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ENTROPY BASED TRANSPORTATION MODEL - A GEOMETRIC PROGRAMMING APPROACH

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Abstract: The entropy model has attached a good deal of attention in transportation analysis, urban and regional planning as well as in other areas. This paper shows the equivalance of entropy maximization models to geometric programs. To provide a better understanding of this entropy based transportation model they are analyzed by geometric programming. Dual mathematical programs and algorithms are also obtained and are supported by an illustrative example.

Keywords: Entropy, primal and dual geometric programming, unconstrained optimization, Lagrange multiplier, sequential minimization.

1. INTRODUCTION

Entropy models are emerging as valuable tools in the study of various social and engineering problems of spatial interaction. In the study of transportation problems or more precisely spatial interaction problems the researcher is often confronted with phenomena, which are a pairing of two locations. These pairing may be, for example, home and business location for a worker, home and school location for a student, home and shopping center locations for a housewife, warehouse and retail shops for a

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company, origin and destination of a central business district of a transport system etc. In general while we may have some idea about the number of people who live, work, go to school or shop in various locations it is very difficult to acquire information on the pairing of locations caused by the various social transactions. Since there are many such pairing compatible with the generally available data it makes sense to choose the most probable set of pairings. This is the 'Principle of Insufficient Reason' of Laplace and the resulting problem is the maximization of entropy with respect to the available information or data.

Wilson, Webber [17], [18] pioneered in the use of entropy models in the study of spatial interaction. Entropy models are commonly used to find the most probable numbers of pairings x_{ij} between locations i and j given the numbers O_i of origins in location i and D_j , of destination in location j, for all locations. In equation form:

$$\sum_j X_{ij} = O_i \ , \ \forall\ i, \tag{1}$$

$$\sum_i X_{ij} = D_j \ , \ \forall j \tag{2}$$

The corresponding entropy, which we want to maximize is:

$$\frac{X!}{\prod_{i,j} X_{ij}!} \tag{3}$$

where $X = \sum_j \sum_i X_{ij} = \sum_i O_i = \sum_j D_j$.

In addition to the O_i and D_j we know the cost of a transaction i to j, c_{ij}. We also add this information to the model in the form of the cost equation:

$$\sum_j \sum_i c_{ij} X_{ij} = C^* \tag{4}$$

where C^* is a fitting parameter to be chosen according to the needs by the model maker. For our purpose, we replace equation (4) with a Lagrange multiplier γ to the logarithm of the objective function (3) so that the fitting parameter C^* can then be dropped and γ can perform its function. Let us introduce a new parameter $\tau_{ij} = e^{-\gamma c_{ij}}$. Combining these conditions the X_{ij} we seek are the solution of the following problem

Problem-I:

Maximize $X! \prod_{i,j} \left(\frac{\tau_{ij}^{X_{ij}}}{X_{ij}!} \right)$

subject to $\sum_j X_{ij} = O_i \quad \forall\ i,$ (5)

$$\sum_i X_{ij} = D_j \quad \forall j, \tag{6}$$

$X_{ij} \geq 0 \quad \forall i,j.$

Dacey and Norcliffe [5] consider a variation of Problem-I. There flexible entropy model relaxes the equations (5) and (6) to inequalities:

$$\sum_j X_{ij} \le O_i \quad \forall\, i, \tag{7}$$

$$\sum_i X_{ij} \le D_j \quad \forall j, \tag{8}$$

It is no longer required that $\sum_j D_j = X$ or $\sum_i O_i = X$ so that for the entropy function the following constraints must be added

$$\sum_j \sum_i X_{ij} = X \tag{9}$$

Replacing (5), (6), (7), (8) and adding (9) the flexible entropy model is:

Problem-II:

Maximize $X! \prod_{i,j} \left(\frac{\tau_{ij}^{X_{ij}}}{X_{ij}!} \right)$

subject to $\sum_j X_{ij} \le O_i \quad \forall\, i,$ (10)

$$\sum_i X_{ij} \le D_j \quad \forall j, \tag{11}$$

$$\sum_j \sum_i X_{ij} = X$$

$$X_{ij} \ge 0 \quad \forall i,j.$$

In fact, instead of replacing both the equalities (5) and (6) by inequalities a more appropriate model might be to replace only one set of equalities by inequalities since the model is symmetric with respect to these sets of equations. Let us replace the first set (5) by inequalities. This will give us the following problem:

