

PRODUCTION INVENTORY POLICY UNDER A DISCOUNTED CASH FLOW

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Received: October 2002 / Accepted: August 2004

Abstract: This paper presents an extended production inventory model in which the production rate at any instant depends on the demand and the inventory level. The effects of the time value of money are incorporated into the model. The demand rate is a linear function of time for the scheduling period. The proposed model can assist managers in economically controlling production systems under the condition of considering a discounted cash flow. A simple algorithm computing the optimal production-scheduling period is developed. Several particular cases of the model are briefly discussed. Through numerical example, sensitive analyses are carried out to examine the effect of the parameters. Results show that the discount rate parameter and the inventory holding cost have a significant impact on the proposed model.

Keywords: EMQ, discounted cash flow, optimal production scheduling.

1. INTRODUCTION

The standard Economic Manufacturing Quantity (EMQ) model assumes a constant and known demand rate over an infinite planning horizon. Mak [16] proposed a production lot size inventory model with a uniform demand rate over a fixed time horizon. However, most items experience a variable demand; they are varied with time. Numerous research efforts have been undertaken to extend the basic EMQ model by relaxing various assumptions so that the model conforms more closely to a real world situation. Bhunia and Maiti [2] and Goswami and Chaudhuri [9][10] relaxed the

assumption of a constant demand. They considered the inventory model of the assumptions that the demand rate changes linearly with time. Hariga [13], Bose et al. [3] and Hong et al. [15] considered the inventory model with time-proportional demand. Wee [26], Mandal and Phaudar [17] and Urban [24] discussed the inventory model with an inventory-level-dependent demand rate. In addition, Su and Lin [22] solved a production inventory model for variable demand and production. In this model, three market demand rates are addressed.

The effect of production rate is so vital in many production inventory systems that it cannot be disregarded. To incorporate the production rate, Goswami and Chaudhuri [10] developed an order-level inventory problem of time dependent deteriorating item with and without backlogged shortages in which the finite production rate is proportional to the time dependent demand rate. Balkhi and Benkherouf [1] considered a production lot size inventory model with arbitrary production and demand rate depending on the time function. Furthermore, Bhunia and Maiti [2] assumed that the production rate is a variable. They also presented inventory models in which the production rate depends on either on-hand inventory or demand. In practice, demand and inventory level may influence the production. The demand decreases (or increases) may cause the manufacturers' decision to decrease (or increase) their production as well. Besides, the production rate may either increase or decrease with the inventory level. Thus, the effect of inventory on production rate warrants further study. In the meantime, Su et al. [21] developed a production inventory model in which considers the dependence of production rate on the demand and inventory level.

Although the assumptions underlying the standard EMQ inventory models seem restrictive, the model has been widely used in practice. However, they almost neglected the time values of cash flow. In other words, the same cash amount will possess different money value at different future time. This situation generally arises in the case of inventories of highly demandable products. When market demand goes up, the customers obviously consume more, thus the managers have to increase the production and inventory. The production scheduling and periodic inventory of the items are in need of operating the system economically.

Sarker et al. [19] developed an optimal payment time under the permissible delay in payment for products with deterioration. Gurnani [12] applied the discounted cash flow (DCF) approach to the finite planning horizon model in which it is a given constant. Trippi and Lewin [23] adopted a DCF over an infinite horizon. Dohi et al. [7] proposed optimal inventory policies for an infinite time span taking account of time value that differs from Trippi and Lewin [23] and Gurnani [12]. Chung and Kim [5] also suggested that the assumption of the infinite planning horizon is not realistic and called for a new model that relaxes the assumption of the infinite planning horizon. To be more realistic, Moon and Yun [18] examined the DCF over the finite planning horizon, which is a random variable. They did not present a production inventory model under the condition of considering a DCF. Hence, the EMQ computed from the standard model would have to be extended to reflect this DCF. To eliminate the cost of holding, the firm may undertake detailed production schemes.

We extended the models of Su and Lin [22] and Bhunia and Maiti [2], allow the time value of money and incorporate a finite production rate which is proportional to both the demand rate and the inventory level. A finite time-horizon production inventory model follows the approach of Su and Lin [22] and Bhunia and Maiti [2] with a linear

time-dependent demand rate. The mathematical formula of the expected cost function is derived. Then, the optimal production scheduling period, maximum inventory level can be easily solved by using Intermediate Value Theorem method and the theory of majorization. A numerical example is given to illustrate the use of the proposed model. Finally, we also briefly discuss the sensitivity of these solutions to changes in underlying parameter values as well as the advantages of the proposed model are addressed in the conclusions.

