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RETAILER'S LOT-SIZING POLICY UNDER TWO WAREHOUSES AND TWO LEVELS OF DELAY PERMITTED USING ALGEBRAIC METHOD

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Abstract: The main purpose of this paper is to modify Huang's model [13] by considering two warehouses. In addition, we try to use algebraic method to determine the optimal lot-sizing policy for the retailer under two warehouses and two levels of delay permitted. This paper provides this algebraic approach that could be used easily to introduce the basic inventory theories to younger students who lack the knowledge of calculus. Furthermore, we develop three easy-to-use theorems to efficiently determine the optimal cycle time and optimal lot sizing for the retailer. As a result, we deduce some previously published results of other researchers as special cases. Finally, a numerical example is given to illustrate these theorems obtained in this paper. In addition, we obtain a lot of managerial insights from this numerical example.

Keywords*:* EOQ, inventory, two warehouses, two levels of delay permitted, algebraic method.

1. INTRODUCTION

In most business transactions, the supplier will allow a specified credit period to the retailer for payment without penalty to stimulate the demand of his/her products. All previously published models discussing permitted delay assumed that the supplier would offer the retailer a delay period but the retailer would not offer the delay period to his/her customer. That is one level of delay permitted. Recently, Huang [13] modified this postulation to assume that the retailer will adopt the delay permitted policy to stimulate his/her customer demand to develop the retailer's replenishment model. These are two levels of delay permitted. This new viewpoint is more matched real-life situations in the supply chain model. Many papers have appeared in the literature that treat inventory problems with varying conditions under one level of delay permitted. Some of the prominent papers are discussed below.

Goyal [11] established a single-item inventory model under permissible delay in payments. Chung [8] developed an alternative approach to determine the economic order quantity under condition of permissible delay in payments. Aggarwal and Jaggi [1] considered the inventory model with an exponential deterioration rate under the condition of permissible delay in payments. Chang et al. [6] extended this issue to the varying rate of deterioration. Liao et al. [18] and Sarker et al. [20] investigated this topic with inflation. Jamal et al. [16] and Chang and Dye [5] extended this issue with allowable shortage. Chang et al. [7] extended this issue with linear trend demand. Hwang and Shinn [15] modelled an inventory system for retailer’s pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payment. Jamal et al. [17] and Sarker et al. [21] addressed the optimal payment time under permissible delay in payment with deterioration. Teng [24] assumed that the selling price was not equal to the purchasing price to modify Goyal’s model [11]. Shinn and Hwang [23] determined the retailer’s optimal price and order size simultaneously under the condition of order-size-dependent delay in payments. They assumed that the length of the credit period is a function of the retailer’s order size, and also the demand rate is a function of the selling price. Chung and Huang [9] extended this problem within the EPQ framework and developed an efficient procedure to determine the retailer’s optimal ordering policy. Huang [13] extended this issue under two levels of trade credit and developed an efficient solution procedure to determine the optimal lot-sizing policy of the retailer. Huang and Chung [14] extended Goyal’s model [11] to cash discount policy for early payment.

Such delay permitted policy is one kind of encouragement of the retailer to order large quantities because a delay of payments indirectly reduces inventory cost. Hence, the retailer may purchase more goods than can be stored in its warehouse. Therefore, these excess quantities are stored in a rented warehouse. In general, the inventory costs for rented warehouse are higher than those for own warehouse. Several researchers have studied this area such as Benkherouf [2], Bhunia and Maiti [3], Goswami and Chaudhuri [10], Pakkala and Achary [19], Sarma [22] and Wu [25].

Therefore, the main purpose of this paper is an attempt to modify Huang’s model [13] by considering two warehouses. In addition, we try to use easier algebraic method to find the optimal solution in this paper. All previously published papers have been derived using differential calculus in order to find the optimal solution and the need to prove optimality condition with second-order derivatives. The mathematical methodology is difficult to many younger students who lack knowledge of calculus. In recent papers, Cárdenas-Barrón [4] and Grubbström and Erdem [12] showed that the formulae for the EOQ and EPQ with backlogging were derived without differential calculus. They mentioned that this approach must be considered as a pedagogical advantage for explaining the basic inventory concepts to students that lack knowledge of derivatives, simultaneous equations and the procedure to construct and examine the Hessian matrix. This algebraic approach could be easily used to introduce the basic inventory theories to younger students who lack knowledge of calculus.

So, this paper tries to deal with the retailer’s lot-sizing problem under two warehouses and two levels of delay permitted using algebraic method. In addition, we develop easy-to-use procedures to find the optimal lot-sizing policy for the retailer under minimizing annual total relevant cost.

2. MODEL FORMULATION

In this section, we want to develop the retailer's inventory model under two warehouses and two levels of delay permitted. For convenience, most notation and assumptions similar to Huang [13] will be used in this paper.