Problem-III:

Maximize $X! \prod_{i,j} \left(\frac{\tau_{ij}^{X_{ij}}}{X_{ij}!} \right)$

subject to $\sum_j X_{ij} \le O_i \quad \forall\, i,$ (12)

$$\sum_i X_{ij} \le D_j \quad \forall j, \tag{13}$$

$$X_{ij} \ge 0 \quad \forall i,j.$$

Now when the quantities O_i, D_j, τ_{ij} are known to us, Problem-III seems to be the best model to handle. To forecast the effects of initial interaction there is no reason to assume that homes O_i or jobs D_j will be fully occupied. This should be left to the model to determine. The model to be here is Problem-II. Again to study the effect of a factory starting up or a new housing development on a community Problem-III is suitable for application.

We will use the theory of geometric programming to analyse these three entropy problems. Duffin, Peterson and Zener [4], Braighter and Philips [2] developed geometric programming to solve a class of problems called Posynomial problems. In the following section we shall represent a brief description of the theory of geometric programming.

2. DISCUSSION OF GEOMETRIC PROGRAMMING

In this section we present an analysis of posynomial programs using the theory of geometric programming. A usual posynomial program has the mathematical form:

Problem-IV:

Minimize $\psi_0(p)$

subject to $\psi_q(p) \le 1$, for q = 1,2,.........,t

$p = (p_1, p_2, \ldots\ldots\ldots, p_m) > 0$

$$\psi_q(p) = \sum_{i \in [q]} c_i \prod_{j=1}^{m} p_j^{l_{ij}}$$

where the coefficients c_i >0 but the exponent l_{ij} are arbitrary real numbers. The index set [q], q = 1,2,.........,t are arbitrary partition of the integers i to n. i.e. the total number of terms in the objective function and constraints. The positivity being required by the coefficients c_i in the functions of posynomial form means that the name posynomial is an abbreviation of positive polynomial.

The dual of **Problem-IV** is:

Problem-V:

$$\text{Maximize } \Phi(\varepsilon) = \prod_{i=1}^{n} \left(\frac{c_i}{\varepsilon_i} \right)^{\varepsilon_i} \prod_{q=1}^{t} \mu_q^{\mu_q}$$

where $\mu_q = \sum_{i \in [q]} \varepsilon_i$ for q = 1,2,...,t

$\varepsilon_i \ge 0$ i =1,2,...,n

$$\sum_{i=1}^{n} l_{ij} \varepsilon_i = 0 \quad \text{for } j = 1,2,\ldots,m$$

$$\sum_{i \in [0]} \varepsilon_i = 1$$

For feasible points of **Problem-IV** and **Problem-V** p and ε , $\Phi(\varepsilon) \le \psi_0(p)$.

At the optimality the solution to **Problem-IV** and **Problem-V** are related by the following equations:

$$\sum_{j=1}^{m} l_{ij} \log p_j = \begin{cases} \log \dfrac{\varepsilon_i \phi(\varepsilon)}{c_i}, & \text{i} \in [0] \\ \log \dfrac{\varepsilon_i}{c_i \mu_q}, & \text{i} \in [\text{q}]\ and\ \mu_q > 0\ \ and\ q \ne 0 \end{cases} \tag{14}$$

3. ANALYSIS OF THE ENTROPY MODELS

To apply geometric programming to the entropy models it is first necessary to apply the Stirling approximation $N! \approx N^N e^{-N}$ to the objective functions and to introduce certain variable transformation to convert Problems I, II and III into the form of Problem-V. When this is done correctly Problem-I becomes:

Problem-VI:

Maximize $\left(\dfrac{1}{\varepsilon_0}\right)^{\varepsilon_0} \prod_{i,j} \left(\dfrac{\tau_{ij}}{X_{ij}}\right)^{X_{ij}} (X)^X$

subject to $\sum_j X_{ij} - O_i \varepsilon_0 = 0 \quad \forall\ i,$

$\sum_i X_{ij} - D_j \varepsilon_0 = 0 \quad \forall j,$

$\varepsilon_0 = 1$

$X_{ij} \ge 0 \quad \forall i,j.$

Problem-II takes the form:

