Yugoslav Journal of Operations Research
14 (2004), Number 2, 155-170

AIRLINE SEAT MANAGEMENT WITH ROUND-TRIP REQUESTS

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Received: February 2003 / Accepted: February 2004

Abstract: Consider a multi-period multi-fare class airline booking problem related to a two-leg airline network. Travel requests include outbound, inbound trip, and round trips. The round-trip refers to a journey comprising both outbound and inbound trips. To develop a dynamic-nested booking decision-making system for the airline network, this study designs a dynamic model that enables the airline reservations system to devise a set of dynamic decision rules for any given booking status. The booking process is found to be controlled by some set of booking thresholds.

Keywords: Airline, inventory, booking, revenue management.

1. INTRODUCTION

Since the fare deregulation of the airline industry in 1978, many airline companies have used discriminatory pricing policies in order to segment potential customers into competitively relevant groups in order to maximize revenues. A common approach is to divide a pool of identical seats in the same cabin of a flight into several fare classes through different restrictions and charge different fares (c.f. Belobaba (1987)).

In circumstances where the capacity of the aircraft is relatively fixed and cannot be changed in short notice, and the marginal cost of carrying an additional passenger proves relatively lower compared to high fixed costs incurred from passengers with reserved bookings, airline companies devise booking schemes in order to fill in vacant seats since those vacant seats upon departure time mean lost revenues.

Airline passengers can roughly be categorized into two groups: reserved passengers and go-show passengers. Reserved passengers book airline seats in advance. They have the right to board the airplane during departure time. Go-show passengers, on the other hand, appear on airline counters without reservations and can only book remaining seats after the check-in time of reserved passengers. Airline companies can raise revenues by opening and closing a variety of fare classes based on reservation status. The reservation status of customers is based on the customer's needs as well as the amount of fare they will pay.

One of the problems faced by airline companies lies in the seat inventory control. The seat inventory control determines the number of seats sold at different fare categories. In practice, customers make reservations randomly over time. Such behavior reflects the stochastic nature of airline passengers. It also prevents airline companies from predetermining future booking requests. If airline companies accept bookings of customers regardless of fare class, they may lose revenues from customers willing to pay higher fares. On the other hand, if airline companies reject most of the lower fare booking requests, they run the risk of flying with many vacant seats.

Reserved passengers who do not show up during departure time are called "no-show passengers". "No-show passengers" fail to use reserved accommodations for reasons such as missed connections and traffic tie-ups. They may also be passengers who suddenly decided to cancel their reservations prior to departure time. The occurrence of cancellations means loss of revenues for the airline companies since companies cannot immediately replace the cancelled booking with another customer.

Faced with such stochastic behavior and the necessity of filling up vacated seats, airline companies overbook flights. The challenge in overbooking flights lies in the extent to which overbooking policies should be employed. Although overbooking policies reduce the likelihood of taking off with much vacant seats, they may also lead to difficult situations when the number of reservations exceeds the available seats at the time of departure. In such cases, airline companies not only lose customers but also have to deal with the fact that they must offer some form of compensation to the customer.

In order to aid the airline's seat inventory control and overbooking policy, airline officials employ the revenue/yield management concept. Revenue management is described as the application of inventory data and pricing strategies to maximize profit from a fixed number of resources (c.f. Weatherford et al., 1992). This paper applies the revenue management concept to develop a round trip seat inventory control problem. Various models of revenue management have been proposed to determine booking policies for various types of seat inventory control problems. The seat inventory control structure can be categorized into the separated structure and the nested structure.

In a separated structure, the booking period is regarded as a single interval. Airline personnel must set a booking limit for every fare class at the start of the booking process. The sum of the booking limits for every fare class must be equal to the total booking capacity of the flight. The weakness of this structure lies in the fact that requests for higher fare classes may be denied even though seats are still available in lower fare classes.

In a nested structure, requests for higher fare classes can be accommodated if seats are available in lower fare classes. The nested structure can be further divided into two: the static nested structure and the dynamic nested structure. In the static nested structure, booking limits are set at the beginning of the booking period. In the dynamic nested structure, booking limits are updated during the booking period depending on the actual booking status.

Various approaches have been proposed to set booking limits. Among them is the one-leg based method. This method controls each flight leg independently. It has the advantage of developing dynamic nested booking control policies. Another approach is the two-leg based method. This method controls two flight legs and generates dynamic nested booking control policies. Generally, this method generates higher revenues. Its revenues are higher than the sum of revenues from independent flight leg bookings. This is also based on the fact that itineraries are composed mostly of one-stop city flights.

The approach that normally produces mathematical programming models is the network-based method. In order to ensure maximum revenues, airline companies must tackle the entire network. Through airline networks, airline companies offer numerous original-destination-fare classes to customers and flight itineraries may comprise multiple-stop city flights. However, such an approach may result in the inability of airline companies to develop dynamic-nested booking policies since the combination of products strictly increases with the number of flight legs.

Airline seat inventory control is a form of revenue management. The literature contains several wonderful introductions to the airline revenue management problems (e.g., Belobaba, 1988; Lautenbacher et al., 1999; McGill et al., 1999;Weatherford et al., 1992). Additionally, various models have been proposed to determine booking policy for di®erent types of seat inventory control problems (e.g. Alstrup et al., 1986; , Belobaba, 1989; Brumelle, 1990; Brumelle, 1993; Curry, 1990; Gerchak et al., 1985; Lee et al., 1993; Littlewood, 1972; Robinson, 1995; Wollmer, 1992; You, 1999; You, 2001).