2. ASSUMPTIONS AND NOTATIONS

The mathematical model of the production inventory problem considered in this paper is developed on the basis of the following assumptions and notations. Additional notations will be introduced later when needed.

1. A single item is considered over a prescribed period of T units of time, where $T = t_1 + t_2$; t_1 and t_2 are the durations of the production scheduling period and after the production period, respectively.
2. The demand rate $D(t)$ changes linearly with time t , i.e. $D(t) = \alpha + \beta t$, where $\alpha, \beta > 0$.
3. At time $t(0 \leq t \leq T)$, the on-hand inventory is $I(t)$.
4. Production rate, $P(t)$, at any instant depends on both the demand and the inventory level. That is at time $t(0 \leq t \leq t_1)$, $P(t) = a + bD(t) - cI(t)$, $a > 0$, $0 \leq b < 1$, and $0 \leq c < 1$.
5. Shortages are not allowed.
6. The inventory system involves only one stocking point; I_m represents the maximum inventory level.
7. A DCF approach is adopted to consider the time value of money. The discount rate r is compounded continuously, $0 \leq r \leq 1$, where the present value of a unit of cost after a time t is e^{-rt} .
8. The relevant costs are the inventory holding cost C_i per unit per time unit and the setup cost C_s per new cycle, which are all known and constant during the period T .

3. THE MATHEMATICAL MODEL

Based on the above assumptions and notations, the inventory level starts at a time $t=0$ and reaches I_m maximum level after t_1 time units have elapsed. Then the production is stopped, the stock level declines continuously and the inventory level becomes zero at time $t_1 + t_2 (= T)$. Our purpose is to find out the optimal values of t_1 , T and I_m that minimize the average cost K over the time horizon $[0, T]$.

The change in the inventory level, $dI(t)$ during a small interval of time dt is a function of the production rate $P(t)$, demand rate $D(t)$, and the remaining inventory. Thus, the expression for the differential equations governing the stock status during period $[0, T]$ can be written as

$$\frac{dI(t)}{dt} = P(t) - D(t) = a + (b-1)(\alpha + \beta t) - cI(t), \quad 0 \leq t \leq t_1, \quad (1)$$

and

$$\frac{dI(t)}{dt} = -(\alpha + \beta t), \quad t_1 \leq t \leq t_1 + t_2. \quad (2)$$

Using the boundary conditions, i.e. $I(t) = 0$ at $t = 0$; $I(t_1) = I_m$ at $t = t_1$; and $I(T) = 0$ at $t = T$. After having adjusted the constants of integration, Eqs. (1) and (2) are clearly equivalent to the following equations

$$I(t) = \frac{\int_0^t [a + (b-1)](\alpha + \beta t) e^{\int ctdt} dt}{e^{\int_0^t ctdt}}$$

$$= \frac{a}{c} [1 - e^{-ct}] + (b-1)e^{-ct} \int_0^t (\alpha + \beta t) e^{ct} dt = M [1 - e^{-ct}] + Nt, \quad 0 \leq t \leq t_1, \quad (3)$$

where $M = \frac{a + (b-1)\alpha}{c} - \frac{(b-1)\beta}{c^2}$, and $N = \frac{(b-1)\beta}{c}$,

and

$$I(t) = -\int_t^{t_1+t_2} (\alpha + \beta t) dt = \alpha(T-t) + \frac{\beta}{2}(T^2 - t^2), \quad t_1 \leq t \leq t_1 + t_2. \quad (4)$$

Again $I(t_1) = I_m$; thus

$$I_m = M [1 - e^{-ct_1}] + Nt_1 = \alpha t_2 + \frac{\beta}{2}(t_2^2 + 2t_1 t_2). \quad (5)$$

The relationship between t_1 and t_2 is defined by the equation

$$t_2 = \frac{-(\alpha + \beta t_1) + \sqrt{(\alpha + \beta t_1)^2 + 2\beta [M(1 - e^{-ct_1}) + Nt_1]}}{\beta}. \quad (6)$$