Problem-VII:

Maximize $\left(\dfrac{1}{\varepsilon_0}\right)^{\varepsilon_0} \prod_{i,j} \left(\dfrac{\tau_{ij}}{X_{ij}}\right)^{X_{ij}} (X)^X$

subject to $\sum_j X_{ij} - O_i \varepsilon_0 + \rho_i = 0 \quad \forall i,$

$\sum_i X_{ij} - D_j \varepsilon_0 + \sigma_j = 0 \quad \forall j,$

$\sum_j \sum_i X_{ij} - X\varepsilon_0 = 0$

$\varepsilon_0 = 1$

$X_{ij} \ge 0\ ,\ \rho_i \ge 0, \sigma_j \ge 0 \quad \forall i,j.$

Problem-III takes the form:

Problem-VIII:

$$\text{Maximize } \left(\frac{1}{\varepsilon_0}\right)^{\varepsilon_0} \prod_{i,j} \left(\frac{\tau_{ij}}{X_{ij}}\right)^{X_{ij}} (X)^X$$

$$\text{subject to } \sum_j X_{ij} - O_i\varepsilon_0 + \rho_i = 0 \quad \forall\, i,$$

$$\sum_i X_{ij} - D_j\varepsilon_0 = 0 \quad \forall j,$$

$$\varepsilon_0 = 1$$

$$X_{ij} \geq 0\,,\ \rho_i \geq 0 \quad \forall i,j.$$

Now we are in a position to determine the geometric programming dual to the entropy models. In fact the dual of Problem-VI is:

Problem-IX:

$$\text{Minimize } \prod_i v_i^{-O_i} \prod_j u_j^{-D_j}$$

$$\text{subject to } \sum_i \sum_j \tau_{ij} v_i u_j \leq 1 \tag{15}$$

$$v_i > 0 \quad \forall\, \text{i}$$

$$u_j > 0 \quad \forall\, \text{j}$$

The dual of Problem-VII is:

Problem-X:

$$\text{Minimize } \prod_i v_i^{-O_i} \prod_j u_j^{-D_j} w^{-X}$$

$$\text{subject to } \sum_i \sum_j \tau_{ij} v_i u_j w \leq 1 \tag{16}$$

$$1 \geq v_i > 0 \quad \forall\, i$$

$$1 \geq u_j > 0 \quad \forall\, j$$

The dual of Problem-VIII is:

Problem-XI:

$$\text{Minimize } \prod_i v_i^{-O_i} \prod_j u_j^{-D_j}$$

$$\text{subject to } \sum_i \sum_j \tau_{ij} v_i u_j \leq 1 \tag{17}$$

$$1 \geq v_i > 0 \quad \forall\, i$$

$$u_j > 0 \quad \forall\, j$$

Now we observe that $\tau_{ij} / \sum_i \sum_j \tau_{ij}$ is the a priori probability of a trip going from i to j. The dual problems IX, X and XI seek maximum likelihood estimators of the a posteriori probabilities of a trip going from i to j with the additional information of the O_i and D_j. Therefore to solve the original entropy problems I, II and III we look at the optimality condition (14) between problems VI, VII, VIII and IX, X, XI. We can easily see that the problems VI and IX and problems VIII, XI are related at the optimality by the relation

$$\frac{X_{ij}}{X} = \tau_{ij} v_i u_j \quad \forall i,j \tag{18}$$

whereas Problems VII, X is joined by

$$\frac{X_{ij}}{X} = \tau_{ij} v_i u_j \, w \quad \forall i,j \tag{19}$$

We can use (18) in (5) and (6) which gives us the following relationship in the u_j's and v_i's

$$\frac{O_i}{X} = v_i \sum_j \tau_{ij} u_j \quad \forall i \tag{20}$$

$$\frac{D_j}{X} = u_j \sum_i \tau_{ij} v_i \quad \forall j \tag{21}$$

Also using (18) in (12) and (13) gives us the following relationships

$$\frac{O_i}{X} = v_i \sum_j \tau_{ij} u_j \quad \text{or} \quad v_i = 1 \ \forall i \tag{22}$$

