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PARADOX IN A NON-LINEAR CAPACITATED TRANSPORTATION PROBLEM

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Abstract: This paper discusses a paradox in fixed charge capacitated transportation problem where the objective function is the sum of two linear fractional functions consisting of variables costs and fixed charges respectively. A paradox arises when the transportation problem admits of an objective function value which is lower than the optimal objective function value, by transporting larger quantities of goods over the same route. A sufficient condition for the existence of a paradox is established. Paradoxical range of flow is obtained for any given flow in which the corresponding objective function value is less than the optimum value of the given transportation problem. Numerical illustration is included in support of theory.

Keywords: Capacitated transportation problem, paradox, fixed charge.

1. INTRODUCTION

The fixed charge transportation problem is an extension of the classical transportation problem in which a fixed cost is incurred for every origin. The fixed charge transportation problem (FCTP) was originally formulated by Hirsch and Dantzig [6]. Sandrock [9] gave a simplex algorithm for solving a FCTP. Basu et.al.[3] gave an algorithm for finding optimal solution of solid-fixed charge transportation problem. Fixed charge transportation problems have been studied by Arora et.al.[2], Thirwani [12] and many others. Many distribution problems in practice can only be modelled as FCTPs. For example, rails, roads and trucks have invariably used freight rates which consists of a fixed cost and a variable cost. The fixed cost may represent the cost of renting a vehicle, landing fees at an airport, set up costs for machines in manufacturing environment etc. Another class of transportation problems, where the objective function to be optimized is a ratio of two linear functions, optimization of a ratio of criteria gives more insight into the situation than the optimization of each criterion. Dinkelbach [5] solved linear fractional programming problem by converting it into a parametric programming problem. Swarup [10] also gave a method to solve a linear fractional transportation problem.

Another important class of transportation problems consists of capacitated transportation problems. Many researchers like Bit et.al.[4], Kssay [7] and Zhang et.al.[14] have contributed in this field.

A paradox arises when a transportation problem admits of a total objective function value which is lower than the optimum and is attainable by shipping larger quantities of the goods over the same routes that were previously designated as optimal. This unusual phenomenon was noted by Szwarc [11]. Later on, Verma et.al. [13] have studied the paradoxical situation in a linear fractional transportation problem and obtained paradoxical range of flow. In 2000, Arora et.al. [1] have studied the paradoxical situation in fixed charge transportation problem which is of the form

$$(P) \quad \min \left[\sum_{j\in J}\sum_{i\in I} c_{ij} x_{ij} + \sum_{i\in I} f_i \right]$$

subject to

$$\sum_{j\in J} x_{ij} \le a_i ; \forall\, i\in I$$

$$\sum_{i\in I} x_{ij} = b_j ; \forall\, j\in J$$

$$x_{ij} \ge 0, \forall\, i\in I,\, j\in J,$$

where

$I = \{1, 2, \ldots, m\}$ is the index set of warehouses,

$J = \{1, 2, \ldots, n\}$ is the index set of destinations,

$x_{ij} =$ the amount transported from the i^{th} warehouse to the j^{th} destination,

$c_{ij} =$ the variable cost per unit amount transported from the i^{th} warehouse to j^{th} destination,

$f_i =$ the fixed charge associated with i^{th} warehouse and is defined as

$$f_i = \sum_{l=1}^{p} \delta_{il} f_{il} ; \quad i = 1, 2, \ldots, m \tag{1}$$

where

$$\delta_{il} = \begin{cases} 1, & \text{if } \sum_{j=1}^{n} x_{ij} > A_{il} ; i\in I, l = 1, 2, \ldots, p. \\ 0, & \text{otherwise.} \end{cases}$$

Here $0 = A_{i1} < A_{i2} < \cdots < A_{ip}$. $A_{i1}, A_{i2}, \cdots, A_{ip}$ $(i \in I)$ are constants and f_{il} $(l = 1, 2, \ldots, p; i \in I)$ are fixed charges. Some practical situations may give rise to different type of fixed charges e.g. f_i, of the form as defined in above problem, can be the rent at the i^{th} warehouse and let $g_i^{'}$ be the space available for storage at i^{th} warehouse. Then $\sum_{i\in I} \mu g_i^{'} = \sum_{i\in I} g_i$ (say $g_i = \mu g_i^{'}$) denotes the total space cost of all the warehouses where μ is the per unit space cost. Then one is interested in paying minimum possible rent for the space of maximum value. In most practical situations there are bounds on the flow of the amount on each route. This gives rise to the problem of the following form

$$(P_1) \quad \min \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij} x_{ij}} + \frac{\sum_{i\in I} f_i}{\sum_{i\in I} g_i} \right]$$

subject to

$$\sum_{j\in J} x_{ij} \le a_i ; \forall\, i \in I \tag{2}$$

$$\sum_{i\in I} x_{ij} = b_j ; \forall\, j \in J \tag{3}$$

$$l_{ij} \le x_{ij} \le u_{ij} ; \forall\, (i, j) \in I \times J \tag{4}$$

where

c_{ij} = per unit pilferage cost when shipment is sent from i^{th} warehouse to j^{th} destination,

d_{ij} = the variable profit per unit amount transported from the i^{th} warehouse to j^{th} destination,

f_i = the fixed rent associated with i^{th} warehouse,

g_i = the fixed space cost associated with i^{th} warehouse, and

I, J, x_{ij} $\forall\, (i, j) \in I \times J$, f_i, g_i $\forall\, i \in I$ are defined as in problem (P).

It is assumed that $\sum_{i\in I}\sum_{j\in J} d_{ij} x_{ij} > 0$ for every feasible solution X satisfying (2), (3), (4) and all upper bounds $u_{ij}, (i, j) \in I \times J$ are finite. l_{ij} and u_{ij} are the minimum and maximum quantities of the goods that can be transported along $(i, j)^{th}$ route and the problem (P_1) has a unique solution.

A sufficient condition for the existence of paradox in the above problem has been developed. The condition so obtained indicates which supply point should be given an increment so that the increment is beneficial in the sense that the same optimal basis starts yielding better results. A paradoxical range of flow is obtained such that on increasing the flow within this range the value of the objective function decreases steadily and rises, if flow is increased beyond this range.

It can be easily seen that the problem (P_1) is equivalent to following balanced problem $(\hat{P}_1)$

$$(\hat{P}_1)\ \min \left[\frac{\sum_{i=1}^{m}\sum_{j=1}^{n+1} c_{ij} x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n+1} d_{ij} x_{ij}} + \frac{\sum_{i=1}^{m} f_i}{\sum_{i=1}^{m} g_i} \right]$$

subject to

$$\sum_{j=1}^{n+1} x_{ij} = a_i;\ \forall\, i = 1, 2, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j;\ \forall\ j = 1, 2, \ldots, n, n+1.$$

$$l_{ij} \le x_{ij} \le u_{ij};\ 0 \le x_{i,n+1}\ \forall\, i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, n+1.$$

$$c_{i,n+1} = d_{i,n+1} = 0;\ \forall\, i = 1, 2, \ldots, m \text{ and } b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j.$$

f_i, g_i, for $i \in I$ are defined as in (P_1).

This paper is organized as follows. In section 2, optimality criterion for problem $(\hat{P}_1)$ is developed. In section 3, condition for existence of paradox is developed and methods to determine the best paradoxical pair and to get a paradoxical solution for a specified flow have been developed. In section 4, numerical illustration is included.