Littlewood (1972) was the first to consider the revenue management problem, using the marginal seat revenue approach to optimize booking policy for a single leg problem with two-fare classes. Meanwhile, Belobaba (1989) built upon Littlewood's work and proposed a general model with multiple fare classes, assuming that the booking process to follow the pattern of lower before higher, that is customers requesting lower value fare was assumed to be booked before customers requesting the higher value fare. By the same assumption, Curry (1990) developed a multiple fare class model using the mathematical programming approach. Meanwhile, Wollmer (1992) addressed a single leg multi-fare class model and introduced an algorithm for optimizing the booking policy. Brumelle, McGill, Oum, Sawaki, and Tretheway (1990) dealt with a multiple fare class problem by formulating a revenue function for both discrete and continuous probability distributions of demand, and the conditions of what exhibited a concave revenue function. Many previous researchers assume a lower before higher pattern in the booking process (e.g., Belobaba, 1989; Brumelle et al., 1993; Curry, 1990; Wollmer, 1992). However, this assumption is not always valid. Robinson (1995) considered a relatively general case in which the customers of any given fare class remain clustered but the order of such clusters may not match that of the increasing fares.

The airlines expect to have a functions in the Computer-Reservation-System which is the function of revising the booking decision based on the actual booking status. Thus, research on setting the dynamic-nested booking strategy can not be ignored. Using the dynamic approach, Gerchak, Parlar and Yee (1985) develop a dynamic-nested booking strategy on a single leg two-fare class model. An important outcome of this study (1985) is that booking policy parameters can be reduced to two types of critical values: critical booking capacity and critical decision periods, and these values are important in reducing computational time and eliminating the need for extensive data storage. Building upon Gerchak's work, Lee, and Hersh (1993) developed a dynamic model for a single flight with multiple fare classes and multiple seat bookings.

The single leg airline seat inventory model above can be applied to determine a booking strategy for each trip in an airline network. However, passengers may simultaneously request multiple flight legs across an airline network. Thus, to maximize revenues the booking strategy for all trips in an airline network must be set simultaneously. Notably, large airline network booking controls are usually ineffective owing to computational barrier, data overflow, and so on. Therefore, current computer technology makes developing small airline network booking control systems preferable to building larger but ineffective systems. This study attempts to develop a booking policy for a two-leg airline network comprising outbound and inbound legs. The travel requests include outbound trip, inbound trip and round trips, where round trip refers to a journey involving outbound and inbound trips.

Developing a means of allowing round trip requests is the key difference between this paper and previous ones. The problem is solved by using the dynamic approach to create a two leg airline seat inventory control model in which demands are modelled as a stochastic process. The proposed dynamic model sets the booking policy for each booking class according to the actual bookings throughout the entire booking process.

This work aims to maximize expected revenue. It is found herein that booking policy can be reduced to a set of critical values, including the following information: which fare classes should be opened for sale within each trip (that is, whether to accept a request for a fare class in each trip).

The rest of the paper is organized as follows: Section 2 outlines all assumptions made and formulate the problem as a dynamic programming model. Section 3 analyze the novel model and determine the optimal booking policy. The analysis reveals that the booking policy can be controlled by using a set of critical booking values, called booking limit. Section 4 then demonstrates the properties of the novel model using a numerical example, and finally, Section 6 presents conclusions.

2. MODEL ASSUMPTIONS AND FORMULATION

Suppose an airline company has the right to fly passengers between two cities and is permitted to sell tickets to customers requesting an outbound trip from city A to city B, an inbound trip from city B to city A, or a round trip that includes both outbound trip and inbound trips. Furthermore, assume that the airline tries to develop an optimal booking policy for two scheduled flights, including an outbound flight with a booking capacity of I_1 and departure time of t_o , and an inbound flight with a booking capacity of I_2 and departure time of $t_i > t_o$. The outbound, inbound, and round trips are denoted by $j = 1$, $j = 2$ and $j = 3$, respectively.

Suppose the airline divides the seats on trip j into L^j fare classes and prices tickets at x_l^j for fare class l in trip j. Herein, the fare classes in each trip j are classified into ordered types l, $l \in \{1, 2, ..., L\}$, where fare class 1 is the most expensive and L^j is the least expensive.

For convenience, the total planning horizon is divided into T decision periods which are sufficiently small that no more than one customer arrives during each period. Additionally, the periods are counted in reverse time sequence and it is assumed that the departure times of the outbound and inbound flights are at the end of periods t_1 and 1, respectively. Let i_1, i_2 and $i_2 = \min\{i_1, i_2\}$ denote the seats available on the outbound, inbound and round trips, respectively. Let λ_{tl}^j represent the probability that a request for fare class l in trip j will arrive during decision period t with $\sum_{l=1}^{3}\sum_{l=1}^{L_j}\lambda_{tl}^j \le 1$. Meanwhile, let $\mathbf{i} = (i_1, i_2)$ and let $v_t(\mathbf{i})$ denote the maximum total expected revenue that can be generated within t periods when i seats remain. Eq. (1) is then produced

$$v_0(\mathbf{i}) = 0. \tag{1}$$

Notable, t has also been used to represent the number of periods remaining before departure time. Let $I_1 = (1,0)$, $I_2 = (0,1)$ and $I_3 = (1,1)$. Then, Eq. (2)

