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EXISTENCE OF A SOLUTION OF THE QUASI-VARIATIONAL INEQUALITY WITH SEMICONTINUOUS OPERATOR

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Abstract. The paper considers quasi-variational inequalities with point to set operator. The existence of a solution, in the case when the operator of the quasi-variational inequality is semi-continuous and the feasible set is convex and compact, is proved.

Keywords: Quasi-variational inequality, existence of a solution, semi continuous operator.

1. INTRODUCTION

Let X be a real Hilbert space, $U \subset X$ convex closed subset of the space X, $Q: U \mapsto 2^U$ point to set mapping from U to its subsets.

The quasi-variational inequality $QVI(F,U,Q)$ is the problem:

Find $u \in U$ such that there exists $y \in F(u)$ satisfying

$u \in Q(u)$ and $(\forall v \in Q(u)) < y, v - u > \geq 0$.

There are many problems which can be formulated as quasi-variational inequalities, for example: equilibrium problems in economics, impulse control problems, etc.

Existence of a solution of the quasi-variational inequality is considered in [1], [3]. In this paper we prove existence of a solution of the quasi-variational inequality $QVI(F,U,Q)$ with semi-continuous operator F and continuous mapping Q.

If $Q(u) = U$ the quasi-variational inequality is variational inequality $VI(F,U)$. In Section 2 some properties of the point to set mapping are considered. In Section 3 some existence theorems are proved.

2. DEFINITIONS, NOTATIONS, PRELIMINARIES

Let X be a real Hilbert space, $U \subset X$ convex, closed subset of the space X, $Q: U \to 2^U$ point to set mapping.

Definition 2.1. *We say that $F: U \to 2^U$ is upper semi continuous at u_0 if for any open set N such that $F(u_0) \in N$ there exists a neighborhood M of u_0 such that $F(M) \subset N$.*

Definition 2.2. *We say that $F: U \to 2^U$ is lower semi continuous at $u_0 \in U$ if for any $y_0 \in F(u_0)$ and any neighborhood $N(y_0)$, there exists a neighborhood $M(u_0)$ of u_0 such that*

$$(\forall u \in M(u_0)) F(u) \cap N(y_0) \neq \emptyset .$$

Definition 2.3. *A set valued map $F: X \to X$ is said to be continuous at u_0 if it is both upper semi continuous and lower semi continuous. It is said to be continuous at X if it is continuous at every point $x \in X$.*

Definition 2.4. *For mapping $q: U \to U$ we say that it is a selection of mapping Q iff*

$$(\forall u \in U) q(u) \in Q(u) .$$

If q is continuous then we say that q is a continuous selection of the mapping Q. The minimal selection of the mapping Q is defined by

$$m(Q(u)) = P_{Q(u)} O ,$$

where P_Q is a metric projector, and O is the origin.

It is known that the following lemma holds.

Lemma 2.1. (Theorem 1.7.1. [2]) *If $Q: U \to 2^U$ is continuous then the minimal selection $q: U \to U$ is a continuous selection.*

From Lemma 2.1 it follows

Lemma 2.2. *Let $w \in X$, and let $Q: U \to 2^U$ be a point to set mapping which is continuous, with closed convex images. Then the mapping $p: U \to U$ defined by*

$$q(u, w) = P_{Q(u)} w$$

is continuous.

Proof: Let $u, u_n \in U$ and let $u_n \to u \ (n \to \infty)$, $w_n, w \in X$ and $w_n \to w \ (n \to \infty)$. Then

$$\| q(u_n, w_n) - q(u, v) \| = \| P_{Q(u_n)} w_n - P_{Q(u)} w \|$$
$$= \| P_{Q(u_n)} w_n - P_{Q(u_n)} w + P_{Q(u_n)} - P_{Q(u)} w \| \leq \| w_n - w \| + \| p(u_n) - p(u) \|$$

Since $w_n \to w \ (n \to \infty)$ and $p(u_n) \to p(u) \ (n \to \infty)$ it follows that

$$\lim_{n \to \infty} q(u_n, w_n) = q(u, w) ,$$

and hence $q(u, w)$ is continuous.