2.1 Notation:

A = ordering cost per order
c = unit purchasing price per item
D = demand rate per year
h = unit stock holding cost per item per year excluding interest charges
I_e = interest earned per \$ per year
I_p = interest charged per \$ in stocks per year by the supplier
k = unit stock holding cost of rented warehouse per item per year, ($k \geq h$)
M = the retailer's trade credit period offered by supplier in years
N = the customer's trade credit period offered by retailer in years
t_w = the rented warehouse time in years, $t_w = \begin{cases} \dfrac{DT - W}{D} & \text{if } DT > W,\ (T > \dfrac{W}{D}) \\ 0 & \text{if } DT \leq W,\ (T \leq \dfrac{W}{D}) \end{cases}$
T = the cycle time in years
W = retailer's storage capacity
$TRC(T)$ = the annual total relevant cost, which is a function of T
T^* = the optimal cycle time of $TRC(T)$
Q^* = the optimal order quantity = DT^*.

2.2 Assumptions:

(1) Demand rate is known and constant.
(2) Shortages are not allowed.
(3) Time horizon is infinite.
(4) Replenishments are instantaneous with a known and constant lead time.
(5) $k \geq h$ and $I_p \geq I_e$.
(6) If the order quantity is larger than retailer's storage capacity W, the retailer will rent the warehouse to storage these exceeding items. And the rented warehouse has unlimited capacity. When the demand occurs, it first is replenished from the warehouse which storages those exceeding items.
(7) When $T \geq M$, the account is settled at $T = M$ and the retailer starts paying for the interest charges on the items in stock with rate I_p. When $T \leq M$, the account is settled at $T = M$ and the retailer does not need to pay any interest charge.

(8) The retailer can accumulate revenue and earn interest after his/her customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier. That is, the retailer can accumulate revenue and earn interest during the period N to M with rate I_e under the condition of trade credit.

2.3. The model

The total annual relevant cost consists of the following elements. Three situations may arise. (I) $M \geq W/D \geq N$, (II) $M \geq N \geq W/D$ and (III) $W/D \geq M > N$.

Case I: Suppose that $\underline{M \geq W/D \geq N}$.

(1) Annual ordering cost = $\frac{A}{T}$.

(2) According to assumption (6), annual stock holding cost (excluding interest charges) can be obtained as follows.

(i) $\underline{W/D < T}$, shown in Figure 1 and Figure 2.

In this case, the order quantity is larger than retailer's storage capacity. So the retailer needs to rent the warehouse to storage the exceeding items. Hence

Annual stock holding cost = annual stock holding cost of rented warehouse + annual stock holding cost of the storage capacity W

$$= \frac{kt_w(DT - W)}{2T} + \frac{h\left[Wt_w + \frac{W(T - t_w)}{2}\right]}{T} = \frac{k(DT - W)^2}{2DT} + \frac{hW(2DT - W)}{2DT}$$

(ii) $\underline{T \leq W/D.}$

In this case, the order quantity is not larger than retailer's storage capacity. So the retailer will not necessary to rent warehouse to storage items. Hence

Annual stock holding cost = $\frac{DTh}{2}$.

(3) According to assumption (7), cost of interest charges for the items kept in stock per year can be obtained as follows.

(i) $\underline{M \leq T}$, shown in Figure 1.

Cost of interest charges for the items kept in stock per year

$$= \frac{cI_p D(T - M)^2}{2T}.$$

(ii) $\underline{T \leq M}.$

In this case, no interest charges are paid for the items kept in stock.

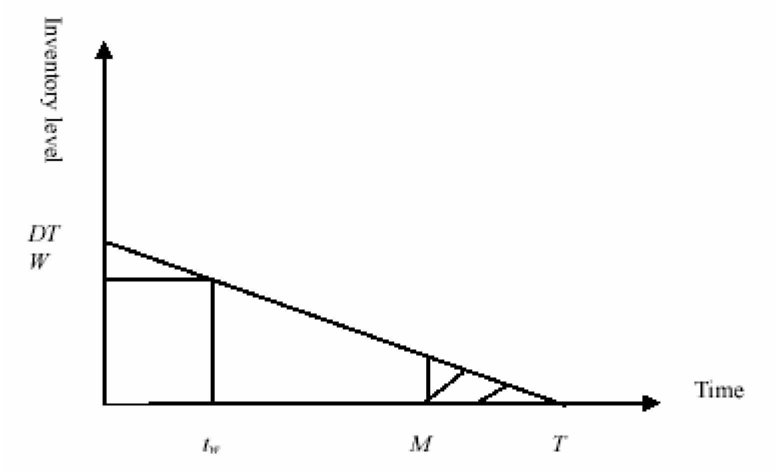

Figure 1: The inventory level and the total accumulation of interest payable when $M \le T$