$$\begin{aligned} v_t(\mathbf{i}) = {} & (1 - \sum_{l=1}^{L^2}\lambda_{tl}^2 + \sum_{l=1}^{L^2}\lambda_{tl}^2 I(i_2 = 0))v_{t-1}(\mathbf{i}) \\ & + \sum_{l=1}^{L^2}\lambda_{tl}^2 \max\{v_{t-1}(\mathbf{i}), x_l^2 + v_{t-1}(\mathbf{i} - I_2)\}\, I(i_2 \ge 1) \end{aligned} \tag{2}$$

exists for $0 < t \le t_1$, while Eq. (3)

$$\begin{aligned} v_t(\mathbf{i}) = {} & (1 - \sum_{j=1}^{3}\sum_{l=1}^{L^j}\lambda_{tl}^j + \sum_{j=1}^{3}\sum_{l=1}^{L^j}\lambda_{tl}^j I(i_j = 0))v_{t-1}(\mathbf{i}) \\ & + \sum_{j=1}^{3}\sum_{l=1}^{L^j}\lambda_{tl}^j \max\{v_{t-1}(\mathbf{i}), x_l^j + v_{t-1}(\mathbf{i} - I_j)\}\, I(i_j \ge 1) \end{aligned} \tag{3}$$

exists for $t > t_1$.

3. DECISION ANALYSIS

This section analyzes the novel model and develops the optimal booking policy. A policy is termed a booking-limit policy when a request for a certain class is accepted if and only if the total number of reservations immediately preceding reservation requests is less than the booking-limit value for that class. The following demonstrates that the optimal booking policy is also a booking-limit policy. First, Eqs. (2) and (3) are rewritten in the following simple form.

$$v_t(\mathbf{i}) = \begin{cases} v_{t-1}(\mathbf{i}) + \sum_{j=1}^{3}\sum_{l=1}^{L^j}\lambda_{tl}^j \max\{0, x_l^j - z_{t-1}^j(\mathbf{i})\}\, I(i_j \ge 1), & t \ge t_1 \\ v_{t-1}(\mathbf{i}) + \sum_{l=1}^{L^j}\lambda_{tl}^j \max\{0, x_l^2 - z_{t-1}^2(\mathbf{i})\}\, I(i_2 \ge 1), & t < t_1 \end{cases} \tag{4}$$

where

$$z_t^j(\mathbf{i}) = v_t(\mathbf{i}) - v_t(\mathbf{i} - I_j)\ . \tag{5}$$

From Eq. (4), booking policy is clearly dependent on the value $z_t^j(\mathbf{i})$, and the following lemmas are needed. To derive the property of $z_t^j(\mathbf{i})$, we need the following lemmas.

Lemma 3.1. *Suppose (i)* $z_t^j(\mathbf{i})$ *is nonincreasing in* i_1 *and (ii)* $z_t^2(\mathbf{i})$ *is nonincreasing in* i_2*, then*

(a) $z_t^1(\mathbf{i})$ *is nondecreasing in* i_2*,*

(b) $z_t^2(\mathbf{i})$ *is nondecreasing in* i_1.

Proof: Statement (a) is equivalent to statement (b) since $z_t^1(i_1,i_2) - z_t^1(i_1,i_2-1)$ $= z_t^2(i_1,i_2) - z_t^2(i_1-1,i_2)$. Consequently, it is only necessary to verify that $z^1(\mathbf{i})$ satisfies the inequality $z_t^2(i_1,i_2) \geq z_t^2(i_1-1,i_2)$. Checking that the inequality is valid for $t=1$ according to Eq. (4). Suppose that the assertion holds for some $t \geq 2$. (4) is used to express $z_t^2(i_1,i_2) - z_t^2(i_1-1,i_2)$ as

$$\begin{aligned}
&z_t^2(i_1,i_2) - z_t^2(i_1-1,i_2)\\
&= z_{t-1}^2(i_1,i_2) - z_{t-1}^2(i_1-1,i_2)\\
&+\sum_{l=1}^{L^1} \lambda_{tl}^1(\max\{0, x_l^1 - z_{t-1}^1(i_1,i_2)\} - \max\{0, x_l^1 - z_{t-1}^1(i_1,i_2-1)\}\\
&-\max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2)\} + \max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2-1)\}\\
&+\sum_{l=1}^{L^2} \lambda_{tl}^2(\max\{0, x_l^2 - z_{t-1}^2(i_1,i_2)\} - \max\{0, x_l^2 - z_{t-1}^2(i_1,i_2-1)\}\\
&-\max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2)\} + \max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2-1)\}\\
&+\sum_{l=1}^{L^3} \lambda_{tl}^3(\max\{0, x_l^3 - z_{t-1}^3(i_1,i_2)\} - \max\{0, x_l^3 - z_{t-1}^3(i_1,i_2-1)\}\\
&-\max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2)\} + \max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2-1)\}.
\end{aligned} \tag{6}$$