Lemma 2.3. *Let* $Q:U\to 2^U$ *satisfy conditions of Lemma 2.2., and let* $f:U\to U$ *be continuous. Then the mapping* $g:U\to 2^U$ *defined by*

$$g(u)=P_{Q(u)}f(u)\ ,$$

is also continuous.

Proof: Since $g(u)=q(u,f(u))$, where q is the mapping defined in Lemma 2.2, from continuity of f and q it follows that g is continuous.

Lemma 2.4. *Let* $G:X\to 2^X$ *be upper semi-continuous with closed images and let* $(u_n),(v_n),(\varepsilon_n)$ *be sequences such that*

$$u_n\in\mathbb{X},\varepsilon_n\in\mathbb{R},\varepsilon_n>0,v_n\in\mathbb{X},v_n\in G(u_n)+\varepsilon_n,\varepsilon_n\to 0\ (n\to\infty)\ .$$

Then from

$$\lim_{n\to\infty}u_n=u\in Dom(F)$$

and

$$\lim_{n\to\infty}v_n=v$$

it follows that $v\in G(u)$.

Proof: Let $\varepsilon>0$. Since G is upper semi continuous and

$$\lim_{n\to\infty}v_n=v$$

it follows that there exists $n_0,n_1,n_2\in\mathbb{N}$ such that

$$(\forall n>n_0)\,\|v_n-v\|<\frac{\varepsilon}{3} \quad (1)$$

$$(\forall n>n_1)G(u_n)\subset G(u)+\frac{\varepsilon}{3}B \quad (2)$$

$$(\forall n>n_2)\,\|\varepsilon\|<\frac{\varepsilon}{3}\ . \quad (3)$$

From (1), (2), (3) it follows

$$v_n\in G(u_n)+\frac{\varepsilon}{3}B\subset G(u)+\frac{\varepsilon}{3}B+\frac{\varepsilon}{3}B=G(u)+\frac{2}{3}\varepsilon B\ .$$

From the last relation and (1), (3) it follows that $v\in G(u)+\varepsilon B$. Since ε is any real number greater then 0, it follows that

$$v\in\bigcap_{\varepsilon>0}G(u)+\varepsilon B=G(u)\ .$$

The best known continuous selection theorem is the following result by Michael [2].

Theorem 2.1. *Let X be a metric space, Y a Banach space. Let the mapping F from X into the closed convex subsets of Y be lower semi continuous. Then there exists $f: X \to Y$,a continuous selection of F .*

Theorem 2.2. *(The Approximate Selection Theorem) Let M be a metric space, Y a Banach space, F a mapping from M into the convex subsets of Y which is upper semi continuous. Then for every $\varepsilon > 0$ there exists a locally Lipschitzean map f_ε from M to Y such that its range is contained in the convex hull of the range of F and*

$$Graph(f_\varepsilon) \subset Graph(F) + \varepsilon B \ .$$

3. EXISTENCE OF A SOLUTION

In Lemma 3.1 we will prove sufficient and necessary conditions for the existence of a solution of $QVI(F,U,Q)$.

Lemma 3.1. *$u \in U$ is a solution of the quasi-variational inequality $QVI(F,U,Q)$ if and only if*

$$u \in P_{Q(u)}(u - \alpha F(u)) \text{ for any } \alpha > 0$$

Proof: $u \in P_{Q(u)}(u - \alpha F(u))$, for any $\alpha > 0$ if and only if it holds

$$(\exists \varepsilon \in F(u)) u = P_{Q(u)}(u - \alpha y) \text{ for any } \alpha > 0 \ .$$

The last statement is equivalent with

$$(\exists y \in F(u))(\forall v \in Q(u)) < u, v - u > \geq < u - \alpha y, v - u >$$

i.e.

$$(\exists y \in F(u))(\forall v \in Q(u)) \quad \alpha < y, v - u > \geq 0,$$

This proves that u is a solution of $QVI(F,U,Q)$.