2. PRELIMINARY RESULTS

Various algorithms have been developed for solving fixed charge transportation problems when the variables are non negative. These algorithms can be easily extended to capacitated fixed charge transportation problems by using the results developed for capacitated transportation problems by Murty [8]. We have the following optimality criterion for the fixed charge transportation problem $(\hat{P}_1)$,

Result 1. *A feasible solution* $X^0 = \{x_{ij}^0\}_{I\times J}$ *of* (P_1) *with objective function value* $Z^0 = N^0/D^0 + F^0/G^0$ *will be a local optimal basic feasible solution iff*

$$\delta_{ij}^1 = \frac{\theta_{ij}[N^0(Z_{ij}^2 - d_{ij}) - D^0(Z_{ij}^1 - c_{ij})]}{D^0[D^0 - \theta_{ij}(Z_{ij}^2 - d_{ij})]} + \frac{G^0\ \Delta F_{ij} - F^0\ \Delta G_{ij}}{G^0(G^0 + \Delta G_{ij})} \ge 0\ \forall\,(i, j) \in N_1,$$

$$\delta_{ij}^2 = -\frac{\theta_{ij}[N^0(Z_{ij}^2 - d_{ij}) - D^0(Z_{ij}^1 - c_{ij})]}{D^0[D^0 + \theta_{ij}(Z_{ij}^2 - d_{ij})]} + \frac{G^0\ \Delta F_{ij} - F^0\ \Delta G_{ij}}{G^0(G^0 + \Delta G_{ij})} \ge 0\ \forall\,(i, j) \in N_2,$$

and if X^0 *is an optimal solution of* $(\hat{P}_1)$ *then* $\delta^1_{ij} \geq 0 \,\forall\, (i,j) \in N_1$ *and* $\delta^2_{ij} \geq 0$ $\forall\, (i,j) \in N_2$, *where* $N^0 = \sum_{i\in I}\sum_{j\in J} c_{ij}x^0_{ij}$, $D^0 = \sum_{i\in I}\sum_{j\in J} d_{ij}x^0_{ij}$, $F^0 = \sum_{i\in I} f_i$, $G^0 = \sum_{i\in I} g_i$, B *denotes the set of cells* (i,j) *which are basic and* N_1, N_2 *denote the set of non-basic cells* (i,j) *which are at their lower bounds and upper bounds respectively.* $u^1_i, u^2_i, v^1_j, v^2_j$ $(i \in I, j \in J)$ *are such that*

$$u^1_i + v^1_j = c_{ij};\forall\, (i,j) \in B \quad \text{and} \quad u^2_i + v^2_j = d_{ij};\forall\, (i,j) \in B,$$

$$Z^1_{ij} = u^1_i + v^1_j \;\forall\, (i,j) \in N_1 \quad \text{and} \quad Z^2_{ij} = u^2_i + v^2_j \;\forall\, (i,j) \in N_2,$$

$\Delta F_{ij}, \Delta G_{ij}$ *are the corresponding changes in* $\sum_{i\in I} f_i$ *and* $\sum_{i\in I} g_i$ *when some non-basic variable* x_{ij} *undergoes change by an amount of* θ_{ij}.

Proof: Let $X^0 = \{x^0_{ij}\}_{I\times J}$ be a basic feasible solution of problem (P_1) with equality constraints. Let Z^0 be the corresponding value of objective function. Then

$$Z^0 = \frac{\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x^0_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} d_{ij}x^0_{ij}} + \frac{\sum\limits_{i\in I} f_i}{\sum\limits_{i\in I} g_i} = \frac{N^0}{D^0} + \frac{F^0}{G^0} \text{ (Say)}$$

$$= \frac{\sum\limits_{i\in I}\sum\limits_{j\in J}(c_{ij} - u^1_i - v^1_j)x^0_{ij} + \sum\limits_{i\in I}\sum\limits_{j\in J}(u^1_i + v^1_j)x^0_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J}(d_{ij} - u^2_i - v^2_j)x^0_{ij} + \sum\limits_{i\in I}\sum\limits_{j\in J}(u^2_i + v^2_j)x^0_{ij}} + \frac{\sum\limits_{i\in I} f_i}{\sum\limits_{i\in I} g_i}$$

$$= \frac{\sum\sum\limits_{(i,j)\in N_1}(c_{ij} - u^1_i - v^1_j)l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(c_{ij} - u^1_i - v^1_j)u_{ij} + \sum\limits_{i\in I}\sum\limits_{j\in J}(u^1_i + v^1_j)x^0_{ij}}{\sum\sum\limits_{(i,j)\in N_1}(d_{ij} - u^2_i - v^2_j)l_{ij} + \sum\sum\limits_{(i,j)\in N_2}(d_{ij} - u^2_i - v^2_j)u_{ij} + \sum\limits_{i\in I}\sum\limits_{j\in J}(u^2_i + v^2_j)x^0_{ij}} + \frac{\sum\limits_{i\in I} f_i}{\sum\limits_{i\in I} g_i}$$

$$Z^0 = \frac{\sum\limits_{i\in I} a'_i u^1_i + \sum\limits_{j\in J} b'_j v^1_j - \sum\sum\limits_{(i,j)\in N_1}(Z^1_{ij} - c_{ij})l_{ij} - \sum\sum\limits_{(i,j)\in N_2}(Z^1_{ij} - c_{ij})u_{ij}}{\sum\limits_{i\in I} a'_i u^2_i + \sum\limits_{j\in J} b'_j v^2_j - \sum\sum\limits_{(i,j)\in N_1}(Z^2_{ij} - d_{ij})l_{ij} - \sum\sum\limits_{(i,j)\in N_2}(Z^2_{ij} - d_{ij})u_{ij}} + \frac{\sum\limits_{i\in I} f_i}{\sum\limits_{i\in I} g_i}$$

Let some non-basic variable $x_{rs} \in N_1$ undergoes change by an amount θ_{rs} where θ_{rs} is given by

$\min\{u_{rs} - l_{rs};\, x^0_{ij} - l_{ij}$, for all basic cells (i,j) with a $-\theta$ entry in the $\theta-$loop;
$u_{ij} - x^0_{ij}$, for all basic cells (i,j) with a $+\theta$ entry in the $\theta-$loop$\}$.

Let ΔF_{rs} and ΔG_{rs} be the corresponding changes in $\sum_{i\in I} f_i$ and in $\sum_{i\in I} g_i$.