The present value of the holding cost during the period $[0, T]$ is obtained by discounting $C_i I(t)$ at a rate of r , i.e. $C_i I(t) e^{-rt}$. According to the above arguments, the present value of the holding cost can be shown as

$$\begin{aligned}
 & C_i \int_0^{t_1} I(t)e^{-rt} dt + C_i \int_{t_1}^{t_1+t_2} I(t)e^{-rt} dt \\
 &= C_i \int_0^{t_1} \left\{ M \left[1 - e^{-ct} \right] + Nt \right\} e^{-rt} dt + C_i \int_{t_1}^{t_1+t_2} \left[\alpha(T-t) + \frac{\beta}{2}(T^2 - t^2) \right] e^{-rt} dt. \quad (7)
 \end{aligned}$$

Hence, the total average cost of the inventory system is

$$\begin{aligned}
 & K = \text{setup cost} + \text{holding cost} \\
 &= \frac{C_s}{t_1+t_2} + \frac{C_i}{t_1+t_2} \left\{ \frac{M}{r} (1 - e^{-rt_1}) - \frac{M}{c+r} [1 - e^{-(c+r)t_1}] + \frac{N}{r^2} (1 - e^{-rt_1}) - \frac{Nt_1}{r} e^{-rt_1} \right\} \\
 &+ \frac{C_i}{t_1+t_2} \left\{ \frac{\alpha + \beta(t_1+t_2)}{r^2} e^{-r(t_1+t_2)} + \frac{\beta}{r^3} [e^{-r(t_1+t_2)} - e^{-rt_1}] + \frac{\alpha t_2}{r} e^{-rt_1} \right. \\
 &\left. - \frac{\alpha + \beta t_1}{r^2} e^{-rt_1} + \frac{\beta(t_2^2 + 2t_1 t_2)}{2r} e^{-rt_1} \right\}. \quad (8)
 \end{aligned}$$

4. SOLUTION PROCEDURE

The above cost function K is a function of two variables t_1 and t_2 . However, they are not independent and are related by Eq. (5). The problem is to determine the optimal value of t_1 that minimizes the total average cost K . We take the first and second derivative of K with respect to t_1 as follows:

$$\begin{aligned}
 \frac{dK}{dt_1} &= -\frac{C_s}{(t_1+t_2)^2} \left[1 + \frac{dt_2}{dt_1} \right] \\
 &+ \frac{C_i}{t_1+t_2} \left\{ (M + Nt_1)e^{-rt_1} - Me^{-(c+r)t_1} - \left[\alpha t_2 + \frac{\beta(t_2^2 + 2t_1 t_2)}{2} \right] e^{-rt_1} \right. \\
 &\left. + \left[\frac{\alpha + \beta(t_1+t_2)}{r} \right] \left(1 + \frac{dt_2}{dt_1} \right) [e^{-rt_1} - e^{-r(t_1+t_2)}] \right\} \\
 &- \frac{C_i}{(t_1+t_2)^2} \left\{ \frac{M}{r} (1 - e^{-rt_1}) - \frac{M}{c+r} [1 - e^{-(c+r)t_1}] + \frac{N}{r^2} (1 - e^{-rt_1}) \right. \\
 &- \frac{Nt_1}{r} e^{-rt_1} + \frac{\alpha + \beta(t_1+t_2)}{r^2} e^{-r(t_1+t_2)} + \frac{\beta}{r^3} [e^{-r(t_1+t_2)} - e^{-rt_1}] \\
 &\left. + \frac{\alpha t_2}{r} e^{-rt_1} - \frac{\alpha + \beta t_1}{r^2} e^{-rt_1} + \frac{\beta(t_2^2 + 2t_1 t_2)}{2r} e^{-rt_1} \right\} \left(1 + \frac{dt_2}{dt_1} \right), \quad (9)
 \end{aligned}$$

and

$$\frac{d^2K}{dt_1^2} > 0. \text{ (The detail of mathematical given in Appendix)}$$

Let $q(t_1) = \frac{dK}{dt_1}$, then q increases with respect to t_1 , and t_1^* is the optimal value

if and only if $q(t_1^*) = 0$. Since K is convex with respect to t_1 , the Newton-Raphson method can be used to find the optimal value of t_1 . However, it may not be easy for a practitioner with limited mathematical knowledge to understand the Newton-Raphson method. In this section, we shall present a simple algorithm to compute the optimal value of t_1 . Before describing the algorithm, we need the following theorem.

Intermediate Value Theorem: *Let q be a continuous function on $[L,U]$, and let $q(L)q(U) < 0$. Then, there exists a number $d \in [L,U]$ such that $q(d) = 0$.*

Since $q(t)$ is strictly increasing, the following algorithm is based on the above theorem and the uniqueness of the root of equation (9). Recall that $q(0) < 0$ and $q(t_U) > 0$. We are in a position to outline the algorithm.