Lemma 3.1 produces inequalities

$$\begin{aligned}
&\max\{0, x_l^1 - z_{t-1}^1(i_1,i_2)\} - \max\{0, x_l^1 - z_{t-1}^1(i_1,i_2-1)\}\\
&-\max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2)\} + \max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2-1)\}\\
&\geq \max\{0, x_l^1 - z_{t-1}^1(i_1,i_2)\} - \max\{0, x_l^1 - z_{t-1}^1(i_1,i_2-1)\}\\
&\geq z_{t-1}^1(i_1,i_2-1) - z_{t-1}^1(i_1,i_2) = z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2)
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\max\{0, x_l^2 - z_{t-1}^2(i_1,i_2)\} - \max\{0, x_l^2 - z_{t-1}^2(i_1,i_2-1)\} \\
&-\max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2)\} + \max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2-1)\} \\
&\geq \max\{0, x_l^2 - z_{t-1}^2(i_1,i_2)\} - \max\{0, x_l^2 - z_{t-1}^2(i_1,i_2-1)\} \\
&\geq z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2),
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\max\{0, x_l^3 - z_{t-1}^3(i_1,i_2)\} - \max\{0, x_l^3 - z_{t-1}^3(i_1,i_2-1)\} \\
&-\max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2)\} + \max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2-1)\} \\
&= \begin{cases}
\begin{aligned}&-z_{t-1}^3(i_1,i_2) + z_{t-1}^3(i_1,i_2-1) + z_{t-1}^3(i_1-1,i_2) - z_{t-1}^3(i_1-1,i_2-1) \\
&= z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2) + z_{t-1}^1(i_1-1,i_2-1) - z_{t-1}^1(i_1-1,i_2-2) \\
&\geq z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2) \\
&\qquad\qquad\qquad\qquad \text{if } x_l^3 \geq z_{t-1}^3(i_1-1,i_2-1)\end{aligned} \\[2ex]
\begin{aligned}&-z_{t-1}^3(i_1,i_2) + z_{t-1}^3(i_1,i_2-1) \\
&= z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2) + z_{t-1}^2(i_1-1,i_2-1) - z_{t-1}^2(i_1-1,i_2) \\
&\geq z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2) \\
&\qquad\qquad \text{if } x_l^3 \geq z_{t-1}^3(i_1,i_2-1) \text{ and } x_l^3 \leq z_{t-1}^3(i_1-1,i_2)\end{aligned} \\[2ex]
\begin{aligned}&-z_{t-1}^3(i_1,i_2) + z_{t-1}^3(i_1-1,i_2) \\
&= z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2) + z_{t-1}^1(i_1-1,i_2-1) - z_{t-1}^1(i_1,i_2-1) \\
&\geq z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2) \\
&\qquad\qquad \text{if } x_l^3 \leq z_{t-1}^3(i_1,i_2-1) \text{ and } x_l^3 \geq z_{t-1}^3(i_1-1,i_2)\end{aligned} \\[2ex]
\geq \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.}
\end{cases}
\end{aligned} \tag{9}$$

Substituting inequalities Eqs. (7)-(9) into (6) obtains

$$z_t^2(i_1,i_2) - z_t^2(i_1-1,i_2) \geq (1 - \sum_{j=1}^{3}\sum_{l=1}^{L^j} \lambda_{tl}^j)(z_{t-1}^2(i_1,i_2) - z_{t-1}^2(i_1-1,i_2) \geq 0 \tag{10}$$

which completes the proof. ♦

The following important theory can now be confirmed.

Theorem 3.1.

(*a*) $z_t^1(\mathbf{i})$ *is nondecreasing in* i_1,

(*b*) $z_t^2(\mathbf{i})$ *is nondecreasing in* i_2,

(*c*) $z_t^1(\mathbf{i})$ *is nonincreasing in* i_2,

(*d*) $z_t^2(\mathbf{i})$ *is nonincreasing in* i_1,

Proof: Statement (a) is proved first. Proving (a) requires first verifying that $z_t^1(\mathbf{i})$ satisfies the inequality $z_t^1(i_1,i_2) \le z_t^1(i_1-1,i_2)$. Notably, the statement for $t=1$ is immediately derived from Eq. (4). Including an inequality in the inductive hypothesis used to establish the general formula is helpful. For $t \ge 2$, Eq. (4) and Lemma 5, we have

$$
\begin{aligned}
& z_t^1(i_1,i_2) - z_t^1(i_1-1,i_2) \\
& = z_{t-1}^1(i_1,i_2) - z_{t-1}^1(i_1-1,i_2) \\
& + \sum_{l=1}^{L^1} \lambda_{tl}^1 (\max\{0, x_l^1 - z_{t-1}^1(i_1,i_2)\} - \max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2)\} \\
& - \max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2)\} + \max\{0, x_l^1 - z_{t-1}^1(i_1-2,i_2)\} \\
& + \sum_{l=1}^{L^2} \lambda_{tl}^2 (\max\{0, x_l^2 - z_{t-1}^2(i_1,i_2)\} - \max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2)\} \\
& - \max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2)\} + \max\{0, x_l^2 - z_{t-1}^2(i_1-2,i_2)\} \\
& + \sum_{l=1}^{L^3} \lambda_{tl}^3 (\max\{0, x_l^3 - z_{t-1}^3(i_1,i_2)\} - \max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2)\} \\
& - \max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2)\} + \max\{0, x_l^3 - z_{t-1}^3(i_1-2,i_2)\} \\
& \le 0
\end{aligned}
\tag{11}
$$