Then new value of the objective function $\hat{Z}$ will be given by

$$\hat{Z} = \frac{N^0 - \theta_{rs}(Z^1_{rs} - c_{rs})}{D^0 - \theta_{rs}(Z^2_{rs} - d_{rs})} + \frac{F^0 + \Delta F_{rs}}{G^0 + \Delta G_{rs}}$$

and

$$\hat{Z} - Z^0 = \left[\frac{N^0 - \theta_{rs}(Z^1_{rs} - c_{rs})}{D^0 - \theta_{rs}(Z^2_{rs} - d_{rs})} - \frac{N^0}{D^0}\right] + \left[\frac{F^0 + \Delta F_{rs}}{G^0 + \Delta G_{rs}} - \frac{F^0}{G^0}\right]$$

$$= \frac{\theta_{rs}[N^0(Z^2_{rs} - d_{rs}) - D^0(Z^1_{rs} - c_{rs})]}{D^0[D^0 - \theta_{rs}(Z^2_{rs} - d_{rs})]} + \frac{G^0\ \Delta F_{rs} - F^0\ \Delta G_{rs}}{G^0(G^0 + \Delta G_{rs})} = \delta^1_{rs}\ (\text{S}ay).$$

Similarly, when some non-basic variable $x_{pq} \in N_2$ undergoes change by an amount θ_{pq}, then

$$\hat{Z} - Z^0 = -\frac{\theta_{pq}[N^0(Z^2_{pq} - d_{pq}) - D^0(Z^1_{pq} - c_{pq})]}{D^0[D^0 + \theta_{pq}(Z^2_{pq} - d_{pq})]} + \frac{G^0 \Delta F_{pq} - F^0 \Delta G_{pq}}{G^0(G^0 + \Delta G_{pq})} = \delta^2_{pq}\ (\text{S}ay).$$

Hence X^0 will be local optimal solution iff

$$\delta^1_{ij} \geq 0\ \forall\,(i, j) \in N_1 \ \text{ and } \ \delta^2_{ij} \geq 0\ \forall\,(i, j) \in N_2.$$

If X^0 is global optimal solution of $(\hat{P}_1)$, then it is locally optimal and hence the result follows.

3. THEORETICAL DEVELOPMENT

Let an optimal basic feasible solution of (P_1) yields value Z^0 of the objective function and $H^0 = \sum_{i \in I} a'_i = \sum_{j \in J} b'_j$ be the corresponding flow where $a'_i \leq a_i,\ i \in I$, $b'_j = b_j,\ j \in J$. A paradox exists if more that H^0 is flown at an objective function value less that Z^0. It may be observed that flow can be increased by an increase of a certain a'_i and b'_j. This gives rise to the following problem (P_2)

$$(P_2) \quad \min \left[\frac{\sum_{i \in I}\sum_{j \in J} c_{ij} x_{ij}}{\sum_{i \in I}\sum_{j \in J} d_{ij} x_{ij}} + \frac{\sum_{i \in I} f_i}{\sum_{i \in I} g_i}\right]$$

subject to

$$\sum_{j \in J} x_{ij} \geq a'_i;\ \forall\, i \in I \tag{5}$$

$$\sum_{i \in I} x_{ij} \geq b'_j;\ \forall\, j \in J \tag{6}$$

$$l_{ij} \le x_{ij} \le u_{ij}; \forall (i,j) \in I \times J \tag{7}$$

where f_i and g_i are defined as in problem (P_1).

Definitions.

(a) Paradoxical Pair: *An objective function -flow pair* (Z,H) *of problem* (P_2) *is called a paradoxical pair if* $Z < Z^0$ *and* $H > H^0$.

(b) Best Paradoxical Pair: *The paradoxical pair* (Z^*,H^*) *is called the best paradoxical pair if forall paradoxical pairs* (Z,H), *either* $Z^* < Z$ *and* $H^* > H$ *or* $Z^* = Z$ *and* $H^* > H$.

(c) Paradoxical Range of Flow: *If on increasing the flow from value* H^0 *to* H^*, *value of objective function decreases steadily from* Z^0 *to* Z^*, *where* Z^* *corresponds to flow* H^* *and further on increasing the flow beyond* H^*, *objective function value starts rising, then interval* $[H^0, H^*]$ *is called 'Paradoxical Range of flow'. All objective function-flow pairs in this range are paradoxical pairs.*

3.1. Sufficient condition for the existence of a paradoxical solution

Let $X^0 = \{x_{ij}\}$ be a basic feasible solution of (P_1) with respect to the variable cost only. Let B denotes the set of cells (i,j) which are basic and N_1, N_2 denote the set of non-basic cell (i,j) which are at their lower bounds and upper bounds respectively. Let $u_i^1, v_j^1, u_i^2, v_j^2$ $(i \in I, j \in J)$ be such that

$$u_i^1 + v_j^1 = c_{ij}; \forall (i,j) \in B$$

and $u_i^2 + v_j^2 = d_{ij}; \forall (i,j) \in B.$

Let this X^0 also be the optimal solution of (P_1). Let Z^0 be the corresponding value of the objective function and $H^0 = \sum_{i \in I} a_i^{'} = \sum_{j \in J} b_j^{'}$ be the corresponding flow where $a_i^{'} \le a_i$, $i \in I$; $b_j^{'} = b_j$; $j \in J$. Then as in Result 1,

$$Z^0 = \frac{\sum_{i \in I} a_i^{'} u_i^1 + \sum_{j \in J} b_j^{'} v_j^1 - \sum \sum_{(i,j) \in N_1} (Z_{ij}^1 - c_{ij}) l_{ij} - \sum \sum_{(i,j) \in N_2} (Z_{ij}^1 - c_{ij}) u_{ij}}{\sum_{i \in I} a_i^{'} u_i^2 + \sum_{j \in J} b_j^{'} v_j^2 - \sum \sum_{(i,j) \in N_1} (Z_{ij}^2 - d_{ij}) l_{ij} - \sum \sum_{(i,j) \in N_2} (Z_{ij}^2 - d_{ij}) u_{ij}} + \frac{\sum_{i \in I} f_i}{\sum_{i \in I} g_i}$$

$$= \frac{N^0}{D^0} + \frac{F^0}{G^0} \text{ (Say)}$$

where $Z_{ij}^1 - c_{ij} = u_i^1 + v_j^1 - c_{ij}$, $Z_{ij}^2 - d_{ij} = u_i^2 + v_j^2 - d_{ij}$.

Now suppose that $a_p^{'}$ is replaced by $a_p^{'}+\lambda$ and $b_q^{'}$ by $b_q^{'}+\lambda$ where $\lambda > 0$ is such that same basis B remains optimal after replacement. Then the new value $Z^{'}$ of the objective function is given by

$$Z^{'} = \frac{N^0+\lambda(u_p^1+v_q^1)}{D^0+\lambda(u_p^2+v_q^2)}+\frac{F^0+\Delta F_{pq}}{G^0+\Delta G_{pq}}$$

where ΔF_{pq}, ΔG_{pq} are the changes in the fixed rent F^0 and the fixed space cost G^0 respectively.

$$Z^{'}-Z^0 = \frac{\lambda[D^0(u_p^1+v_q^1)-N^0(u_p^2+v_q^2)]}{D^0[D^0+\lambda(u_p^2+v_q^2)]}+\frac{G^0\Delta F_{pq}-F^0\Delta G_{pq}}{G^0[G^0+\Delta G_{pq}]}$$

$$= \frac{\begin{bmatrix}\{\lambda[D^0(u_p^1+v_q^1)-N^0(u_p^2+v_q^2)]\}G^0[G^0+\Delta G_{pq}]+ \\ (G^0\Delta F_{pq}-F^0\Delta G_{pq})\{D^0[D^0+\lambda(u_p^2+v_q^2)]\}\end{bmatrix}}{D^0[D^0+\lambda(u_p^2+v_q^2)]G^0[G^0+\Delta G_{pq}]}$$

