is achieved by stochastic modeling of the number of trees and the number of felled trees and by solving the partial differential equation.

Keywords: Random walk, Itô´s lemma, lognormal distribution, Itô´s stochastic differential equation, partial differential equation.

1. INTRODUCTION

The number of trees in selection stands naturally increases in time. If the conditions were ideal, if there were no natural or artificial (due to an anthropogenic impact) removals of trees, the increase would be exponential in the time from t_0 to t, i.e. the dependence would be:

$$x(t) = x_0 \cdot e^{\mu (t - t_0)} \tag{1}$$

where $x(t)$ is the number of trees in the selection stand per hectare depending on time t; μ is the factor of constant growth, x_0 is the initial number of trees in the stand at the moment t_0 .

Let $S(x,t)$ be the number of felled trees, i.e. the planned impact of the anthropogenic factor on the number of trees x in the selection stand. There is a problem how to organise and limit the number of trees for felling in a time period $t = T$, without

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disturbing the present resources. In other words, the question is what capacity of the number of trees in the selection stand can be counted on and planned for harvesting at a definite future moment, so as to maintain the present number of trees in the stand. The significance of the problem is manifold: environmental and economic, bearing in mind that taking care of environmental conditions results in long-term economic gains.

In this paper, the dependencies $x(t)$ and $S(x,t)$ are modeled by Itô´s stochastic differential equations and the distribution for the random process $x(t)$ is given. Then the decision on the allowable cut is based on the requirement that the ratio of the changed number of trees in the selection stand and the changed number of felled trees is balanced. This is achieved by solving the derived partial differential equation of the second order, with two boundary and one final condition.

2. MODELING THE NUMBER OF TREES IN THE STAND

If we observe a relative change $\frac{dx(t)}{x(t)}$ of $x(t) > 0$ in a short time interval, we may conclude that it is broken down into the predictable, deterministic part and the random part, with white noise in its base. This is the standard way to explain many natural processes of growth. In other words, the continuous random process $x(t)$ can be represented by Itô´s stochastic differential equation:

$$dx = \mu x\, dt + \sigma x\, dw, \tag{2}$$

where μ is the drift, the expected rate of return, σ is the volatility, the measure of standard deviation, and w is the standard Wiener process. The first term in the above relation is deterministic and the second one is characterised by the presence of the Wiener process as the carrier of randomness. If $\sigma = 0$ (there were no removals of trees in selection stand), from the differential equation $dx = \mu x dt$, we can easily get the solution (1) for the process x.

Assume that μ and σ are constants in the definite time period. The parameters μ and σ are given in time units, mainly for a year. Often, μ is not changeable for one kind of selection stand. In the spatial model they are taken to be constant values, previously determined by statistical methods based on field data (look at statistical data from [3]).

For the standard Wiener process we take:

$$dw = z\sqrt{t}\ ,$$

where $z \in N(0,1)$. Hence, the expected value $E(dw) = 0$, and the variance $D(dw) = dt$, and:

$$E(dx) = E(\mu x\, dt + \sigma x\, dw) = \mu x\, dt\ , \quad D(dx) = E(\sigma^2 x^2 dw^2) = \sigma^2 x^2 dt.$$

In the case of discrete time with known μ and σ, we can simulate the process x by:

$$\Delta x = \mu x\, \Delta t + \sigma x\, z\sqrt{\Delta t}\ .$$

Example. Assume that the initial value is $x(t_0) = x_0 = 10$, the drift is 5%, ($\mu = 0.05$), and the standard deviation is 20%, ($\sigma = 0.2$) by year, $\Delta t = 1$. Then

$$\Delta x = (0.05 + 0.2\, z) \cdot x \, .$$

For 9 realised (casually chosen) values of the stochastic variable z, we get the table:

Table 1.

x	z	Δx
10	-0.175	0.15
10.15	1.44	3.43
13.59	-0.83	- 1.58
12.01	1.31	3.75
15.76	-0.69	-1.39
14.37	-0.21	0.11
14.48	0.97	3.53
18.01	-1.13	- 3.17
14.84	0.35	1.78
16.62	- 0.47	- 0.73

At the end of simulation we read the value $x = 16.62 - 0.73 = 15.89$, from the last row, which is valid for $x(t)$ after 10 years.