$$\frac{D_j}{X} = u_j \sum_i \tau_{ij} v_i \quad \forall j \tag{23}$$

For the problems VIII and X we combine (19) with (10) and (11) to generate the following relationships

$$\frac{O_i}{X} = v_i w \sum_j \tau_{ij} u_j \quad \text{or} \quad v_i = 1 \ \forall i \tag{24}$$

$$\frac{D_j}{X} = u_j w \sum_i \tau_{ij} v_i \quad \text{or} \quad u_j = 1 \ \forall j \tag{25}$$

$$w = \left[\sum_i \sum_j \tau_{ij} v_i u_j \right]^{-1} \tag{26}$$

If the inequalities (12) are equalities at the optimum for the problem-III, the relationships reduce to those for the problem-VI and problem-IX. At the other extreme if all inequalities hold strictly at the optimum, then

$$v_i = 1 \ \forall i \quad \text{and} \quad \frac{D_j}{X} = u_j \sum_i \tau_{ij} \ \forall j.$$

Therefore

$$u_j = \frac{D_j}{X \sum_i \tau_{ij}} \quad \forall j \tag{27}$$

Using (27) in (18) the solution is

$$X_{ij} = \frac{\tau_{ij} D_j}{\sum_i \tau_{ij}} \quad \forall\ i,j$$

Exactly in the similar manner with the pair of Problem-VII and Problem-X if the inequalities (10), (11) in Problem-II hold with equality the relationships (24)-(26) reduce to (20),(21) the conditions for problem-I. If all inequalities are strict then we have

$$v_i = 1 \qquad \forall i$$

$$u_j = 1 \qquad \forall j$$

$$w = \left[\sum_{i,j} \tau_{ij} \right]^{-1}$$

So that the solution is

$$X_{ij} = \frac{\tau_{ij} X}{\sum_{i,j} \tau_{ij}}$$

To determine the solutions for the entropy models in general we must concentrated to algorithm which is easier to work with the u_j's and v_i 's rather than the X_{ij}'s and to use equations (18),(19) to determine X_{ij}'s from the knowledge of u_j's and v_i 's. The primal problem has only one constraint and through duality its Lagrange multiplier is found. Thus the primal problem is equivalent to an unconstrained optimization problem.

Algorithm I: To solve Problem-I we may use the following algorithm

Step-1: Set n = 0 and $u_j^n = \frac{D_j}{X}$

Step-2: Set n to n+1 and $v_i^n = \dfrac{O_i}{X\left(\sum_j \tau_{ij} u_j^{n-1}\right)}$

Step-3: $u_j^n = \dfrac{D_j}{X\left(\sum_i \tau_{ij} v_i^n\right)}$

Step-4: Test of feasibility- $x v_i^n \sum_j \tau_{ij} u_j^n = O_i + e_i^n \ \ \forall i$

where e_i^n is the error in the i^{th} constraint. If $|\ e_i^n\ |$ is less then a predetermined tolerance for all *i*, then go to step-5 otherwise go to step-6.

Step-5: Test of optimality: $d^n = \sum_i e_i^n \log v_i^n$

where *d* is the difference between the logarithms of the objective functions for Problem-IX and Problem-VI. If $|\ d^n\ |$ is less than a predetermined tolerance, stop. Otherwise go to step-6.

Step-6: Set *n* to *n*+1 and $u_j^n = \dfrac{u_j^{n-1}}{\sum_j u_j^{n-1}}$ and repeat step-2.

Algorithm II: For Problem-X we developed the following algorithm

Step-1: Set $n = 0$ and $v_i^n = 1\ \forall i, \quad w^n = 1$

Step-2: Set n to n+1 and $u_j^n = \min_{\forall j}\left\{ 1, \frac{D_j}{Xw^n\left(\sum_i \tau_{ij} v_i^{n-1}\right)} \right\}$

Step-3: $v_i^n = \min_{\forall i}\left\{ 1, \frac{O_i}{Xw^n\left(\sum_j \tau_{ij} u_j^{n-1}\right)} \right\}$

Step-4: $w^n = \frac{1}{\sum_i \sum_j \tau_{ij} v_i u_j}$

Step-5: Cheek the feasibility-

$$Xv_i^n w^n \sum_j \tau_{ij} u_j^n \le O_i + e_i^n \quad \forall i$$

$$Xu_j^n w^n \sum_i \tau_{ij} v_i^n \le D_j + f_j^n \quad \forall j$$

where $e_i^n \ge 0$ & $f_j^n \ge 0$ are the error in the i^{th} and j^{th} inequality respectively. If e_i^n , f_j^n are less than a predetermined tolerance then go to step-5 otherwise go to step-6.