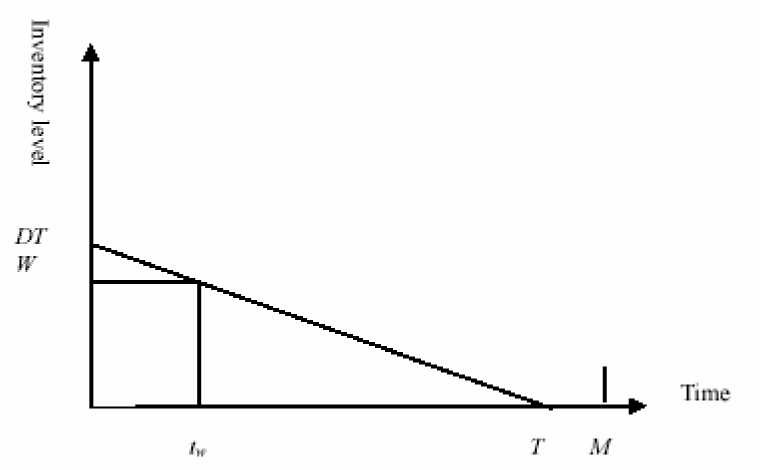

Figure 2: The inventory level when $W/D < T \le M$

(4) According to assumption (8), interest earned per year can be obtained as follows.

(i) $\underline{M \le T}$, shown in Figure 3.

Annual interest earned =

$$= cI_e\left[\frac{(DN + DM)(M - N)}{2}\right]/T = cI_e D(M^2 - N^2)/2T .$$

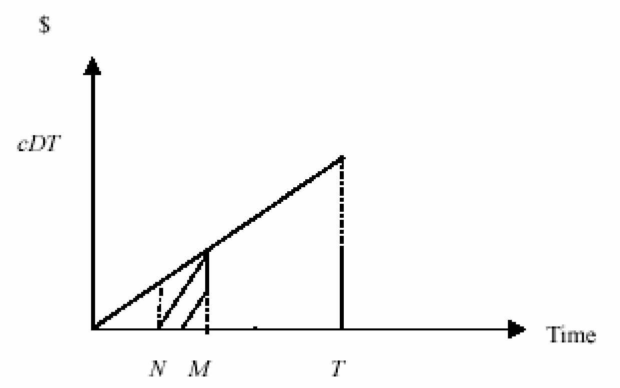

Figure 3: The total accumulation of interest earned when $M \le T$

(ii) $N \le T \le M$, shown in Figure 4.

Annual interest earned =

$$= cI_e\left[\frac{(DN+DT)(T-N)}{2} + DT(M-T)\right]/T$$

$$= cI_e D(2MT - N^2 - T^2)/2T.$$

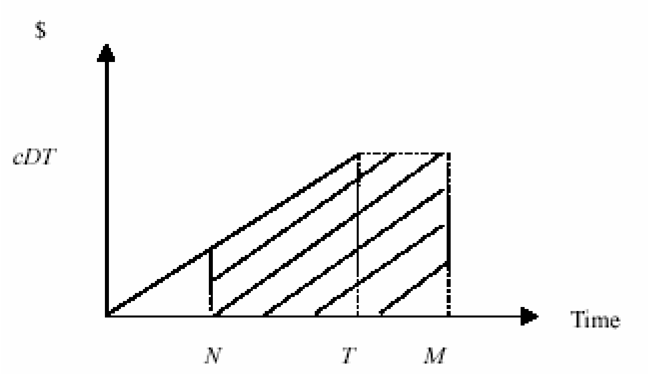

Figure 4: The total accumulation of interest earned when $N \le T \le M$

(iii) $0 < T \le N$, shown in Figure 5.

Annual interest earned = $cI_e DT(M-N)/T$.

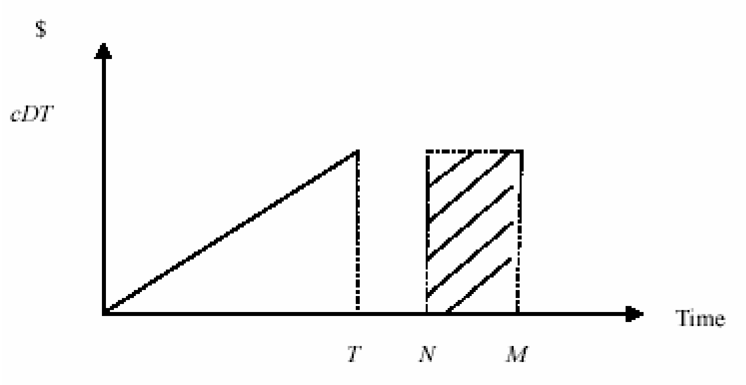

Figure 5: The total accumulation of interest earned when $T \le N$

From the above arguments, the annual total relevant cost for the retailer can be expressed as

$$TRC(T) = \text{ordering cost} + \text{stock-holding cost} + \text{ interest payable} - \text{interest earned}$$