since

$$
\begin{aligned}
& \max\{0, x_l^1 - z_{t-1}^1(i_1,i_2)\} - \max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2)\} \\
& - \max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2)\} + \max\{0, x_l^1 - z_{t-1}^1(i_1-2,i_2)\} \\
& \le \max\{0, x_l^1 - z_{t-1}^1(i_1,i_2)\} - \max\{0, x_l^1 - z_{t-1}^1(i_1-1,i_2)\} \\
& \le z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2),
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
& \max\{0, x_l^2 - z_{t-1}^2(i_1,i_2)\} - \max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2)\} \\
& - \max\{0, x_l^2 - z_{t-1}^2(i_1-1,i_2)\} + \max\{0, x_l^2 - z_{t-1}^2(i_1-2,i_2)\} \\
& \begin{cases}
= z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1,i_2) + z_{t-1}^2(i_1-1,i_2) - z_{t-1}^2(i_1-2,i_2) \\
= z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2) + z_{t-1}^1(i_1,i_2-1) - z_{t-1}^1(i_1-1,i_2-1) \\
\qquad\qquad\qquad\qquad \text{if } x_l^2 \ge z_{t-1}^2(i_1,i_2) \\
\le z_{t-1}^2(i_1-2,i_2) + z_{t-1}^2(i_1-1,i_2) \\
= z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2) + z_{t-1}^3(i_1,i_2) - z_{t-1}^3(i_1-1,i_2) \\
\le z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2) \\
\qquad\qquad\qquad\qquad \text{if } x_l^2 \le z_{t-1}^2(i_1,i_2),
\end{cases}
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&\max\{0, x_l^3 - z_{t-1}^3(i_1,i_2)\} - \max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2)\} \\
&-\max\{0, x_l^3 - z_{t-1}^3(i_1-1,i_2)\} + \max\{0, x_l^3 - z_{t-1}^3(i_1-2,i_2)\} \\
&\begin{cases}
-z_{t-1}^3(i_1,i_2) + z_{t-1}^3(i_1-1,i_2) + z_{t-1}^3(i_1-1,i_2) - z_{t-1}^3(i_1-2,i_2) \\
= z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2) + z_{t-1}^1(i_1-1,i_2-1) - z_{t-1}^1(i_1-2,i_2-1) \\
\le z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2) \\
\qquad\qquad\qquad\qquad \text{if } x_l^3 \ge z_{t-1}^3(i_1-2,i_2) \\
\\
-z_{t-1}^3(i_1,i_2) + z_{t-1}^3(i_1-1,i_2) \\
= z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2) + z_{t-1}^1(i_1-1,i_2-1) - z_{t-1}^1(i_1-1,i_2) \\
\le z_{t-1}^1(i_1-1,i_2) - z_{t-1}^1(i_1,i_2) \\
\qquad\qquad\qquad\qquad \text{if } x_l^3 \le z_{t-1}^3(i_1-2,i_2).
\end{cases}
\end{aligned}
\tag{14}
$$

A similar approach can be used to demonstrate that $z_t^2(i_1,i_2) \le z_t^2(i_1,i_2-1)$, and thus statement (b) holds. Since statements (a) and (b) hold, assertions (c) and (d) follow from Lemma 3.1. ♦

Theorem 3.1(a) and (b) imply that some critical booking capacities $\{m_{tl}^1(i_2)\}$ and $\{m_{tl}^2(i_1)\}$ exist such that $x_l^1 \ge z_{t-1}^1(\mathbf{i})$ and $x_l^1 \ge z_{t-1}^2(\mathbf{i})$ if and only if $i \ge m_{tl}^1(i_2)$ and $i \ge m_{tl}^2(i_1)$. Consequently, the booking limit policy is the optimal booking policy for trips 1 and 2.

- *Booking-limit policy for trips 1 and 2:*

I. A request for fare class l in trip 1 and period t should be accepted if and only if $i_1 \le \{m_t^l(1)\}$ (Theorem 3.1(c));

II. A request for fare class l in trip 1 and period t should be accepted if and only if $i_2 \le \{m_t^l(2)\}$ (Theorem 3.1(d)).

Theorem 3.2. $z_t^3(\mathbf{i})$ *is nonincreasing in* i_1 *and* i_2

Proof: The proof will be completed by demonstrating that $z_t^3(i_1,i_2) - z_t^3(i_1-1,i_2) \le 0$ and $z_t^3(i_1,i_2) - z_t^3(i_1,i_2-1) \le 0$. From Eq. (4) we have

$$
\begin{aligned}
&z_t^3(i_1,i_2)-z_t^3(i_1-1,i_2)\\
&=z_{t-1}^3(i_1,i_2)-z_{t-1}^3(i_1-1,i_2)\\
&+\sum_{l=1}^{L^1}\lambda_{tl}^1(\max\{0,x_l^1-z_{t-1}^1(i_1,i_2)\}-\max\{0,x_l^1-z_{t-1}^1(i_1-1,i_2-1)\}\\
&-\max\{0,x_l^1-z_{t-1}^1(i_1-1,i_2)\}+\max\{0,x_l^1-z_{t-1}^1(i_1-2,i_2-1)\}\\
&+\sum_{l=1}^{L^2}\lambda_{tl}^2(\max\{0,x_l^2-z_{t-1}^2(i_1,i_2)\}-\max\{0,x_l^2-z_{t-1}^2(i_1-1,i_2-1)\}\\
&-\max\{0,x_l^2-z_{t-1}^2(i_1-1,i_2)\}+\max\{0,x_l^2-z_{t-1}^2(i_1-2,i_2-1)\}\\
&+\sum_{l=1}^{L^3}\lambda_{tl}^3(\max\{0,x_l^3-z_{t-1}^3(i_1,i_2)\}-\max\{0,x_l^3-z_{t-1}^3(i_1-1,i_2-1)\}\\
&-\max\{0,x_l^3-z_{t-1}^3(i_1-1,i_2)\}+\max\{0,x_l^3-z_{t-1}^3(i_1-2,i_2-1)\}
\end{aligned}
\tag{15}
$$