Step-6: Cheek the optimality:

$$d^n = \sum_j \left(u_j^n w^n \left(\sum_i \tau_{ij} v_i^n \right) - D_j \right) \log u_j^n + \sum_i \left(v_i^n w^n \left(\sum_j \tau_{ij} u_j^n \right) - O_i \right) \log v_i^n$$

where d^n is the difference between the logarithm of the objective functions of Problem-X and Problem-VIII. If e_i^n , f_j^n and d^n are less than a predetermined tolerance, stop. Otherwise go to step-2.

The algorithm for the solution of other entropy problem discussed in this paper is similar. Now we are in a position to prove the convergence for the algorithm to solve the entropy problem. Let us prove the convergence to solve the entropy problem. Let us prove the convergence to solve the Problem-X. The proofs of the convergence for the other are similar.

The Lagrangian of the logarithm of Problem-X is

$$L(u, v, w ; \lambda) = -\sum_i O_i \log v_i - \sum_j D_j \log u_j - X \log w + \lambda\left(\sum_i \sum_j \tau_{ij} v_i u_j w - 1 \right)$$

The partial derivative of the Lagrangian is

$$\frac{\partial L}{\partial v_i} = -\frac{O_i}{v_i} + \lambda \sum_j \tau_{ij} u_j$$

$$\frac{\partial L}{\partial u_j} = -\frac{D_j}{u_j} + \lambda \sum_i \tau_{ij} v_i$$

$$\frac{\partial L}{\partial w} = -\frac{X}{w} + \lambda \sum_i \sum_j \tau_{ij} u_j v_i$$

The optimality conditions resulting from the duality theory of geometric programming show that $\lambda = X$, sequential minimization of the Lagrangian over u then *v* then *w* produces a convergent algorithm. The minimization over u keeping *v*, *w* fixed gives

$$u_j = \min_{\forall j} \left\{ 1, \frac{D_j}{Xw \left(\sum_i \tau_{ij} v_i \right)} \right\} \tag{28}$$

The minimum over *v* for fixed *u*, *w* is attained at *v* for

$$v_i = \min_{\forall i} \left\{ 1, \frac{O_i}{Xw \left(\sum_j \tau_{ij} u_j \right)} \right\} \tag{29}$$

Also the minimum over w for fixed *u*, *v* is attained at

$$w = \frac{1}{\sum_i \sum_j \tau_{ij} v_i u_j} \tag{30}$$

(28), (29) and (30) are 2, 3, 4 of the algorithm. We use conditions developed by geometric programming to determine when we are close to optimality. These are Step 5 and 6.

4. NUMERICAL RESULTS

To illustrate the preceding any models (take model-I) consider the example

$$\text{Maximize } \left(\frac{1}{\varepsilon_0}\right)^{\varepsilon_0} \prod_{i,j} \left(\frac{\tau_{ij}}{X_{ij}}\right)^{X_{ij}} (X)^X$$

subject to $\sum_{i\neq 1} X_{i1} = 3$, $\sum_{i\neq 2} X_{i2} = 3$, $\sum_{i\neq 3} X_{i3} = 2$, $\sum_{i\neq 4} X_{i4} = 2$

$\sum_{j\neq 1} X_{1j} = 2$, $\sum_{j\neq 2} X_{2j} = 2$, $\sum_{j\neq 3} X_{3j} = 3$, $\sum_{j\neq 4} X_{4j} = 3$

$\varepsilon_0 = 1$, $X_{ij} \geq 0$ for $i, j =$ 1,2,3,4

where $\tau_{21} = 2$, $\tau_{31} = 4$, $\tau_{41} = 3$,

$\tau_{12} = 7$, $\tau_{32} = 3$, $\tau_{42} = 4$,

$\tau_{13} = 6$, $\tau_{23} = 3$, $\tau_{43} = 8$,

$\tau_{14} = 4$, $\tau_{24} = 2$, $\tau_{34} = 5$.