We show that the annual total relevant cost, $TRC(T)$, is given by

$$TRC(T) = \begin{cases} TRC_1(T) & \text{if} \quad T \ge M & \text{(1a)} \\ TRC_2(T) & \text{if} \quad W/D < T \le M & \text{(1b)} \\ TRC_3(T) & \text{if} \quad N \le T \le W/D & \text{(1c)} \\ TRC_4(T) & \text{if} \quad 0 < T \le N & \text{(1d)} \end{cases}$$

where

$$TRC_1(T) = \frac{A}{T} + \frac{k(DT-W)^2}{2DT} + \frac{hW(2DT-W)}{2DT} + \frac{cI_p D(T-M)^2}{2T} - \frac{cI_e D(M^2-N^2)}{2T} \tag{2}$$

$$TRC_2(T) = \frac{A}{T} + \frac{k(DT-W)^2}{2DT} + \frac{hW(2DT-W)}{2DT} - \frac{cI_e D(2MT-N^2-T^2)}{2T} \tag{3}$$

$$TRC_3(T) = \frac{A}{T} + \frac{DTh}{2} - \frac{cI_e D(2MT-N^2-T^2)}{2T} \tag{4}$$

and

$$TRC_4(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e D(M-N) \tag{5}$$

Since $TRC_1(M)=TRC_2(M)$, $TRC_2(W/D)=TRC_3(W/D)$ and $TRC_3(N)=TRC_4(N)$, $TRC(T)$ is continuous and well defined on $T > 0$. All $TRC_1(T)$, $TRC_2(T)$, $TRC_3(T)$, $TRC_4(T)$ and $TRC(T)$ are defined on $T > 0$.

Case II: Suppose that $\underline{M \geq N \geq W/D}$.

If $M \geq N \geq W/D$, equations 1(a, b, c, d) will be modified as

$$TRC(T) = \begin{cases} TRC_1(T) & \text{if} \quad T \geq M & \text{(6a)} \\ TRC_2(T) & \text{if} \quad N \leq T \leq M & \text{(6b)} \\ TRC_5(T) & \text{if} \quad W/D < T \leq N & \text{(6c)} \\ TRC_4(T) & \text{if} \quad 0 < T \leq W/D & \text{(6d)} \end{cases}$$

When $W/D < T \leq N$, the annual total relevant cost, $TRC_5(T)$, consists of the following elements.

(1) Annual ordering cost = $\dfrac{A}{T}$.

(2) In this case, the order quantity is larger than retailer's storage capacity. So the retailer needs to rent the warehouse to storage the exceeding items. Hence

Annual stock holding cost = $\dfrac{k(DT-W)^2}{2DT} + \dfrac{hW(2DT-W)}{2DT}$.

(3) In this case, no interest charges are paid for the items kept in stock.

(4) Annual interest earned = $cI_e DT(M-N)/T$.

Combining the above elements, we get

$$TRC_5(T) = \frac{A}{T} + \frac{k(DT-W)^2}{2DT} + \frac{hW(2DT-W)}{2DT} - cI_e D(M-N) \tag{7}$$

Since $TRC_1(M) = TRC_2(M)$, $TRC_2(N) = TRC_5(N)$ and $TRC_5(W/D) = TRC_4(W/D)$, $TRC(T)$ is continuous and well defined on $T > 0$. All $TRC_1(T)$, $TRC_2(T)$, $TRC_5(T)$, $TRC_4(T)$ and $TRC(T)$ are defined on $T > 0$.

Case III: Suppose that $\underline{W/D \geq M \geq N}$.

If $W/D \geq M \geq N$, equations 1(a, b, c, d) will be modified as

$$TRC(T) = \begin{cases} TRC_1(T) & \text{if} \quad T > W/D & \text{(8a)} \\ TRC_6(T) & \text{if} \quad M \leq T \leq W/D & \text{(8b)} \\ TRC_3(T) & \text{if} \quad N \leq T \leq M & \text{(8c)} \\ TRC_4(T) & \text{if} \quad 0 < T \leq N & \text{(8d)} \end{cases}$$

When $M \leq T \leq W/D$, the annual total relevant cost, $TRC_6(T)$, consists of the following elements.

(1) Annual ordering cost = $\dfrac{A}{T}$.

(2) In this case, the order quantity is not larger than retailer's storage capacity. So the retailer will not necessary to rent warehouse to storage items. Hence

Annual stock holding cost = $\dfrac{DTh}{2}$.

(3) Cost of interest charges for the items kept in stock per year = $\dfrac{cI_p D(T-M)^2}{2T}$.

(4) Annual interest earned = $cI_e D(M^2 - N^2)/2T$.

Combined above elements, we get

$$TRC_6(T) = \frac{A}{T} + \frac{DTh}{2} + \frac{cI_p D(T-M)^2}{2T} - \frac{cI_e D(M^2 - N^2)}{2T} \tag{9}$$

Since $TRC_1(W/D)=TRC_6(W/D)$, $TRC_6(M)=TRC_3(M)$ and $TRC_3(N)=TRC_4(N)$, $TRC(T)$ is continuous and well defined on $T > 0$. All $TRC_1(T)$, $TRC_6(T)$, $TRC_3(T)$, $TRC_4(T)$ and $TRC(T)$ are defined on $T > 0$.