while from inequalities

$$
\begin{aligned}
&\max\{0,x_l^1-z_{t-1}^1(i_1,i_2)\}-\max\{0,x_l^1-z_{t-1}^1(i_1-1,i_2-1)\}\\
&-\max\{0,x_l^1-z_{t-1}^1(i_1-1,i_2)\}+\max\{0,x_l^1-z_{t-1}^1(i_1-2,i_2-1)\}\\
&\le\max\{0,x_l^1-z_{t-1}^1(i_1,i_2)\}-\max\{0,x_l^1-z_{t-1}^1(i_1-1,i_2)\}\\
&\le z_{t-1}^1(i_1-1,i_2)-z_{t-1}^1(i_1,i_2)\\
&=z_{t-1}^3(i_1-1,i_2)-z_{t-1}^3(i_1,i_2)+z_{t-1}^1(i_1-1,i_2)-z_{t-1}^1(i_1-1,i_2-1)\\
&\le z_{t-1}^3(i_1-1,i_2)-z_{t-1}^3(i_1,i_2),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&\max\{0,x_l^2-z_{t-1}^2(i_1,i_2)\}-\max\{0,x_l^2-z_{t-1}^2(i_1-1,i_2-1)\}\\
&-\max\{0,x_l^2-z_{t-1}^2(i_1-1,i_2)\}+\max\{0,x_l^2-z_{t-1}^2(i_1-2,i_2-1)\}\\
&\begin{cases}
=-z_{t-1}^2(i_1,i_2)+z_{t-1}^2(i_1-1,i_2-1)+z_{t-1}^2(i_1-1,i_2)-z_{t-1}^2(i_1-2,i_2-1)\\
=z_{t-1}^3(i_1-1,i_2)-z_{t-1}^3(i_1,i_2)+z_{t-1}^3(i_1,i_2-1)-z_{t-1}^3(i_1-1,i_2-1)\\
\le z_{t-1}^3(i_1-1,i_2)-z_{t-1}^3(i_1,i_2)\\
\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_l^2\ge z_{t-1}^2(i_1-2,i_2-1)\\
\\
\le z_{t-1}^2(i_1,i_2)+z_{t-1}^2(i_1-1,i_2)\\
\le z_{t-1}^3(i_1-1,i_2)-z_{t-1}^3(i_1,i_2)+z_{t-1}^1(i_1,i_2-1)-z_{t-1}^1(i_1-1,i_2-1)\\
\le z_{t-1}^3(i_1-1,i_2)-z_{t-1}^3(i_1,i_2)\\
\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_l^2\le z_{t-1}^2(i_1-2,i_2-1),
\end{cases}
\end{aligned}
\tag{17}
$$

and

$$\begin{aligned}&\max\{0, x_l^3 - z_{t-1}^3(i_1, i_2)\} - \max\{0, x_l^3 - z_{t-1}^3(i_1 - 1, i_2 - 1)\}\\&- \max\{0, x_l^3 - z_{t-1}^3(i_1 - 1, i_2)\} + \max\{0, x_l^3 - z_{t-1}^3(i_1 - 2, i_2 - 1)\}\\&\le z_{t-1}^3(i_1 - 1, i_2) - z_{t-1}^3(i_1, i_2),\end{aligned} \tag{18}$$

we have inequality

$$z_t^3(i_1, i_2) - z_t^3(i_1 - 1, i_2) \le (1 - \sum_{j=1}^{3}\sum_{l=1}^{L^j} \lambda_{tl}^j)(z_{t-1}^3(i_1, i_2) - z_{t-1}^3(i_1 - 1, i_2). \tag{19}$$

Thus, $z_t^3(\mathbf{i})$ is nonincreasing in i_1 . Similarly, it can be shown that $z_{t-1}^3(i_1, i_2)$ $\le z_{t-1}^3(i_1, i_2 - 1)$, thus completing the proof. ♦

Theorem 3.2 implies that some critical booking capacities $\{m_{tl}^3(i_1)\}$ $(\{m_{tl}^3(i_2)\})$ exist such that $x_l^1 \ge z_{t-1}^3(\mathbf{i})$ if and only if $i \ge m_{tl}^3(i_1)$ $(i \ge m_{tl}^3(i_2))$. Thus, the optimal booking policy for trip 3 is thus also a booking-limit policy.

- Booking-limit policy for trip 3:

I. A request for fare class l in trip 3 in period t should be accepted if and only if $i_1 \le \{m_t^l(i_2)\}$ (Theorem 3.2); or

II. A request for fare class l in trip 3 in period t should be accepted if and only if $i_2 \le \{m_t^l(i_1)\}$ (Theorem 3.2).

Theorem 3.3. $z_t^3(\mathbf{i})$ *is nondecreasing in t.*

Proof: From Eq. (4) we have

$$\begin{aligned}z_t^3(\mathbf{i}) - z_{t-1}(\mathbf{i}) = \sum_{j=1}^{3}\sum_{l=1}^{L^j} \lambda_{tl}^j \max\{0, x_l^j - z_{t-1}^j(i_1, i_2)\}\ I(i_j \ge 1)\\ - \max\{0, x_l^j - z_{t-1}^j(i_1 - 1, i_2 - 1)\}\ I(i_j \ge 2),\end{aligned} \tag{20}$$