Step 1. Let $\delta > 0$.

Step 2. Let $t_L = 0$ and $t_U = t_1$.

Step 3. Let $t = \frac{t_L + t_U}{2}$.

Step 4. If $|q(t)| < \delta$, go to Step 6. Otherwise, go to Step 5.

Step 5. If $q(t) > 0$, set $t_U = t$. If $q(t) < 0$, set $t_L = t$. Then, go to Step 3.

Step 6. $t_1^* = t$ and exit the optimal value.

We obtain the optimal value of t_1 by Intermediate Value Theorem method using a computer. The optimal values of T , I_m and the minimum total average cost K can be obtained from equations (5) and (8) respectively.

Special case

In this section, we study some important cases that follow from the problem considered in the previous sections.

Case A. If we assume $r \rightarrow 0$, that is ignoring the time value of money. We then obtain the model which is the same as that given by Su and Lin's [22] increasing demand pattern (growth market), the total average cost of system during $[0,T]$ is

$$K = \frac{C_s}{T} + \frac{C_i}{T} \left\{ \left[\frac{a + (b-1)\alpha}{c} - \frac{(b-1)\beta}{c^2} \right] \left(t_1 + \frac{e^{-ct_1} - 1}{c} \right) + \frac{(b-1)\beta t_1^2}{2c} + \frac{\alpha t_2^2}{2} + \frac{\beta t_1 t_2^2}{2} + \frac{\beta t_2^3}{3} \right\}. \tag{10}$$

Case B. If we assume $\beta = 0$ in case A, the model changes to an inventory system with the uniform demand pattern; the model is the same as Su and Lin's [22] uniform demand pattern (maturity market).

$$K = \frac{C_s}{T} + \frac{C_i}{T} \left\{ \frac{a + (b-1)\alpha}{c} \left(t_1 + \frac{e^{-\alpha t_1} - 1}{c} \right) + \frac{\alpha t_2^2}{2} \right\}. \tag{11}$$

Case C. If we assume $b = 0$ in case A, the model changes to an inventory system where the production rate depends on the on-hand inventory. The model is the same as Bhunia and Maiti's [2] first model, i.e., the production rate varies depending on the amount stocked in the go down.

$$K = \frac{C_s}{T} + \frac{C_i}{T} \left\{ \left[\frac{a - \alpha}{c} + \frac{\beta}{c^2} \right] \left(t_1 + \frac{e^{-\alpha t_1} - 1}{c} \right) - \frac{\beta t_1^2}{2c} + \frac{\alpha t_2^2}{2} + \frac{\beta t_1 t_2^2}{2} + \frac{\beta t_2^3}{3} \right\}, \tag{12}$$

where $c \rightarrow 0$, we then obtain the model the same as that given by Bhunia and Maiti's [2] second model, that is the model reducing to an inventory system where the production rate depends on demand.

$$K = \frac{C_s}{T} + \frac{C_i}{T} \left\{ \frac{[a + (b-1)\alpha] t_1^2}{2} + \frac{(b-1)\beta t_1^3}{6} + \frac{\alpha t_2^2}{2} + \frac{\beta t_1 t_2^2}{2} + \frac{\beta t_2^3}{3} \right\}. \tag{13}$$

Case D. Again, if we assume $b = c = 0$ in case B, that is the model changes to an EMQ model with uniform production and constant demand. In this situation, the cost function becomes

$$K = \frac{C_s}{T} + \frac{C_i}{T} \left\{ \frac{(a - \alpha) t_1^2}{2} + \frac{\alpha t_2^2}{2} \right\}. \tag{14}$$

5. NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS

To illustrate the results so far, we use the following example, which is adapted from the example of Bhunia and Maiti [2]. For this model, let $a = 200$ units/month, $b = 0.3$, $c = 0.3$, $C_s = \$100$ for each new cycle, $C_i = \$1$ /unit/month, $\alpha = 100$ units, $\beta = 20$, and $r = 0.2$. The optimum values of t_1 and T , along with minimum total average cost per month K and optimum values of I_m , are calculated for the model. Next, the values are compared with different situations, as shown in Table 1. For our model, the optimal production scheduling period $t_1 = 1.3589$ months, the maximum inventory level $I_m = 133.75$ units and the total average cost $K = \$99.20$ can be used to assist project managers in marking production scheduling period decisions.