3. DECISION RULE OF THE OPTIMAL CYCLE TIME *T**

In this section, we shall determine optimal cycle time for above three cases under minimizing annual total relevant cost using algebraic method.

Case I: Suppose that $\underline{M \geq W/D \geq N}$.

Then, we can rewrite

$$\begin{aligned} TRC_1(T) &= \frac{2A + \frac{W^2}{D}(k-h) + cD[M^2(I_p - I_e) + N^2 I_e]}{2T} + \frac{DT(k+cI_p)}{2} \\ &-[W(k-h)+cDMI_p] = \frac{D(k+cI_p)}{2T}\left[T - \sqrt{\frac{2A + \frac{W^2}{D}(k-h) + cD[M^2(I_p - I_e) + N^2 I_e]}{D(k+cI_p)}}\right]^2 \\ &+\left\{\sqrt{D(k+cI_p)\left[2A + \frac{W^2}{D}(k-h) + cD(M^2(I_p - I_e) + N^2 I_e)\right]} - [W(k-h)+cDMI_p]\right\}. \end{aligned} \tag{10}$$

Equation (10) represents that the minimum of $TRC_1(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_1* is

$$T_1^* = \sqrt{\frac{2A + \frac{W^2}{D}(k-h) + cD[M^2(I_p - I_e) + N^2 I_e]}{D(k+cI_p)}} \tag{11}$$

Therefore, equation (10) has a minimum value for the optimal value of T_1^* reducing $TRC_1(T)$ to

$$TRC_1(T_1^*) = \left\{ \sqrt{D(k+cI_p)\left[2A + \frac{W^2}{D}(k-h) + cD(M^2(I_p - I_e) + N^2 I_e)\right]} - [W(k-h) + cDMI_p] \right\} \tag{12}$$

Similarly, we can derive $TRC_2(T)$ without derivatives as follows.

$$\begin{aligned} TRC_2(T) &= \frac{2A + \frac{W^2}{D}(k-h) + cDN^2 I_e}{2T} + \frac{DT(k+cI_e)}{2} \\ &- [W(k-h) + cDMI_e] = \frac{D(k+cI_e)}{2T}\left[T - \sqrt{\frac{2A + \frac{W^2}{D}(k-h) + cDN^2 I_e}{D(k+cI_e)}}\right]^2 \\ &+ \left\{ \sqrt{D(k+cI_e)\left[2A + \frac{W^2}{D}(k-h) + cDN^2 I_e\right]} - [W(k-h) + cDMI_e] \right\}. \end{aligned} \tag{13}$$

Equation (13) represents that the minimum of $TRC_2(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_2^* is

$$T_2^* = \sqrt{\frac{2A + \frac{W^2}{D}(k-h) + cDN^2 I_e}{D(k+cI_e)}} \tag{14}$$

Therefore, equation (13) has a minimum value for the optimal value of T_2^* reducing $TRC_2(T)$ to

$$TRC_2(T_2^*) = \left\{ \sqrt{D(k+cI_e)\left[2A + \frac{W^2}{D}(k-h) + cDN^2 I_e\right]} - [W(k-h) + cDMI_e] \right\} \tag{15}$$

Likewise, we can derive $TRC_3(T)$ algebraically as follows.

$$\begin{aligned} TRC_3(T) &= \frac{2A + cDN^2 I_e}{2T} + \frac{DT(h+cI_e)}{2} - cDMI_e \\ &= \frac{D(h+cI_e)}{2T}\left[T - \sqrt{\frac{2A + cDN^2 I_e}{D(h+cI_e)}}\right]^2 + \left[\sqrt{D(h+cI_e)(2A + cDN^2 I_e)} - cDMI_e\right]. \end{aligned} \tag{16}$$

Equation (16) represents that the minimum of $TRC_3(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_3^* is

$$T_3^* = \sqrt{\frac{2A + cDN^2 I_e}{D(h + cI_e)}} \tag{17}$$

Therefore, equation (16) has a minimum value for the optimal value of T_3^* reducing $TRC_3(T)$ to

$$TRC_3(T_3^*) = \left[\sqrt{D(h + cI_e)(2A + cDN^2 I_e)} - cDMI_e\right] \tag{18}$$

At last, we can derive $TRC_4(T)$ algebraically as follows.

$$\begin{aligned} TRC_4(T) &= \frac{A}{T} + \frac{DTh}{2} - cI_e D(M - N) \\ &= \frac{Dh}{2T}\left[T - \sqrt{\frac{2A}{Dh}}\right]^2 + \left[\sqrt{2ADh} - cDI_e(M - N)\right]. \end{aligned} \tag{19}$$