while Lemma 3.1 produces inequality $\max\{0, x_l^1 - z_{t-1}^1(i_1, i_2)\}\ I(i_j \ge 1) - \max\{0, x_l^1 - -z_{t-1}^1(i_1 - 1, i_2 - 1)\} \ge 0$ due to $z_{t-1}^1(i_1, i_2) - z_{t-1}^1(i_1 - 1, i_2 - 1) = z_{t-1}^3(i_1, i_2) - z_{t-1}^3(i_1 - 1, i_2) \le 0;$ $\max\{0, x_l^2 - z_{t-1}^2(i_1, i_2)\}\ I(i_j \ge 1) - \max\{0, x_l^2 - z_{t-1}^2(i_1 - 1, i_2 - 1)\} \ge 0$ due to $z_{t-1}^2(i_1, i_2) - - z_{t-1}^2(i_1 - 1, i_2 - 1) = z_{t-1}^3(i_1, i_2) - z_{t-1}^3(i_1, i_2 - 1) \le 0;$ and $\max\{0, x_l^3 - z_{t-1}^3(i_1, i_2)\}\ I(i_j \ge 1) - - \max\{0, x_l^3 - z_{t-1}^3(i_1 - 1, i_2 - 1)\} \ge 0$ due to $z_{t-1}^3(i_1, i_2) - z_{t-1}^3(i_1 - 1, i_2 - 1) = z_{t-1}^3(i_1, i_2) - -z_{t-1}^3(i_1 - 1, i_2 - 1) \le 0,$ so the right hand side of Eq. (20) is not less than 0. Hence, $z_t^3(\mathbf{i}) - z_{t-1}(\mathbf{i}) \ge 0$ as required. ♦

Theorem 3.3 implies that the dynamic-nested booking-limit for trip 3 is a piecewise-constant functions of the time to flight departure. Data storage can thus be reduced by storing only the critical-booking-period $\{\sigma_l^j(i_1, i_2)\}$.

4. NUMERICAL EXAMPLE

To illustrate the proposed model and booking policies, an example is given below. Assume an outbound flight is departing after T = 450 periods and an inbound flight is be departing after T = 500 periods and that the maximum booking capacity for the outbound and inbound flights is $I_1 = 100$ and $I_2 = 100$, respectively. Furthermore, assume that the airline has previously specified L = 4 fare classes for both flights with corresponding ticket prices x_l^j and arriving probabilities λ_t^l as listed in Tables I and II, respectively.

Table I: the revenues x_l^j

	l			
j	1	2	3	4
1	300	200	150	100
2	500	400	350	300
3	700	550	450	350

Table II: Request Probabilities λ_{tl}^j

	j = 1				*j* = 2				*j* = 3			
t / *l*	1	2	3	4	1	2	3	4	1	2	3	4
1:100	0.02	0.03	0.08	0.03	0.03	0.10	0.06	0.03	0.03	0.06	0.05	0.03
101:200	0.03	0.08	0.07	0.06	0.05	0.09	0.08	0.02	0.04	0.08	0.07	0.02
201:300	0.06	0.06	0.05	0.05	0.06	0.08	0.05	0.05	0.03	0.08	0.07	0.05
301:400	0.02	0.05	0.02	0.06	0.07	0.05	0.04	0.06	0.07	0.05	0.04	0.06
401:500	0.07	0.06	0.02	0.05	0.08	0.04	0.02	0.07	0.08	0.04	0.02	0.07

Tables III, IV and V respectively list some booking limit values for trips 1, 2 and 3 given t = 300. Table III displays the values for $i_2 = 50$ and t = 300. Application of the Tables can be interpreted as follows.

Table III Booking limit for trip 1

i_2	l = 1	2	3	4
0	0	26	56	90
1	0	26	57	91
2	0	26	58	91
3	0	26	47	67
4	0	26	48	68
5	0	27	49	69
6	0	27	50	70
7	0	27	51	71
8	1	28	52	72
9	1	28	53	73
10	1	29	54	74
11	1	29	55	75
12	1	30	56	76
13	1	30	57	77
14	1	31	58	78
15	1	31	59	79
16	1	32	60	80
17	1	32	61	81
18	1	32	62	81
19	1	33	63	82
20	1	33	63	83
21	1	34	64	84
22	1	34	65	85
23	1	35	65	85
24	1	35	65	86
25	1	35	66	86
26	1	36	66	86
27	1	36	66	87
28	1	37	66	87
29	1	37	66	87
30	1	38	66	87
31	2	38	66	87
32	2	39	66	87
33	2	39	66	87
34	2	40	66	87
35	3	40	66	87
36	3	41	66	87
37	3	41	66	87
38	4	41	66	87
39	4	42	66	87
40	5	42	66	87
41	5	43	67	87
42	6	43	67	87
43	6	43	67	88
44	7	44	67	88
45	7	44	67	88
46	8	44	67	89
47	9	45	68	89
48	10	45	68	89
49	11	45	68	90
50	12	45	68	91

Table IV Booking period for trip 2

i_1	l = 1	2	3	4
0	0	25	47	59
1	0	25	48	60
2	0	25	49	61
3	1	26	50	62
4	1	26	50	63
5	1	26	51	64
6	1	27	52	65
7	1	27	53	66
8	2	28	53	67
9	2	28	54	67
10	2	29	54	68
11	2	29	54	68
12	3	30	54	68
13	3	30	54	68
14	4	31	54	69
15	4	31	54	69
16	5	31	55	69
17	5	32	55	69
18	6	32	55	70
19	6	33	55	70
20	7	33	56	71
21	7	33	56	72
22	8	33	57	72
23	8	33	57	73
24	8	33	58	74
25	9	33	58	75
26	9	33	59	76
27	10	33	59	77
28	10	33	60	78
29	10	34	60	79
30	11	34	61	80
31	11	34	62	81
32	11	34	62	82
33	11	34	63	82
34	12	34	64	83
35	12	34	65	84
36	12	35	65	84
37	12	35	66	85
38	12	35	66	85
39	12	35	67	86
40	12	36	67	86
41	12	36	68	86
42	13	37	68	87
43	13	37	69	87
44	13	38	69	87
45	13	38	70	87
46	13	39	70	87
47	13	40	71	87
48	13	40	71	87
49	13	41	71	88
50	13	42	72	88