Table 1: Results of the numerical example

Cases	t_1	T	I_m	K
The proposed model	1.3589	2.3355	133.75	99.20
Su and Lin's [16] increasing demand pattern	0.9734	1.7862	103.71	110.32
Bhunia and Maiti's [2] first model	1.3589	2.0694	95.41	100.45
Bhunia and Maiti's [2] second model	0.8535	1.6969	105.86	112.87

With the above numerical example, the optimal values of t_1 , T , I_m , and the total average inventory cost K for the fixed set $\phi = \{\beta, b, c, r, C_i\}$ of parametric values are denoted by t_1^0 , T^0 , I_m^0 and K^0 , respectively. Therefore, $t_1^0 = 1.3589$, $T^0 = 2.3355$, $I_m^0 = 133.75$, and $K^0 = 99.20$. Now, when only one of the parameters in the set of parametric values changes by a fixed proportion and all other parameters remain unchanged, let t_1^* , T^* , I_m^* and K^* denote the corresponding optimal values, respectively. Then we calculate the following sensitivity measures for 30% changes in the parameters either side.

S.P.P. = Sensitivity of the optimum production scheduling period

$$= \left(\frac{t_1^*}{t_1^0} - 1 \right) \times 100 ;$$

S.P.T. = Sensitivity of the optimum production cycle time

$$= \left(\frac{T^*}{T^0} - 1 \right) \times 100 ;$$

S.M.I. = Sensitivity of the maximum inventory level

$$= \left(\frac{I_m^*}{I_m^0} - 1 \right) \times 100 ; \text{ and}$$

S.T.C. = Sensitivity of the optimum total average cost

$$= \left(\frac{K^*}{K^0} - 1 \right) \times 100 .$$

Table 2 summarizes these results. The increase in the parameter is indicated by the “+” sign and the decrease by the “-” sign attached to it. Based on the sensitivity analysis, we can infer as following:

1. The optimal production scheduling period t_1 is insensitive to changes in the parameter β , slightly sensitive to changes in b and c and quite sensitive to changes in r and C_i ;
2. The optimal production cycle time T is insensitive to changes in the parameter β , moderately sensitive to changes in b and c and highly sensitive to changes in r and C_i ;

3. The maximum inventory level I_m is insensitive to changes in the parameters b , c and β , quite sensitive to changes in r and C_i ;
4. The optimal total average cost K is slightly sensitive to changes in the parameters β , b and c , quite sensitive to changes in C_i ; and
5. The results indicate that the performance of the proposed model is significantly affected by the discount rate r and the inventory holding cost C_i . The larger the discount rate (or the smaller the inventory holding cost), the greater the production-scheduling period and the smaller the optimum total average cost.

Table 2: Sensitivity analysis

Parameters	t_1^*	T^*	I_m^*	K^*	S.P.P. %	S.P.T. %	S.M.I. %	S.T.C. %
β +:26	1.4367	2.3443	135.38	100.23	5.73	0.38	1.22	1.04
β -:14	1.3059	2.3648	133.10	97.82	-3.90	1.25	-0.49	-1.39
b +:0.39	1.2113	2.2043	133.21	101.84	-10.86	-5.62	-0.40	2.67
b -:0.21	1.5910	2.5520	135.92	96.07	17.08	9.27	1.62	-3.16
c +:0.39	1.5428	2.5168	136.94	97.72	13.53	7.76	2.38	-1.49
c -:0.21	1.2421	2.2239	132.21	100.46	-8.60	-4.78	-1.15	1.27
r +:0.26	1.8259	2.9315	163.16	94.82	34.37	25.52	21.99	-4.42
r -:0.14	1.1878	2.0992	121.10	102.88	-12.59	-10.12	-9.46	3.71
C_i +:1.3	1.0686	1.9277	111.65	114.82	-21.36	-17.46	-16.53	15.74
C_i -:0.7	2.4348	3.6276	191.60	80.01	79.17	55.33	43.25	-19.34

6. CONCLUSIONS

This paper studies the effect of an extended production inventory system under a discounted cash flow, and the production rate at any instant depends on the demand and the inventory level. The demand rate is a linear function of time for the scheduling period. The total average cost for such a system is derived, which is a modification of the standard EMQ formula. Such a production cost is found to be smaller than that of Su and Lin's [22] increasing demand pattern and Bhunia and Maiti's [2]. Using Intermediate Value Theorem method can easily solve the developed model. The sensitivity of the solution to change the values of different parameters has been discussed. According to those results, the proposed model is quite sensitive with respect to the discount rate parameter r and the inventory holding cost C_i , slightly sensitive to the parameters b and c , and insensitive to the parameter β .