Equation (19) represents that the minimum of $TRC_4(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_4^* is

$$T_4^* = \sqrt{\frac{2A}{Dh}} \tag{20}$$

Therefore, equation (19) has a minimum value for the optimal value of T_4^* reducing $TRC_4(T)$ to

$$TRC_4(T_4^*) = \left[\sqrt{2ADh} - cDI_e(M - N)\right] \tag{21}$$

Equation (11) implies that the optimal value of T for the case of $T \geq M$, that is $T_1^* \geq M$. We substitute equation (11) into $T_1^* \geq M$, then we can obtain that

if and only if $-2A - \frac{W^2}{D}(k - h) + DM^2(k + cI_e) - cDN^2 I_e \leq 0$.

Similarly, equation (14) implies that the optimal value of T for the case of $W/D < T \leq M$, that is $W/D < T_2^* \leq M$. We substitute equation (14) into $W/D < T_2^* \leq M$, then we can obtain that

if and only if $-2A - \frac{W^2}{D}(k - h) + DM^2(k + cI_e) - cDN^2 I_e \geq 0$

and

$$\text{if and only if } -2A+\frac{W^2}{D}(h+cI_e)-cDN^2I_e<0\ .$$

Likewise, equation (17) implies that the optimal value of T for the case of $N \le T \le W/D,$ that is $N \le T_3^* \le W/D$. We substitute equation (17) into $N \le T_3^* \le W/D,$ then we can obtain that

$$\text{if and only if } -2A+\frac{W^2}{D}(h+cI_e)-cDN^2I_e \ge 0$$

and

$$\text{if and only if } -2A+DN^2h \le 0\ .$$

Finally, equation (20) implies that the optimal value of T for the case of $T \le N$, that is $T_4^* \le N$. We substitute equation (20) into $T_4^* \le N,$ then we can obtain that

$$\text{if and only if } -2A+DN^2h \ge 0\ .$$

Furthermore, we let

$$\Delta_1=-2A-\frac{W^2}{D}(k-h)+DM^2(k+cI_e)-cDN^2I_e\ , \tag{22}$$

$$\Delta_2=-2A+\frac{W^2}{D}(h+cI_e)-cDN^2I_e \tag{23}$$

and

$$\Delta_3=-2A+DN^2h\ . \tag{24}$$

Equations (22), (23) and (24) imply that $\Delta_1 \ge \Delta_2 \ge \Delta_3$. From the above arguments, we can summarize the following results.

Theorem 1: *Suppose that* $M \ge W/D \ge N$ *, then*
(A) *If* $\Delta_3 \ge 0,$ *then* $TRC(T^*)=TRC(T_4^*)$ *and* $T^*=T_4^*$.
(B) *If* $\Delta_2 \ge 0$ *and* $\Delta_3 < 0,$ *then* $TRC(T^*)=TRC(T_3^*)$ *and* $T^*=T_3^*$.
(C) *If* $\Delta_1 > 0$ *and* $\Delta_2 < 0,$ *then* $TRC(T^*)=TRC(T_2^*)$ *and* $T^*=T_2^*$.
(D) *If* $\Delta_1 \le 0,$ *then* $TRC(T^*)=TRC(T_1^*)$ *and* $T^*=T_1^*$.

Case II: Suppose that $\underline{M \ge N \ge W/D}$.

If $M \ge N \ge W/D,$ we know $TRC(T)$ as follows from equations 6(a, b, c, d).

$$TRC(T)=\begin{cases} TRC_1(T) & \text{if} \quad T\geq M \\ TRC_2(T) & \text{if} \quad N\leq T\leq M \\ TRC_5(T) & \text{if} \quad W/D<T\leq N \\ TRC_4(T) & \text{if} \quad 0<T\leq W/D \end{cases}$$

From equation (7), we can derive $TRC_5(T)$ without derivatives as follows.

$$\begin{aligned} TRC_5(T) &= \frac{2A+\frac{W^2}{D}(k-h)}{2T}+\frac{kDT}{2}-[W(k-h)+cDI_e(M-N)] \\ &= \frac{kD}{2T}\left[T-\sqrt{\frac{2A+\frac{W^2}{D}(k-h)}{kD}}\right]^2 \\ &+\left\{\sqrt{kD\left[2A+\frac{W^2}{D}(k-h)\right]}-[W(k-h)+cDI_e(M-N)]\right\}. \end{aligned} \tag{25}$$

Equation (25) represents that the minimum of $TRC_5(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_5^* is

$$T_5^*=\sqrt{\frac{2A+\frac{W^2}{D}(k-h)}{kD}}\ . \tag{26}$$

Therefore, equation (25) has a minimum value for the optimal value of T_5^* reducing $TRC_5(T)$ to

$$TRC_5(T_5^*)=\left\{\sqrt{kD\left[2A+\frac{W^2}{D}(k-h)\right]}-[W(k-h)+cDI_e(M-N)]\right\}. \tag{27}$$