Now $Z^{'} < Z^0$ if

$$\begin{bmatrix}\{\lambda[D^0(u_p^1+v_q^1)-N^0(u_p^2+v_q^2)]\}G^0[G^0+\Delta G_{pq}] \\ +(G^0\Delta F_{pq}-F^0\Delta G_{pq})\{D^0[D^0+\lambda(u_p^2+v_q^2)]\}\end{bmatrix} < 0 \qquad (8)$$

Thus if there exists a cell (p,q) which satisfies condition (8), then the new value $Z^{'}$ of the objective function is less than Z^0. Hence the flow is increased by λ but objective function value is reduced that is a paradox exists. This result can be stated as:

Theorem 1. *Let X^0 be an optimal basic feasible solution of problem (P_1) with objective value $Z^0 = N^0/D^0+F^0/G^0$. If there exists a cell (p,q) such that on changing a_p by $a_p+\lambda$ and b_q by $b_q+\lambda$, for $\lambda > 0$ and basis remaining the same, the condition*

$$\begin{bmatrix}\{\lambda[D^0(u_p^1+v_q^1)-N^0(u_p^2+v_q^2)]\}G^0[G^0+\Delta G_{pq}]+ \\ (G^0\Delta F_{pq}-F^0\Delta G_{pq})\{D^0[D^0+\lambda(u_p^2+v_q^2)]\}\end{bmatrix} < 0,$$

is satisfied, then there exists a paradox.

Remark 1. As $\lambda > 0, D^0, D^0+\lambda(u_p^2+v_q^2), G^0, G^0+\Delta G_{pq}$ are positive, condition (8) implies that to obtain paradoxical solution we consider only those cells (p,q) for which either $[D^0(u_p^1+v_q^1)-N^0(u_p^2+v_q^2)] < 0$ or $(G^0\Delta F_{pq}-F^0\Delta G_{pq}) < 0$ or both.

3.2. Algorithm to find a 'paradoxical solution'

Step 1. Find a basic feasible solution of (P_1) with respect to variable cost only.

Step 2. Find the corresponding fixed cost. Let it be denoted by F (current)/ G (current), where

$$F(\text{current}) = \sum_{i\in I} f_i,\ G(\text{current}) = \sum_{i\in I} g_i.$$

Also find,

$$A_{ij}^1 = \theta_{ij}(Z_{ij}^1 - c_{ij}), \text{ where } Z_{ij}^1 - c_{ij} = u_i^1 + v_j^1 - c_{ij}, \forall (i, j) \notin B,$$
$$A_{ij}^2 = \theta_{ij}(Z_{ij}^2 - d_{ij}), \text{ where } Z_{ij}^2 - d_{ij} = u_i^2 + v_j^2 - d_{ij}, \forall (i, j) \notin B,$$

B being the current basis, A_{ij}^1 is the change in numerator variable cost that occurs when a non-basic cell (i, j) undergoes a change equal to θ_{ij}. Similarly, A_{ij}^2 is the change in denominator variable cost when a non-basic variable undergoes change.

Step 3. (a) Find $\Delta F_{ij} = F_{ij}(NB) - F$ (current), where $F_{ij}(NB)$ is the total fixed cost obtained when some non-basic cell (i, j) undergoes change. Also find $\Delta G_{ij} = G_{ij}(NB) - G$ (current).

(b) Find $\Delta_{ij} = N^0(Z_{ij}^2 - d_{ij}) - D^0(Z_{ij}^1 - c_{ij})$ for all $(i, j) \notin B$. If

$$\delta_{ij}^1 = \left(\frac{\theta_{ij}\Delta_{ij}}{D^0(D^0 - \theta_{ij}(Z_{ij}^2 - d_{ij}))} + \frac{G^0\ \Delta F_{ij} - F^0\ \Delta G_{ij}}{G^0(G^0 + \Delta G_{ij})} \right) \ge 0, \forall (i, j) \in N_1 \tag{9}$$

$$\delta_{ij}^2 = \left(-\frac{\theta_{ij}\Delta_{ij}}{D^0(D^0 + \theta_{ij}(Z_{ij}^2 - d_{ij}))} + \frac{G^0\ \Delta F_{ij} - F^0\ \Delta G_{ij}}{G^0(G^0 + \Delta G_{ij})} \right) \ge 0, \forall (i, j) \in N_2 \tag{10}$$

then current solution is the optimal solution to (P_1). To test for the existence of paradox go to step 4. Otherwise, some $(i, j) \in N_1$ which does not satisfy (9) or some $(i, j) \in N_2$ which does not satisfy (10) undergoes change. Go to step 2.

Step 4. Let $H^0 = \sum_{i\in I} a_i^{'} = \sum_{j\in J} b_j^{'}$ be the optimal flow where $a_i^{'} \le a_i, i \in I; b_j^{'} = b_j, j \in J$. Choose a cell (p, q) for which at least one of the quantity $D^0(u_p^1 + v_q^1) - N^0(u_p^2 + v_q^2)$, $G^0\ \Delta F_{pq} - F^0 \Delta G_{pq}$ is negative, so that on increasing the flow along this route by $\lambda, \lambda > 0$ condition (8) is satisfied with same optimal basis, then corresponding to this basic feasible solution the value of the objective function reduces and the flow increases i.e. a paradox exists.

Remark 2. The approach to solve the problem (P_1) and (P_2) may result in a local minimum instead of a global minimum. One is still happy because in real world one seeks satisfying solutions that are close to optimum and that are realistic.

Best Paradoxical Pair

If a paradox exists, one would obviously be interested in the `Best Paradoxical Pair'. Let $H^0 = \sum_{i\in I} a_i^{'} = \sum_{j\in J} b_j^{'}$ be the flow corresponding to the optimal basic feasible solution X^0 of (P_1) where $a_i^{'} \le a_i; i \in I, b_j^{'} = b_j; j \in J$. Also, let H^* be the flow corresponding to the optimal basic feasible solution X^* of (P_2). Then $[H^0, H^*]$ is the `Paradoxical Range of Flow'. Theorem 2 below proves that the optimal basic feasible solution of problem (P_2) yields the best paradoxical pair.

Theorem 2. *Optimal basic feasible solution of* (P_2) *yields the best paradoxical pair.*

Proof: Let $X^\alpha = \{x_{ij}^\alpha\}$ be an optimal feasible solution of problem (P_2). Let corresponding to this solution, we have

$$\sum_{j\in J} x_{ij}^\alpha = a_i^\alpha \ge a_i^{'} : i \in I$$

$$\sum_{i\in I} x_{ij}^\alpha = b_j^\alpha \ge b_j^{'} : j \in J$$

Let Z^α and H^α be the optimal value of the objective function and the corresponding optimal flow respectively.

Consider the following problem (P_3)

$$(P_3) \quad \min \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij} x_{ij}} + \frac{\sum_{i\in I} f_i}{\sum_{i\in I} g_i} \right]$$

subject to

$$\sum_{j\in J} x_{ij} = a_i^\alpha + p_i \ge a_i^{'}; \forall\, i \in I$$

$$\sum_{i\in I} x_{ij} = b_j^\alpha + q_j \ge b_j^{'}; \forall\, j \in J$$

$$l_{ij} \le x_{ij} \le u_{ij}; \forall\, (i,j) \in I \times J$$

where $\sum_{i\in I} p_i = 0 = \sum_{j\in J} q_j$.