Theorem 3.1. *Let X be a real Hilbert space, $\emptyset \neq U \subset X$ a convex closed subset of the space X, $F: U \to 2^X \setminus \emptyset$ a point to set mapping with convex and compact images which is upper semi continuous, $Q: U \to 2^U \setminus \emptyset$ is a continuous point to set mapping with convex and compact $Q(u)$, and $Dom(F) \supset U$, $Dom(Q) = U$.*

Then the quasi-variational inequality $QVI(F,U,Q)$ has a solution.

Proof: Let $G: U \to 2^X$ be defined by

$$G(u) = u - F(u) \ .$$

From Lemma 2.1 [4] and properties of F it follows that G is upper semi continuous and $G(u)$ is convex and compact. From Theorem 2.2 it follows that for every $\varepsilon > 0$ there exists a locally Lipschitien mapping $g_\varepsilon: U \to X$ such that

$$Graph(g_\varepsilon) \subset Graph(G) + \varepsilon B \ .$$

Let (ε_n) satisfies $\varepsilon_n > 0$, $\lim_{n\to\infty} \varepsilon_n = 0$ and let

$$f_{\varepsilon_n}(u) = P_{Q(u)} g_{\varepsilon_n}(u) .$$

Since g_{ε_n} is continuous, from Lemma 2.2 it follows that f_{ε_n} is continuous too. Since U is convex and compact, from Schauder's theorem it follows that there exists $u_n \in U$ such that

$$u_n = f_{\varepsilon_n}(u_n) = P_{Q(u_n)} g_{\varepsilon_n}(u_n) \in P_{Q(u_n)}(G(u_n) + \varepsilon_n B) \subset P_{Q(u_n)} G(u_n) + \varepsilon_n B .$$

Since U is compact there exist $u \in U$ such that $\lim_{n\to\infty} u_n = u$. From Lemma 2.4 it follows

$$u \in H(u) = P_{Q(u)} G(u) = P_{Q(u)}(u - F(u)) .$$

From Lemma 3.1 it follows that u is a solution of $QVI(F,U,Q)$.

Theorem 3.2. *Let X be a real Hilbert space, $\emptyset \neq U \subset X$ a convex and compact subset of the space X, $F: U \to 2^X \setminus \emptyset$ a point to set mapping which is lower semi continuous with convex and compact images $F(u)$, $Q: U \to 2^U$ a continuous point to set mapping with convex and compact $Q(u)$, $Dom(Q) = U$. Then the quasi-variational inequality $QVI(F,U,Q)$ has a solution.*

Proof: Let $G: U \to 2^X$ be defined by

$$G(U) = u - F(u) .$$

From Lemma 2.3 ([4]) it follows that G is lower semi-continuous. From Theorem 2.1 it follows that G has a continuous selection, i.e. that there exists continuous $g: U \to X$ such that

$$(\forall u \in U) g(u) \in G(u) .$$

Let

$$f(u) = P_{Q(u)} g(u) .$$

From Lemma 2.3 it follows that f is continuous. Since U is convex and compact, from Schauder's theorem it follows that there exists $u \in U$ such that

$$u = P_{Q(u)} g(u) \in P_{Q(u)}(u - F(u)) .$$

From Lemma 3.1 it follows that u is a solution of $QVI(F,U,Q)$.

4. CONCLUSIONS

In this paper we have proved the existence of a solution of the quasy-variational inequality with point to set operator which is semi-continuous. If $Q(u) = U$ for all $u \in U$ the quasy-variational inequality reduces to the variational inequality and Theorems 3.1 and 3.2 reduce to the theorem on existence of a solution of the variational inequality.

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