It is similar as above procedure in Case I. We substitute equation (11) into $T_1^*\geq M$, and then we can obtain that

$$\text{if and only if } -2A-\frac{W^2}{D}(k-h)+DM^2(k+cI_e)-cDN^2I_e\leq 0.$$

Substitute equation (14) into $N\leq T_2^*\leq M$, then we can obtain that

$$\text{if and only if } -2A-\frac{W^2}{D}(k-h)+DM^2(k+cI_e)-cDN^2I_e\geq 0$$

and

$$\text{if and only if } -2A-\frac{W^2}{D}(k-h)+DN^2k \le 0.$$

Substitute equation (26) into $W/D < T_5^* \le N$, then we can obtain that

$$\text{if and only if } -2A-\frac{W^2}{D}(k-h)+DN^2k \ge 0$$

and

$$\text{if and only if } -2A+\frac{W^2}{D}h < 0.$$

Substitute equation (20) into $T_4^* \le W/D$, then we can obtain that

$$\text{if and only if } -2A+\frac{W^2}{D}h \ge 0.$$

Furthermore, we let

$$\Delta_4 = -2A-\frac{W^2}{D}(k-h)+DN^2k \tag{28}$$

and

$$\Delta_5 = -2A+\frac{W^2}{D}h \tag{29}$$

Equations (22), (28) and (29) imply that $\Delta_1 \ge \Delta_4 \ge \Delta_5$. From above arguments, we can summarize following results.

Theorem 2: *Suppose that* $M \ge N \ge W/D$, *then*

(A) *If* $\Delta_5 \ge 0$, *then* $TRC(T^*) = TRC(T_4^*)$ *and* $T^* = T_4^*$.

(B) *If* $\Delta_4 \ge 0$ *and* $\Delta_5 < 0$ *then* $TRC(T^*) = TRC(T_5^*)$ *and* $T^* = T_5^*$.

(C) *If* $\Delta_1 > 0$ *and* $\Delta_4 < 0$, *then* $TRC(T^*) = TRC(T_2^*)$ *and* $T^* = T_2^*$.

(D) *If* $\Delta_1 \le 0$, *then* $TRC(T^*) = TRC(T_1^*)$ *and* $T^* = T_1^*$.

Case III: Suppose that $\underline{W/D \ge M \ge N}$**.**

If $W/D \ge M \ge N$, we know $TRC(T)$ as follows from equations 8(a, b, c, d)

$$TRC(T) = \begin{cases} TRC_1(T) & \text{if} \quad T > W/D \\ TRC_6(T) & \text{if} \quad M \le T \le W/D \\ TRC_3(T) & \text{if} \quad N \le T \le M \\ TRC_4(T) & \text{if} \quad 0 < T \le N \end{cases}$$

From equation (9), we can derive $TRC_6(T)$ without derivatives as follows.

$$\begin{aligned} TRC_6(T) &= \frac{2A + cD[M^2(I_p - I_e) + N^2 I_e]}{2T} + \frac{DT(h + cI_p)}{2} - cDMI_p \\ &= \frac{D(h+cI_p)}{2T}\left[T - \sqrt{\frac{2A + cD[M^2(I_p - I_e) + N^2 I_e]}{D(h+cI_p)}}\right]^2 \\ &+ \left\{\sqrt{D(h+cI_p)[2A + cD(M^2(I_p - I_e) + N^2 I_e)]} - cDMI_p\right\}. \end{aligned} \tag{30}$$

Equation (30) represents that the minimum of $TRC_6(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_6^* is

$$T_6^* = \sqrt{\frac{2A + cD[M^2(I_p - I_e) + N^2 I_e]}{D(h+cI_p)}}\ . \tag{31}$$

Therefore, equation (30) has a minimum value for the optimal value of T_6^* reducing $TRC_6(T)$ to

$$TRC_6(T_6^*) = \left\{\sqrt{D(h+cI_p)[2A + cD(M^2(I_p - I_e) + N^2 I_e)]} - cDMI_p\right\}. \tag{32}$$

It is similar as the above procedures in Case I and Case II. We substitute equation (11) into $T_1^* > W/D$, then we can obtain that

if and only if $-2A + \frac{W^2}{D}(h + cI_p) - cD[M^2(I_p - I_e) + N^2 I_e] < 0.$

Substitute equation (31) into $M \le T_6^* \le W/D$, then we can obtain that

if and only if $-2A + \frac{W^2}{D}(h + cI_p) - cD[M^2(I_p - I_e) + N^2 I_e] \ge 0$

and

if and only if $-2A + DM^2(h + cI_e) - cDN^2 I_e \le 0.$

Substitute equation (17) into $N \le T_3^* \le M$, then we can obtain that

if and only if $-2A + DM^2(h + cI_e) - cDN^2 I_e \ge 0$

and

if and only if $-2A + DN^2 h \le 0.$

Substitute equation (20) into $T_4^* \le N$, then we can obtain that

if and only if $-2A + DN^2 h \ge 0.$

Furthermore, we let

$$\Delta_6 = -2A + \frac{W^2}{D}(h + cI_p) - cD[M^2(I_p - I_e) - N^2 I_e], \tag{33}$$

and

$$\Delta_7 = -2A + DM^2(h + cI_e) - cDN^2 I_e. \tag{34}$$

Equations (24), (33) and (34) imply that $\Delta_6 \geq \Delta_7 \geq \Delta_3$. From the above arguments, we can summarize the following results.