3. DISTRIBUTION OF STOCHASTIC PROCESS *x*(*t*)

Let f be a smooth function of x and t, where the process x is given by (2), then applying Itô´s lemma, we have:

$$df = \frac{\partial f}{\partial t} dt + \mu x \frac{\partial f}{\partial x} dt + \sigma x \frac{\partial f}{\partial x} dw + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} dt \ ,$$

(see [1]). For $f(x) = \ln x$, it is:

$$df = (\mu - \frac{1}{2} \sigma^2)\, dt + \sigma\, dw \, ,$$

with solution:

$$f = f_0 + (\mu - \frac{1}{2} \sigma^2)\,(t - t_0) + \sigma(w(t) - \mathrm{w}(t_0)).$$

Specially, for $t_0 = 0$, $w(t_0) = 0$, $x(0) = x_0$, we have:

$$\ln x - \ln x_0 = (\mu - \frac{1}{2} \sigma^2)\, t + \sigma\, w(t) \, ,$$

Now, from that we conclude $\ln\frac{x}{x_0}$ has a normal distribution with:

$$E(\ln\frac{x}{x_0}) = (\mu - \frac{1}{2}\sigma^2)\, t, \qquad D(\ln\frac{x}{x_0}) = \sigma^2 t\,.$$

It means that $\frac{x}{x_0}$ has a lognormal distribution with density:

$$\frac{x_0}{\sigma x\sqrt{2\pi t}}\ \exp\Big(-\frac{(\ln\frac{x}{x_0} - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}\Big), \qquad x > 0.$$

4. REQUIREMENT DERIVATIVES

As $S(x,t)$ depends on $x(t)$, $t > 0$, let us suppose that the dependence is expressed by a continuous function with continuous partial derivations $\frac{\partial S}{\partial t}$, $\frac{\partial S}{\partial x}$ and $\frac{\partial^2 S}{\partial x^2}$. Then by applying Itô´s lemma:

$$dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial x}dx + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 S}{\partial x^2}dt\ ,$$

By this we get the stochastic differential equation of random walk, which represents the process of the number of felled trees.

Let us consider the relation $S(x,t)$ and $x(t)$, by introducing a new function, the requirement Π:

$$\Pi = S - \Omega \cdot x,$$

where $\Omega = \frac{\partial S}{\partial x}$ is chosen (rate of change $S(x,t)$ compared to $x(t)$). Such an interpretation of Ω in this model, aims at neutralising the stochastic component in the following step. From the natural assumption that in a short time interval dt, the number of trees in the selection stand is not disturbed and that the ratio of the number of trees and the number of felled trees is constant, we get:

$$d\Pi = dS - \Omega \cdot dx = dS - \frac{\partial S}{\partial x}dx = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial x}dx + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 S}{\partial x^2}dt - \frac{\partial S}{\partial x}dx =$$
$$= \Big(\frac{\partial S}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 S}{\partial x^2}\Big)\, dt = 0.$$

Therefore:

$$\frac{\partial S}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 S}{\partial x^2} = 0, \tag{3}$$

and the determination of $S(x,t)$ means the solution of (3), a linear partial differential equation of the second order PDE, parabolic type. To be solved, it should be transformed into the canonical form, with set boundary conditions and a final condition.

First, if for a t, $x(t) = 0$, then also from (2), $dx = 0$, i.e. $x(t)$ is an invariable process. This means that $\sigma = 0$, and from $x(t) = x_0 \cdot e^{\mu(t - t_0)}$, for $x_0 = 0$, we get $x(t) = 0$, therefore:

$$S(0, t) = 0 . \tag{4}$$

On the other hand it is clear that:

$$S(x, t) \approx x , \tag{5}$$

when x increases, i.e. when x tends to ∞. These are boundary conditions.