Let $X^{\alpha'} = \{x_{ij}^{\alpha'}\}$ be the optimal solution of problem (P_3). Then $X^{\alpha'}$ will be a feasible solution of (P_2). But X^α is the optimal solution of (P_2). Therefore, $Z^{\alpha'} \ge Z^\alpha$ where $Z^{\alpha'}$ is the value of the objective function of problem (P_2) at the feasible solution $X^{\alpha'}$. This implies that no optimal solution of (P_3) can yield the objective function value less than Z^α. Thus there does not exist any solution of problem (P_3) which gives value less than Z^{α} and flow greater than H^{α}. Hence, optimal solution of (P_2) yields the best paradoxical pair.

To solve (P_2), we construct and solve the related fixed charge transportation problem (P_4) with an additional supply point and an additional destination.

$$(P_4) \quad \min \left[\frac{\sum_{j\in J'}\sum_{i\in I'} c_{ij}u_{ij} + \sum_{j\in J'}\sum_{i\in I'} c'_{ij}w_{ij}}{\sum_{j\in J'}\sum_{i\in I'} d_{ij}u_{ij} + \sum_{j\in J'}\sum_{i\in I'} d'_{ij}w_{ij}} + \frac{\sum_{i\in I'} f'_i}{\sum_{i\in I'} g'_i} \right]$$

subject to

$$\sum_{j\in J'} w_{ij} = A'_i ; i \in I' \tag{11}$$

$$\sum_{i\in I'} w_{ij} = B'_j ; j \in J' \tag{12}$$

$$0 \le w_{ij} \le u_{ij} - l_{ij} ; (i,j) \in I \times J, w_{m+1,j}, w_{i,n+1} \ge 0, \forall i \in I, j \in J, w_{m+1,n+1} \ge 0, \tag{13}$$

where

$$\left.\begin{array}{l} f'_i = \sum_{i=1}^{p} \delta_{il} f_{il} ; \quad g'_i = \sum_{i=1}^{p} \delta_{il} g_{il} ; \quad f'_{m+1} = g'_{m+1} = 0 \\ \delta_{il} = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} w_{ij} < \sum_{j=1}^{n} u_{ij} - A_{il}, i \in I, l = 1,2,\ldots,p. \\ 0 & otherwise \end{cases} \\ I' = \{1,2,\ldots,m,m+1\}, J' = \{1,2,\ldots,n,n+1\} \end{array}\right\} \tag{14}$$

$$\left.\begin{array}{l} A'_i = \sum_{j\in J} u_{ij} - a'_i, i \in I ; \quad A'_{m+1} = \sum_{j\in J}\sum_{i\in I} u_{ij} - \sum_{j\in J} b'_j \\ B'_j = \sum_{i\in I} u_{ij} - b'_j, j \in J ; \quad B'_{n+1} = \sum_{j\in J}\sum_{i\in I} u_{ij} - \sum_{i\in I} a'_i \end{array}\right\} \tag{15}$$

$$\left.\begin{array}{l} c_{m+1,j} = d_{i,n+1} = 0, i \in I', j \in J' ; c'_{ij} = -c_{ij}, d'_{ij} = -d_{ij}, i \in I, j \in J ; \\ c'_{m+1,j} = d'_{i,n+1} = 0, i \in I', j \in J'. \end{array}\right\} \tag{16}$$

can be easily proved that problem (P_2) and (P_4) are equivalent.

Lemma 1. *There is a one-to-one correspondence between the feasible solutions of problem (P_2) and (P_4).*

Proof: Let $\{x_{ij}\}_{I\times J}$ be a feasible solution of problem (P_2). Therefore, $x_{ij}, i \in I, j \in J$ satisfy relations (5) to (7). Define $w_{ij}, i \in I', j \in J'$ by the following transformation

$$w_{ij} = u_{ij} - x_{ij}, i \in I, j \in J \tag{17}$$

$$w_{i,n+1} = \sum_{j\in J} x_{ij} - a'_i, i \in I \tag{18}$$

$$w_{m+1,j} = \sum_{i\in I} x_{ij} - b_j^{'}, \; j \in J \tag{19}$$

$$w_{m+1,n+1} = \sum_{j\in J}\sum_{i\in I} u_{ij} - \sum_{j\in J}\sum_{i\in I} x_{ij} \tag{20}$$

$$\left.\begin{array}{l}\text{Relations (7) and (17) imply that } 0 \le w_{ij} \le u_{ij} - l_{ij}; i \in I, j \in J \text{ and relations} \\ \text{(18) to (20) and (5), (6) imply that } w_{i,n+1}, w_{m+1,j}, w_{m+1,n+1} \ge 0; i \in I, j \in J.\end{array}\right\} \tag{21}$$

Also, for $i \in I$

$$\begin{aligned}\sum_{j\in J^{'}} w_{ij} &= \sum_{j\in J} w_{ij} + w_{i,n+1} \\ &= \sum_{j\in J}(u_{ij} - x_{ij}) + (\sum_{j\in J} x_{ij} - a_i^{'}) \\ &= \sum_{j\in J} u_{ij} - a_i^{'} = A_i^{'}\end{aligned} \tag{22}$$

Also, for $i = m+1$,

$$\begin{aligned}\sum_{j\in J^{'}} w_{m+1,j} &= \sum_{j\in J} w_{m+1,j} + w_{m+1,n+1} \\ &= \sum_{j\in J}\left[\sum_{i\in I} x_{ij} - b_j^{'}\right] + \sum_{j\in J}\sum_{i\in I} u_{ij} - \sum_{j\in J}\sum_{i\in I} x_{ij} \\ &= \sum_{j\in J}\sum_{i\in I} u_{ij} - \sum_{j\in J} b_j^{'} = A_{m+1}^{'}\end{aligned} \tag{23}$$

Similarly, it can be shown that

$$\sum_{i\in I^{'}} w_{ij} = B_j^{'}, \; j \in J^{'} \tag{24}$$

Relations (21) to (24) show that $\{w_{ij}\}_{I^{'}\times J^{'}}$, as defined above is a feasible solution of problem (P_4).

Conversely, let $\{w_{ij}\}_{I^{'}\times J^{'}}$ be a feasible solution to (P_4). Define $x_{ij}, i \in I, j \in J$ by the following transformation,

$$x_{ij} = u_{ij} - w_{ij}; \forall\, i \in I, j \in J. \tag{25}$$

(13) and (25) imply that

$$l_{ij} \le x_{ij} \le u_{ij}; \forall\, i \in I, j \in J. \tag{26}$$

Now, for $i \in I$, the source constraints in (P_4) give

$$\sum_{j\in J^{'}} w_{ij} = A_i^{'} = \sum_{j\in J} u_{ij} - a_i^{'}$$

Therefore $\sum_{j\in J} w_{ij} \le \sum_{j\in J} u_{ij} - a_i^{'}$, because $w_{i,n+1} \ge 0$. Hence using relation (25)

$$\sum_{j\in J} x_{ij} \ge a_i^{'}, \forall\, i \in I. \tag{27}$$

Similarly, for $j \in J$

$$\sum_{i\in I} x_{ij} \ge b_j^{'}, \forall\, j \in J. \tag{28}$$

Relation (26) and (28) show that $\{x_{ij}\}_{I\times J}$ defined as above is a feasible solution of problem (P_2).