Table V Booking limit for trip 3

i_1	l = 1	2	3	4
0	100	0	0	0
1	34	100	100	100
2	26	100	100	100
3	23	100	100	100
4	21	100	100	100
5	20	100	100	100
6	19	100	100	100
7	19	100	100	100
8	18	100	100	100
9	18	100	100	100
10	17	100	100	100
11	17	100	100	100
12	16	100	100	100
13	15	100	100	100
14	14	100	100	100
15	13	100	100	100
16	13	100	100	100
17	12	80	100	100
18	11	76	100	100
19	10	74	100	100
20	10	73	100	100
21	9	73	100	100
22	8	72	100	100
23	7	72	100	100
24	7	71	100	100
25	6	71	100	100
26	4	70	100	100
27	3	70	100	100
28	2	69	100	100
29	2	67	100	100
30	1	65	100	100
31	1	60	100	100
32	1	57	100	100
33	1	55	100	100
34	0	54	100	100
35	0	53	100	100
36	0	52	100	100
37	0	51	100	100
38	0	50	100	100
39	0	50	100	100
40	0	49	100	100
41	0	49	100	100
42	0	48	100	100
43	0	48	100	100
44	0	48	100	100
45	0	48	100	100
46	0	47	100	100
47	0	47	100	100
48	0	46	100	100
49	0	46	100	100
50	0	45	100	100

Table VI Booking period for trip 3

i_1	l = 1	2	3	4
0	0	0	0	0
1	378	72	61	56
2	405	93	71	63
3	423	109	81	70
4	435	123	90	76
5	445	137	100	83
6	453	151	107	90
7	461	163	114	96
8	467	174	121	102
9	472	182	127	107
10	477	188	134	112
11	482	191	140	117
12	486	194	146	122
13	489	197	150	127
14	491	201	154	131
15	493	206	156	135
16	496	210	158	137
17	499	214	160	139
18	500	219	161	141
19	500	223	162	143
20	500	227	164	144
21	500	230	164	145
22	500	234	165	146
23	500	237	166	146
24	500	240	166	147
25	500	242	167	147
26	500	244	168	148
27	500	246	169	148
28	500	249	170	149
29	500	251	171	149
30	500	254	172	150
31	500	257	173	150
32	500	259	173	151
33	500	262	174	151
34	500	265	175	152
35	500	268	176	153
36	500	271	178	153
37	500	274	179	154
38	500	277	180	154
39	500	280	182	155
40	500	283	183	156
41	500	286	185	157
42	500	289	186	158
43	500	292	188	159
44	500	295	190	160
45	500	298	192	161
46	500	301	194	162
47	500	304	196	163
48	500	308	197	164
49	500	311	199	165
50	500	315	201	166

From Table III, if $i_2 = 20$ exist, a request for fare classes 1, 2, 3 and 4 in trip 1 at period 300 is accepted if and only if $i_1 > 1$, $i_1 > 33$, $i_1 > 63$ and $i_1 > 83$, respectively. From Table IV, if $i_1 = 40$ seats remain, a request for fare classes 1, 2, 3 and 4 in trip 2 at period 300 is accepted if and only if $i_2 > 12$, $i_2 > 36$, $i_2 > 67$ and $i_2 > 86$, respectively. Additionally, from Tables V, if $i_2 = 30$ seats and $t = 300$ periods are available, a request for fare classes 1, 2, 3 and 4 in trip 3 is accepted if and only if $i_1 > 1$, $i_1 > 65$, $i_1 > 100$ and $i_1 > 10$, respectively.

Besides the booking limit, the critical booking period can also be used in controlling the booking process for trip 3. Table VI lists some critical booking period values for $i_2 = 30$. If a request for fare classes 1, 2, 3 and 4 in trip 3 appears when $i_1 = 30$ and $i_2 = 40$ remain, the request should be accepted if and only if the arrival time is $t \le 500$, $t \le 254$, $t \le 172$, $t \le 150$, respectively.

In this example, since the total number of periods exceeds total booking capacities and the booking classes by 5 times, using the critical booking period to control the booking process for trip 3 is more efficient than using the booking limit policy.

5. CONCLUSION

This investigation studied a seat inventory problem for multiple-fare classes on a simple airline network comprising outbound and inbound. Numerous noteworthy models have dealt with multiple flight leg problem. However, these have rarely considered round trip requests. Since customers can request round trips in real life situations, a dynamic model was proposed herein to deal with this problem.

This study aimed to develop optimal booking decisions-making that allow an airline reservation system to make timely decisions on whether to accept or reject a request. The analytical results demonstrate that the optimal booking policy is the booking limit policy, implying that data storage can be reduced.

The booking policy is that booking for each trip can be controlled using a set of critical booking capacities. Additionally, the booking policy for the round trip can be controlled using a set of critical booking periods, with the capability of further reducing data storage.

The novel model could be extended to include overbookings, no-shows, go-shows and cancellations, and such extensions would be worthy directions for future research.

Acknowedgement: The author would like to thank reviewer for his/her helpful comments and suggestions that greatly improved this paper and the National Science Council of the Republic of China for financially supporting this research under Contract No. NSC93-2213-E-415-003-.

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