Inventory and DCF of the production system are inherent characteristics in all manufacturing industries. An understanding of the relationship among production, demand, inventory and DCF for such systems will help managers to maintain efficient and economic control of operations. A future study should incorporate more realistic assumptions into the proposed model, such as relaxing a terminal condition of zero inventories at the end of the production cycle.

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APPENDIX

From equations (9), we derive

$$\frac{dK}{dt_1} = -C_s f(t_1) + \frac{C_i}{t_1 + t_2} h(t_1) + C_i k(t_1) q(t_1) - C_i f(t_1) g(t_1), \quad [\text{A1}]$$

where $f(t_1) = \frac{1}{(t_1 + t_2)^2} \left[1 + \frac{dt_2}{dt_1} \right]$

$$k(t_1) = \frac{1}{t_1 + t_2} \left[1 + \frac{dt_2}{dt_1} \right]$$

$$g(t_1) = \frac{M}{r} (1 - e^{-rt_1}) - \frac{M}{c+r} \left[1 - e^{-(c+r)t_1} \right] + \frac{N}{r^2} (1 - e^{-rt_1}) - \frac{Nt_1}{r} e^{-rt_1}$$

$$+ \frac{\alpha + \beta(t_1 + t_2)}{r^2} e^{-r(t_1+t_2)} + \frac{\beta}{r^3} \left[e^{-r(t_1+t_2)} - e^{-rt_1} \right] + \frac{\alpha t_2}{r} e^{-rt_1}$$

$$- \frac{\alpha + \beta t_1}{r^2} e^{-rt_1} + \frac{\beta(t_2^2 + 2t_1 t_2)}{2r} e^{-rt_1}$$

$$h(t_1) = (M + Nt_1) e^{-rt_1} - M e^{-(c+r)t_1} - \left[\alpha t_2 + \frac{\beta(t_2^2 + 2t_1 t_2)}{2} \right] e^{-rt_1}$$

$$q(t_1) = \left[\frac{\alpha + \beta(t_1 + t_2)}{r} \right] \left[e^{-rt_1} - e^{-r(t_1+t_2)} \right]$$

Take the second derivative of K with respect to t_1 , we obtain

$$\frac{d^2 K}{dt_1^2} = -C_s f'(t_1) - C_i \left[f(t_1) h(t_1) - \frac{h'(t_1)}{t_1 + t_2} - k'(t_1) q(t_1) - k(t_1) q'(t_1) + f'(t_1) g(t_1) - f(t_1) g'(t_1) \right]$$

where $f'(t_1) = -2f(t_1)k(t_1) + \frac{1}{(t_1 + t_2)^2} \frac{d^2 t_2}{dt_1^2}$

$$k'(t_1) = -f(t_1) \left(1 + \frac{dt_2}{dt_1} \right) + \frac{1}{t_1 + t_2} \frac{d^2 t_2}{dt_1^2}$$

$$g'(t_1) = M \left(e^{-rt_1} - e^{-(c+r)t_1} \right) + Nt_1 e^{-rt_1} - \alpha t_2 e^{-rt_1}$$

$$\begin{aligned}
& + \frac{\alpha + \beta(t_1 + t_2)}{r} [e^{-rt_1} - e^{-r(t_1 + t_2)}] \left[1 + \frac{dt_2}{dt_1} \right] - \frac{\beta(t_2^2 + 2t_1 t_2)}{2} e^{-rt_1} \\
h'(t_1) & = [(1-r)N - rM] e^{-rt_1} + (c+r)M e^{-(c+r)t_1} + \alpha t_2 e^{-rt_1} - (\alpha + \beta t_1) e^{-rt_1} \frac{dt_2}{dt_1} \\
& - \beta t_2 e^{-rt_1} \left(1 + \frac{dt_2}{dt_1} \right) - \beta t_1 e^{-rt_1} \frac{dt_2}{dt_1} \\
q'(t_1) & = -[\alpha + \beta(t_1 + t_2)] \left[e^{-rt_1} - e^{-r(t_1 + t_2)} \left(1 + \frac{dt_2}{dt_1} \right) \right] \\
& + \frac{\alpha + \beta(t_1 + t_2)}{r} [e^{-rt_1} - e^{-r(t_1 + t_2)}]
\end{aligned}$$