Lemma 2. *The value of the objective function of (P_4) at a feasible solution is equal to the objective function of (P_2) at its corresponding feasible solution and conversely.*

Proof: The value of the objective function of (P_4) at the feasible solution $\{w_{ij}\}_{I^{'}\times J^{'}}$ is

$$= \frac{\sum_{j\in J^{'}}\sum_{i\in I^{'}} c_{ij}u_{ij} + \sum_{j\in J^{'}}\sum_{i\in I^{'}} c_{ij}^{'} w_{ij}}{\sum_{j\in J^{'}}\sum_{i\in I^{'}} d_{ij}u_{ij} + \sum_{j\in J^{'}}\sum_{i\in I^{'}} d_{ij}^{'} w_{ij}} + \frac{\sum_{i\in I^{'}} f_i^{'}}{\sum_{i\in I^{'}} g_i^{'}}$$

$$= \frac{\sum_{j\in J}\sum_{i\in I} c_{ij}u_{ij} + \sum_{j\in J}\sum_{i\in I} (-c_{ij})(u_{ij} - x_{ij})}{\sum_{j\in J}\sum_{i\in I} d_{ij}u_{ij} + \sum_{j\in J}\sum_{i\in I} (-d_{ij})(u_{ij} - x_{ij})} + \frac{\sum_{i\in I} f_i}{\sum_{i\in I} g_i} \quad \text{[using (14), (15) and (16)]}$$

$$= \frac{\sum_{j\in J}\sum_{i\in I} c_{ij}x_{ij}}{\sum_{j\in J}\sum_{i\in I} d_{ij}x_{ij}} + \frac{\sum_{i\in I} f_i}{\sum_{i\in I} g_i}$$

= The value of the objective function of (P_2) at the corresponding feasible solution $\{x_{ij}\}_{I\times J}$

The converse can be proved similarly.

Lemma 3. *There is a one-to-one correspondence between the optimal solution to (P_2) and optimal solution to (P_4).*

Proof: let $\{x_{ij}^0\}_{I\times J}$ be an optimal solution to (P_2) yielding value Z^0 and $\{w_{ij}^0\}_{I^{'}\times J^{'}}$ be the corresponding feasible solution to (P_4). The value yielded by $\{w_{ij}^0\}_{I^{'}\times J^{'}}$ is Z^0 (refer to Lemma 2). If possible, let $\{w_{ij}^0\}_{I^{'}\times J^{'}}$ be not an optimal feasible solution to (P_4). Therefore, there exists a feasible solution $\{w_{ij}^{'}\}_{I^{'}\times J^{'}}$, say, to (P_4) with the value $Z^{'} < Z^0$. Let $\{x_{ij}^{'}\}_{I\times J}$ be the corresponding feasible solution to (P_2). Then, by Lemma 2,

$$\frac{\sum_{j\in J}\sum_{i\in I} c_{ij} x_{ij}^{'}}{\sum_{j\in J}\sum_{i\in I} d_{ij} x_{ij}^{'}} + \frac{\sum_{i\in I} f_i}{\sum_{i\in I} g_i} = Z^{'}$$

which is a contradiction to the assumption that Z^0 is the optimal solution of (P_2) as $Z^{'} < Z^0$. Similarly, an optimal solution of (P_4) will give an optimal solution to (P_2).

Theorem 3. *Optimizing* (P_2) *is equivalent to optimizing* (P_4), *provided both problems have feasible solution.*

Proof: As (P_2) has a feasible solution, by lemma 1, there exists a feasible solution to (P_4). Hence by Lemma 2 and Lemma 3, and optimal solution to (P_2) can be obtained.

We now discuss how to find a paradoxical solution for a specified flow in a given paradoxical range of flows.

Paradoxical solution for a specified flow in $[H^0, H^*]$

Quite often, finding the best objective function value for a given flow in $[H^0, H^*]$ is of great importance to the decision maker. Let the specified flow be $H \in [H^0, H^*]$. The `Paradoxical solution' for H is given by the optimal solution of problem (P_5)

$$(P_5) \quad \min \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij} x_{ij}} + \frac{\sum_{i\in I} f_i}{\sum_{i\in I} g_i} \right]$$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i^{'};\ \forall\, i\in I$$

$$\sum_{i\in I} x_{ij} \geq b_j^{'};\ \forall\, j\in J$$

$$\sum_{j\in J}\sum_{i\in I} x_{ij} = H \quad \left(H > \sum_{i\in I} a_i^{'} = \sum_{j\in J} b_j^{'} \right)$$

$$l_{ij} \leq x_{ij} \leq u_{ij};\ \forall\, (i,j)\in I\times J$$

Note that due to flow constraint problem (P_5) is different from (P_2). To solve (P_5) we consider the following related problem (P_6) with an additional supply point and an additional destination.

$$(P_6) \quad \min \left[\frac{\sum_{i\in I^{'}}\sum_{j\in J^{'}} c_{ij} u_{ij} + \sum_{i\in I^{'}}\sum_{j\in J^{'}} c_{ij}^{'} w_{ij}}{\sum_{i\in I^{'}}\sum_{j\in J^{'}} d_{ij} u_{ij} + \sum_{i\in I^{'}}\sum_{j\in J^{'}} d_{ij}^{'} w_{ij}} + \frac{\sum_{i\in I^{'}} f_i}{\sum_{i\in I^{'}} g_i} \right]$$

subject to

$$\sum_{j\in J'} w_{ij} = A_i';\, i\in I' \tag{29}$$

$$\sum_{i\in I'} w_{ij} = B_j';\, j\in J' \tag{30}$$

$$0 \le w_{ij} \le u_{ij} - l_{ij},\, i\in I,\, j\in J,\, w_{m+1,j},\, w_{i,n+1},\, w_{m+1,n+1} \ge 0.$$

$$\sum_{i\in I}\sum_{j\in J} w_{ij} = \sum_{i\in I}\sum_{j\in J} u_{ij} - H \tag{31}$$

$$\left.\begin{aligned} A_i' &= \sum_{j\in J} u_{ij} - a_i',\, \forall i\in I,\, A_{m+1}' = H - \sum_{j\in J} b_j' \\ B_j' &= \sum_{i\in I} u_{ij} - b_j',\, \forall j\in J,\, B_{n+1}' = H - \sum_{i\in I} a_i' \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} &c_{m+1,j} = d_{i,n+1} = c_{m+1,n+1} = d_{m+1,n+1} = 0, i\in I, j\in J, c_{ij}' = -c_{ij}, i\in J, j\in J, \\ &c_{m+1,j}' = d_{i,n+1}' = 0, i\in I, j\in J, d_{ij}' = -d_{ij}, i\in I, j\in J, \\ &c_{m+1,n+1}' = M, d_{m+1,n+1}' = 0, \text{ where } M \text{ is a large positive number.} \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} &f_i' = \sum_{l=1}^{p} \delta_{il} f_{il},\, g_i' = \sum_{l=1}^{p} \delta_{il} g_{il},\ \text{for } i\in I \\ &\delta_{il} = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} w_{ij} < \sum_{j\in J} u_{ij} - A_{il};\forall i\in I \\ 0 & otherwise \end{cases} \\ &f_{m+1}' = g_{m+1}' = 0 \end{aligned}\right\} \tag{34}$$

Definition. *A feasible solution* $\{w_{ij}\}, i\in I', j\in J'$ *to* (P_6) *is called a corner feasible solution (cfs) if* $w_{m+1,n+1} = 0$.