Finally, we have to determine the final condition. If the felling was performed at the moment t_0, after which $x(t)$ was measured, then the next future number of felled trees will depend on that number of trees in the stand x_0 and the number of trees in future time T, i.e. there will be a difference between the future theoretical number (the maximal number which would be realised in ideal conditions) and the initial number of trees in the stand:

$$S(x, T) = x(T) - x(t_0) = x(t_0) \cdot \left(\frac{x(T)}{x(t_0)} - 1\right) = x_0 \cdot \left(e^{\mu(T - t_0)} - 1\right) . \tag{6}$$

It is clear that for $T > t_0$ and $\mu > 0$, $S(x,T) > 0$. Condition (6) is the final condition for PDE.

5. SOLVING PDE

First, we have to transform PDE (3) in the canonical form taking a function u:

$$u(y, \tau) = S(x, t),$$

where $y = \ln x$, and $\tau = \frac{1}{2}\sigma^2(T - t)$. According to partial derivations:

$$\frac{\partial S}{\partial t} = -\frac{1}{2}\sigma^2\frac{\partial u}{\partial \tau}, \qquad \frac{\partial S}{\partial x} = \frac{1}{x}\cdot\frac{\partial u}{\partial y}, \qquad \frac{\partial^2 S}{\partial x^2} = -\frac{1}{x^2}\cdot\frac{\partial u}{\partial y} + \frac{1}{x^2}\cdot\frac{\partial^2 u}{\partial y^2},$$

we have the new PDE, for $-\infty < y < \infty$:

$$\frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary conditions $u(-\infty, \tau) = 0$, and $u(\infty, \tau) = \infty$.

Now, we choose the next form of $u(y, \tau)$:

$$u(y , \tau) = e^{\alpha y + \beta \tau}\, v(y, \tau), \tag{7}$$

taking $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{4}$. Putting the derivatives

$$\frac{\partial u}{\partial \tau} = (-\frac{1}{4}v + \frac{\partial v}{\partial \tau})\, e^{\frac{1}{2}y-\frac{1}{4}\tau}, \quad \frac{\partial u}{\partial y} = (\frac{1}{2}v + \frac{\partial v}{\partial y})\, e^{\frac{1}{2}y-\frac{1}{4}\tau}, \quad \frac{\partial^2 u}{\partial y^2} = (\frac{1}{4}v + \frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2})\, e^{\frac{1}{2}y-\frac{1}{4}\tau}$$

into the previous equation, we have finally PDE in canonical form:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2}, \quad -\infty < y < \infty, \ \tau > 0. \tag{8}$$

It is well known that the solution of PDE (8), called diffusion equation, is:

$$v(y, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} v_0(p) e^{\frac{-(y-p)^2}{4\tau}} dp \ ,$$

(see [4, 5]), where the function $v_0(y)$ is the beginning condition given by the final condition (6), and $u(y, 0) = S(x, T)$, for $\tau = 0$:

$$v_0(y) = v(y, 0) = e^{-y/2} u(y, 0) = e^{-y/2} \cdot e^c \cdot (e^{\mu(T-t_0)} - 1) \tag{9}$$

where $c = \ln x_0$.

Taking $\frac{p-y}{\sqrt{2\tau}} = q$, we get:

$$v(y, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_0(q\sqrt{2\tau} + y) e^{\frac{-q^2}{2}} dq \ .$$

According to (9), the integral is:

$$v(y, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{y}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(q\sqrt{2\tau}+y)-\frac{q^2}{2}+c} (e^{\mu(T-t_0)} - 1) dq$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2+c} (e^{\mu(T-t_0)} - 1) \int_{-\frac{y}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(q+\frac{1}{2}\sqrt{2\tau})^2+\frac{\tau}{4}} dq$$

$$= \frac{1}{\sqrt{2\pi}} (e^{\mu(T-t_0)} - 1)\, e^{-\frac{1}{2}y+\frac{1}{4}\tau+c} \int_{-\frac{y}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(q+\frac{1}{2}\sqrt{2\tau})^2} dq$$

$$= \frac{1}{\sqrt{2\pi}} (e^{\mu(T-t_0)} - 1)\, e^{-\frac{1}{2}y+\frac{1}{4}\tau+c} \int_{-\frac{y}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}r^2} dr$$

$$= \frac{1}{\sqrt{2\pi}} (e^{\mu(T-t_0)} - 1)\, e^{-\frac{1}{2}y+\frac{1}{4}\tau+c} \int_{-\infty}^{d} e^{-\frac{1}{2}r^2} dr \ .$$