Use **Algorithm I** to solve the above numerical problem is as follows:

First iteration result:

$u_1^0 = 0.3$, $v_1^1 = 0.04878$, $u_1^1 = 0.34078$,
$u_2^0 = 0.3$, $v_2^1 = 0.12500$, $u_2^1 = 0.24405$,
$u_3^0 = 0.2$, $v_3^1 = 0.09677$, $u_3^1 = 0.12899$,
$u_4^0 = 0.2$, $v_4^1 = 0.08108$, $u_4^1 = 0.18432$,
$X_{12} = 0.83335$, $X_{21} = 0.85195$, $X_{31} = 1.31909$, $X_{41} = 0.82891$,
$X_{13} = 0.35404$, $X_{23} = 0.48372$, $X_{32} = 0.70852$, $X_{42} = 0.79152$,
$X_{14} = 0.35965$, $X_{24} = 0.46081$, $X_{34} = 0.87256$, $X_{43} = 0.83672$,
$e_1^1 = 0.45295$, $e_2^1 = 0.20353$, $e_3^1 = 0.09984$, $e_4^1 = 0.54285$, $d^1 = -3.38829$

Similarly second iteration, we get

$u_1^2 = 0.37942$, $v_1^3 = 0.05579$, $u_1^3 = 0.34327$,
$u_2^2 = 0.27173$, $v_2^3 = 0.12499$, $u_2^3 = 0.19722$,
$u_3^2 = 0.143624$, $v_3^3 = 0.08931$, $u_3^3 = 0.14074$,
$u_4^2 = 0.20522$, $v_4^3 = 0.08891$, $u_4^3 = 0.21746$,
$X_{12} = 1.06122$, $X_{21} = 0.94845$, $X_{31} = 1.35548$, $X_{41} = 1.01204$,
$X_{13} = 0.48078$, $X_{23} = 0.53853$, $X_{32} = 0.72806$, $X_{42} = 0.96638$,
$X_{14} = 0.45799$, $X_{24} = 0.51301$, $X_{34} = 0.91645$, $X_{43} = 1.02157$
$e_1^2 = -0.00000089$, $e_2^2 = -0.0000007$, $e_3^2 = -0.00000013$, $e_4^2 = -0.00000011$.

and third iteration

$u_1^4 = 0.37942$, $v_1^5 = 0.05579$, $u_1^5 = 0.34327$,
$u_2^4 = 0.27173$, $v_2^5 = 0.12499$, $u_2^5 = 0.29582$,
$u_3^4 = 0.14362$, $v_3^5 = 0.08931$, $u_3^5 = 0.14674$,
$u_4^4 = 0.20522$, $v_4^5 = 0.08891$, $u_4^5 = 0.21746$,
$X_{12} = 1.15531$, $X_{21} = 0.85868$, $X_{31} = 1.22631$, $X_{41} = 0.91560$,
$X_{13} = 0.47115$, $X_{23} = 0.52774$, $X_{32} = 0.79262$, $X_{42} = 1.05207$,
$X_{14} = 0.48530$, $X_{24} = 0.54359$, $X_{34} = 0.97109$, $X_{43} = 1.00110$
$e_1^3 = -0.11177$, $e_2^3 = 0.07508$, $e_3^3 = 0.009968$, $e_4^3 = 0.03122$.

5. CONCLUSION

In this paper we have analyzed variety entropy models by geometric programming. In all cases the effect of the constraints on the solution x_{ij} was multiplicative in terms of the dual variables v_i and u_j. The parameter τ_{ij} which capture the a priori probability of an interaction between *i* and *j* is the other important key factor. If there are $m \times n$ variable τ_{ij} in the entropy problem there are $m+n$ variables in the geometric dual problems. Therefore duality provides us with a substantial saving in the number of variables and hence much simpler problem to solve. Finally the algorithm is developed using pair of primal and dual problems and are relatively simple to program.

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