Theorem 4. *A non corner feasible solution to* (P_6) *can not provide a feasible solution to* (P_5).

Proof: Let $\{\overline{w}_{ij}\}$ be a non corner feasible solution to (P_6). Therefore, $\overline{w}_{m+1,n+1} = \lambda\,(>0)$.

Thus, $\sum_{i\in I}\overline{w}_{i,n+1} = (H - \sum_{i\in I} a_i') - \lambda = H - (\sum_{i\in I} a_i' + \lambda).$

Now, for $i\in I$,

$$\sum_{j\in J'} \overline{w}_{ij} = A_i' = \sum_{j\in J} u_{ij} - a_i'$$

Therefore

$$\sum_{j\in J'}\sum_{i\in I} \overline{w}_{ij} = \sum_{j\in J}\sum_{i\in I} u_{ij} - \sum_{i\in I} a_i'$$

Hence

$$\sum_{j\in J}\sum_{i\in I}\bar{w}_{ij} = \sum_{j\in J}\sum_{i\in I}u_{ij} - \sum_{i\in I}a_i^{'} - H + \sum_{i\in I}a_i^{'} + \lambda = (\sum_{j\in J}\sum_{i\in I}u_{ij} - H) + \lambda$$

This means that the quantity transported from the sources in I to the destinations in J is $(\sum_{j\in J}\sum_{i\in I}u_{ij} - H) + \lambda$ which is greater than $\sum_{j\in J}\sum_{i\in I}u_{ij} - H$, which shows that $\{\bar{w}_{ij}\}$ cannot provide a feasible solution to (P_5) .

Remark 3. If (P_6) has a corner feasible solution, then, from the definition of $c^{'}_{m+1,n+1}$, it follows that no non corner feasible solution can be its optimal solution.

Remark 4. It is easy to verify that problems (P_5) and (P_6) are equivalent using the transformation

$$w_{ij} = u_{ij} - x_{ij}; \forall i\in I, j\in J$$
$$w_{i,n+1} = \sum_{j\in J}x_{ij} - a_i^{'}; \forall i\in I$$
$$w_{m+1,j} = \sum_{i\in I}x_{ij} - b_j^{'}; \forall j\in J$$
$$w_{m+1,n+1} = 0.$$

Concluding Remarks

If the condition that u_{ij}'s are finite is relaxed, then algorithm discussed in Section 3.2 may not be directly applicable and this gives rise to unbalanced capacitated fixed charge transportation problem with mixed type of bounds.

4. NUMERICAL ILLUSTRATION

Consider the problem (P_1) for $m = 2, n = 3$. Table I gives the values of $c_{ij}, d_{ij}, (i = 1,2;\ j = 1,2,3)$ and the values of $a_i\,(i = 1,2)$ and $b_j\,(j = 1,2,3)$

Table I: Values of c_{ij}, d_{ij}, a_i, b_j

				$a_i \downarrow$
$c_{ij} \rightarrow$ / $d_{ij} \rightarrow$	2 / 3	3 / 4	1 / 5	40
	1 / 4	2 / 4	2 / 6	30
$b_j \rightarrow$	20	10	20	

$0 \le x_{11} \le 20, 0 \le x_{12} \le 10, 0 \le x_{13} \le 20, 0 \le x_{21} \le 10, 0 \le x_{22} \le 20, 0 \le x_{23} \le 30$.

The fixed rents f_i 's and space costs g_i 's for all $i \in I$ are given by

$$f_i = \sum_{l=1}^{3} \delta_{il} f_{il};\, i = 1, 2 \text{ and } g_i = \sum_{l=1}^{3} \delta_{il} g_{il};\, i = 1, 2,$$

where $f_{11} = 20, f_{12} = 10, f_{13} = 10, \quad g_{11} = 20, g_{12} = 15, g_{13} = 15,$

$f_{21} = 10, f_{22} = 5, f_{23} = 10, \quad g_{21} = 15, g_{22} = 10, g_{23} = 5.$

$$\delta_{i1} = \begin{cases} 1, & if \sum_{j=1}^{3} x_{ij} > 0, i = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{i2} = \begin{cases} 1, & if \sum_{j=1}^{3} x_{ij} > 20, i = 1, 2 \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

$$\delta_{i3} = \begin{cases} 1, & if \sum_{j=1}^{3} x_{ij} > 30, i = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

As $\sum_{i=1}^{2} a_i > \sum_{j=1}^{3} b_j$, we add a dummy destination in Table I with $c_{i4} = d_{i4} = 0, i = 1, 2$. A basic feasible solution of the related balanced problem $(\hat{P}_1)$ is given in Table II.

Table II: Basic feasible solution of $(\hat{P}_1)$

					u_i^1	u_i^2	f_i	g_i
	2 **10** 3	3 $\underline{0}$ 4	1 **20** 5	0 **10** 0	0	0	30	35
	1 $\overline{10}$ 4	2 **10** 4	2 $\underline{0}$ 6	0 **10** 0	0	0	10	15
v_j^1	2	2	1	0				
v_j^2	3	4	5	0				

Note: In above table entries in bold face represent allocations in basic cells and entries of the form $\underline{a}$ and $\overline{b}$ represent the allocations in non-basic cells which are at their lower bounds and upper bounds respectively.

$$N^0 = 70, D^0 = 210, F^0 = 40, G^0 = 50 \text{ and } Z^0 = 1.133333, H^0 = 50 .$$

On applying step 2 and step 3, we get the values of $\theta_{ij}, A_{ij}^1, A_{ij}^2, \Delta F_{ij}, \Delta G_{ij}, \Delta_{ij}, \delta_{ij}^1, \delta_{ij}^2$, which are displayed in Table III.

Table III: Values of $\theta_{ij}, A_{ij}^1, A_{ij}^2, \Delta F_{ij}, \Delta G_{ij}, \Delta_{ij}, \delta_{ij}^1, \delta_{ij}^2$

(i, j)	(1,2)	(2,1)	(2,3)
θ_{ij}	10	10	10
A_{ij}^1	-10	10	-10
A_{ij}^2	0	-10	-10
ΔF_{ij}	10	10	-5
ΔG_{ij}	15	15	-5
Δ_{ij}	210	-280	140
δ_{ij}^1	23/1365	-	4/495
δ_{ij}^2	-	64/2145	-

As $\delta_{ij}^1, \delta_{ij}^2 \geq 0 \; \forall (i, j) \notin B$, the solution in Table II is an optimal solution of $(\hat{P}_1)$ and hence yields optimal solution of (P_1). Here $a_1^{'} = 30, a_2^{'} = 20$.

Suppose, we increase the flow along $(1,2)$ route by λ where λ can vary between 1 and 10. Let $\lambda = 10$. Then $G^0 \Delta F_{12} - F^0 \Delta G_{12} = -150 < 0$ and

$$\begin{bmatrix} \lambda[D^0(u_1^1 + v_2^1) - N^0(u_1^2 + v_2^2)]G^0[G^0 + \Delta G_{12}] + \\ (G^0 \Delta F_{12} - F^0 \Delta G_{12})D^0[D^0 + \lambda(u_1^2 + v_2^2)] \end{bmatrix} = -3675000 < 0.$$

Thus a paradox exists in this case.