So,

$$v(y, \tau) = e^{-\frac{1}{2}y+\frac{1}{4}\tau+c} \cdot (e^{\mu(T-t_0)} - 1) \cdot N(d),$$

where $d = \frac{y}{\sqrt{2\tau}} - \frac{1}{2}\sqrt{2\tau}$, and N is the function of normal distribution.
Hence, from (7)

$$u(y, \tau) = e^{\frac{1}{2}y-\frac{1}{4}\tau} \cdot e^{-\frac{1}{2}y+\frac{1}{4}\tau+c} (e^{\mu(T-t_0)} - 1) \cdot N(d) = e^{c} (e^{\mu(T-t_0)} - 1) \cdot N(d) .$$

where $d = \frac{\ln x - \sigma^2 (T-t)/2}{\sigma\sqrt{T-t}}$. Finally, we get the explicit expression for $S(x, t)$ in time T:

$$S(x, T) = x_0 \cdot (e^{\mu(T-t_0)} - 1) \cdot N(d) , \tag{10}$$

which represents the maximal number of wood allowed to cut in time T. The value of $N(d)$ usually equals to one. In the case of very small x, it is less than one.

Note. If $T - t = 0$, then $N(d) = 1$, and the formulas (6) and (10) are the same.

6. EXAMPLES

Based on statistical data measured in the Management Unit "Tara" - National Park "Tara", Bajina Basta, Serbia, between 1960 and 1990, several times the number of felled trees amounted up to even 20% of the number of trees in the stand. Also, by comparing only the numbers of trees in the stand in a ten-year period, it was calculated that these values were statistically equal (see [3]), which leads to a conclusion that the method of selection forests management applied in the forests of Tara in the past decades was relatively successful. In other words, the initial resource of the number of trees was maintained, and the quantity selected for felling was only the percentage that accumulated during ten years, like an "interest".

Example. Let us take $x(t) = 100$, expected rate of return $\mu = 0.02$, and volatility or measure of standard deviation $\sigma = 0.2$, by year, and $T - t = 10$ years. Then,

$$d = \frac{\ln 100 - 0.2^2 \cdot 10/2}{0.2\sqrt{10}} = 6.965, \qquad N(6.965) = 1,$$

and the forecast for $S(x, T)$ by (10): $S(x, T) = 22.14$.

Example. In 1970, the number of trees in the selection stand, Compartment 51, Management Unit "Tara", was $x(t) = 557.4$. The number of trees per hectare is not integer number. On the same sample plot in 1980 the number of felled trees was 109.1, (see [3]) which is less than the maximal number of trees selected for felling, calculated by theoretical means.

Namely, if we take the expected rate of return $\mu = 0.02$, standard deviation $\sigma = 0.2$, and $T - t = 10$ years, then $d = \dfrac{\ln 557.4 - 0.2^2 \cdot 10/2}{0.2\sqrt{10}} = 9.68$, $N(9.68) = 1$, and for $S(x, T)$ we get:

$$S(x, T) = 123.41 .$$

Example. In 1960, the number of trees in the selection stand, Compartment 62, Management Unit "Tara", was $x(t) = 664.0$. On the same sample plot in 1970 the number of felled trees was 126.4, (see [3]) which is less than the maximal number of trees selected for felling, calculated by theoretical means.

Namely, if we take the expected rate of return $\mu = 0.02$, standard deviation $\sigma = 0.2$, and $T - t = 10$ years, then, $d = \dfrac{\ln 664 - 0.2^2 \cdot 10/2}{0.2\sqrt{10}} = 9.96$, $N(9.96) = 1$, and for $S(x, T)$ we get:

$$S(x, T) = 147.01.$$

The above examples point out that the field data on the number of felled trees, correspond completely to the theoretical upper boundaries. This means that the calculated forecast (10) for $S(x,T)$ can also be applied in future for the assessment of the upper permissible limit of the number of trees for felling.

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