Best Paradoxical pair is found by solving the problem (P_2) for $m = 2, n = 3$. Values of $c_{ij}, d_{ij}, a_i^{'}, b_j^{'}$ are given in Table IV.

Table IV: Values of $c_{ij}, d_{ij}, a_i^{'}, b_j^{'}$

			$a_i^{'} \downarrow$
2 3	3 4	1 5	$\geq$ 30
1 4	2 4	2 6	$\geq$ 20
$b_j^{'} \rightarrow$ $\geq$ 20	$\geq$ 10	$\geq$ 20	

$0 \leq x_{11} \leq 20, 0 \leq x_{12} \leq 10, 0 \leq x_{13} \leq 20, 0 \leq x_{21} \leq 10, 0 \leq x_{22} \leq 20, 0 \leq x_{23} \leq 30$.

Optimal solution of problem (P_2) is obtained by solving the corresponding problem (P_4)

$$(P_4)\ \min\left[\frac{\sum_{i\in I'}\sum_{j\in J'}c'_{ij}w_{ij}+200}{\sum_{i\in I'}\sum_{j\in J'}d'_{ij}w_{ij}+500}+\frac{\sum_{i\in I'}f'_i}{\sum_{i\in I'}g'_i}\right]$$

$$\sum_{j\in J'}w_{ij}=A'_i;\ i\in I'$$

$$\sum_{i\in I'}w_{ij}=B'_j;\ j\in J'$$

$$0\le w_{11}\le 20, 0\le w_{12}\le 10, 0\le w_{13}\le 20, 0\le w_{21}\le 10, 0\le w_{22}\le 20,$$
$$0\le w_{23}\le 30, w_{i4}, w_{3j}, w_{34}\ge 0, i\in I, j\in J.$$

Values of $c'_{ij}, d'_{ij}, A'_i, B'_j$ for $i\in I'=\{1,2,3\}, j\in J'=\{1,2,3,4\}$ are given in Table V,

Table V: Values of $c'_{ij}, d'_{ij}, A'_i, B'_j$

					$A'_i \downarrow$
$c'_{ij} \rightarrow$ / $d'_{ij} \rightarrow$	-2 / -3	-3 / -4	-1 / -5	0 / 0	20
	-1 / -4	-2 / -4	-2 / -6	0 / 0	40
	0 / 0	0 / 0	0 / 0	0 / 0	60
$B'_j \rightarrow$	10	20	30	60	

The fixed rents f'_i's and space costs g'_i's for all $i\in I'$ are given by

$$f'_i=\sum_{l=1}^{3}\delta_{il}f_{il};\ i=1,2 \text{ and } g'_i=\sum_{l=1}^{3}\delta_{il}g_{il};\ i=1,2,;\ f'_3=g'_3=0,$$

where

$$f_{11}=20, f_{12}=10, f_{13}=10, \quad g_{11}=20, g_{12}=15, g_{13}=15,$$

$$f_{21}=10, f_{22}=5, f_{23}=10, \quad g_{21}=15, g_{22}=10, g_{23}=5.$$

$$\delta_{11} = \begin{cases} 1, & if \sum_{j=1}^{3} w_{1j} < 50, \\ 0, & \text{otherwise} \end{cases} \quad \delta_{21} = \begin{cases} 1, & if \sum_{j=1}^{3} w_{2j} < 60, \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{12} = \begin{cases} 1, & if \sum_{j=1}^{3} w_{1j} < 30, \\ 0, & \text{otherwise} \end{cases} \quad \delta_{22} = \begin{cases} 1, & if \sum_{j=1}^{3} w_{2j} < 40, \\ 0, & \text{otherwise} \end{cases} \qquad (36)$$

$$\delta_{13} = \begin{cases} 1, & if \sum_{j=1}^{3} w_{1j} < 20, \\ 0, & \text{otherwise} \end{cases} \quad \delta_{21} = \begin{cases} 1, & if \sum_{j=1}^{3} w_{2j} < 30, \\ 0, & \text{otherwise} \end{cases}$$

The optimal solution of problem (P_4) is given in Table VI.

Table VI: Optimal solution of (P_4)

					u_i^1	u_i^2	$f_i^{'}$	$g_i^{'}$
	-2 / $\underline{0}$ / -3	-3 / $\underline{0}$ / -4	-1 / $\underline{0}$ / -5	0 / **20** / 0	0	0	40	50
	-1 / **10** / -4	-2 / **20** / -4	-2 / $\underline{0}$ / -6	0 / **10** / 0	0	0	15	25
	0 / $\underline{0}$ / 0	0 / $\underline{0}$ / 0	0 / **30** / 0	0 / **30** / 0	0	0	0	0
v_j^1	-1	-2	0	0				
v_j^2	-4	-4	0	0				

On making the transformation, the optimal solution to problem (P_2) is given in Table VII.

Table VII: Optimal solution of problem (P_2)

			f_i	g_i
2 / **20** / 3	3 / **10** / 4	1 / **20** / 5	40	50
1 / **0** / 4	2 / **0** / 4	2 / **30** / 6	15	25

Here the objective function value $Z^* = 1.1280702$ and flow $H^* = 80$. Thus the paradoxical range of flow is $[H^0, H^*] = [50, 80]$.

Consider the `Paradoxical Solution' for a specified flow $H = 60$. It is obtained by solving the problem (P_6). $f_i^{'}, g_i^{'}\ i \in I^{'}$ are defined as in (1.36). Values of $c_{ij}^{'}, d_{ij}^{'}, A_i^{'}, B_j^{'}$ are given in Table VIII,

Table VIII: Values of $c_{ij}^{'}, d_{ij}^{'}, A_i^{'}, B_j^{'}$

(Each cell: $c_{ij}^{'}$ (top left) / $d_{ij}^{'}$ (bottom right))

					$A_i^{'} \downarrow$
$c_{ij}^{'} \rightarrow$ / $d_{ij}^{'} \rightarrow$	-2 / -3	-3 / -4	-1 / -5	0 / 0	20
	-1 / -4	-2 / -4	-2 / -6	0 / 0	40
	0 / 0	0 / 0	0 / 0	M / 0	10
$B_j^{'} \rightarrow$	10	20	30	10	

Optimal solution of problem (P_6) is given in Table IX.

Table IX: Optimal solution of (P_6)

(Each cell: $c_{ij}^{'}$ / flow / $d_{ij}^{'}$)

					u_i^1	u_i^2	$f_i^{'}$	$g_i^{'}$
	-2 / **10** / -3	-3 / **10** / -4	-1 / $\underline{0}$ / -5	0 / $\underline{0}$ / 0	-1	0	30	35
	-1 / $\underline{0}$ / -4	-2 / **10** / -4	-2 / **20** / -6	0 / **10** / 0	0	0	15	25
	0 / $\underline{0}$ / 0	0 / $\underline{0}$ / 0	0 / **10** / 0	M / $\underline{0}$ / 0	2	6	0	0
v_j^1	-1	-2	-2	0				
v_j^2	-3	-4	-6	0				

On making the transformation, the optimal solution to problem (P_2) for specified flow $H = 60$ is given in Table X.

Table X: Paradoxical solution for flow $H = 60$

			f_i	g_i
2 / **10** / 3	3 / **0** / 4	1 / **20** / 5	30	35
1 / **10** / 4	2 / **10** / 4	2 / **10** / 6	15	25

Here $Z = 1.133333$ and $H = 60$.

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