Theorem 3: *Suppose that* $W/D \geq M \geq N$, *then*

(A) *If* $\Delta_3 \geq 0$, *then* $TRC(T^*) = TRC(T_4^*)$ *and* $T^* = T_4^*$.

(B) *If* $\Delta_7 \geq 0$ *and* $\Delta_3 < 0$, *then* $TRC(T^*) = TRC(T_3^*)$ *and* $T^* = T_3^*$.

(C) *If* $\Delta_6 \geq 0$ *and* $\Delta_7 < 0$, *then* $TRC(T^*) = TRC(T_6^*)$ *and* $T^* = T_6^*$.

(D) *If* $\Delta_6 < 0$, *then* $TRC(T^*) = TRC(T_1^*)$ *and* $T^* = T_1^*$.

4. SPECIAL CASES

(I) Huang’s model

When $k = h$, it means that the unit stock holding cost of the rented warehouse and the unit stock holding cost of the retailer himself are equal. It implies that the retailer’s storage capacity is unlimited. Let

$$TRC_7(T) = \frac{A}{T} + \frac{DTh}{2} + cI_p D(T - M)^2 / 2T - cI_e D(M^2 - N^2)/2T, \tag{35}$$

$$TRC_8(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e D(2MT - N^2 - T^2)/2T \tag{36}$$

and

$$TRC_9(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e D(M - N). \tag{37}$$

Equations 1(a, b, c, d), 6(a, b, c, d) and 8(a, b, c, d) will be reduced as follows:

$$TRC(T) = \begin{cases} TRC_7(T) & \text{if} \quad T \geq M & (38a) \\ TRC_8(T) & \text{if} \quad N \leq T \leq M & (38b) \\ TRC_9(T) & \text{if} \quad 0 < T \leq N & (38c) \end{cases}$$

Equations 38(a, b, c) will be consistent with equations 1(a, b, c) in Huang [13], respectively. Hence, Huang [13] will be a special case of this paper.

(II) Goyal's model

When $N = 0$, it means that the supplier would offer the retailer a delay period but the retailer would not offer the delay period to his/her customer. That is one level of delay permitted. Therefore, when $k = h$ and $N = 0$, let

$$TRC_{10}(T) = \frac{A}{T} + \frac{DTh}{2} + cI_p\left[\frac{D(T-M)^2}{2}\right]/T - cI_e\left(\frac{DM^2}{2}\right)/T \tag{39}$$

and

$$TRC_{11}(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e\left(\frac{DT^2}{2} + DT(M-T)\right)/T. \tag{40}$$

Equations 1(a, b, c, d), 6(a, b, c, d) and 8(a, b, c, d) will be reduced as follows:

$$TRC(T) = \begin{cases} TRC_{10}(T) \text{ if } M \le T & \text{(41a)} \\ TRC_{11}(T) \text{ if } 0 < T \le M & \text{(42b)} \end{cases}$$

Equations 41(a, b) will be consistent with equations (1) and (4) in Goyal [11], respectively. Hence, Goyal [11] will be a special case of this paper.

5. NUMERICAL EXAMPLE

To illustrate the results obtained in this paper, let us apply the proposed method to efficiently solve the following numerical example. For convenience, the numbers of the parameters are selected randomly.

From Table 1, we can observe the optimal cycle time with various parameters of W, k and c, respectively. The following inferences can be made based on Table 1.

(1) The optimal cycle time for the retailer will increase when retailer's storage capacity W is increased. The retailer will order more quantity since the retailer owns larger storage space to store a bigger number of items.
(2) When the unit stock holding cost of rented warehouse k is increasing, the optimal cycle time for the retailer will not increase. The retailer will order less quantity to avoid renting expensive warehouse to store these exceeding items.
(3) And lastly, we can find the optimal cycle time for the retailer will decrease when the unit purchasing cost c is increasing. This result implies that the retailer will order less quantity to take the benefits of the delay permitted more frequently.

Table 1: The optimal cycle time with various values of W, k and c

Example: Let D=2000units/year, A=\$100/order, h=\$3/unit/year, I_p=\$0.15/\$/year, I_e=\$0.1/\$/year, M=0.1year, N=0.07year.

W=50 units, (W/D =0.025 year)												
k(\$/unit/year)	c=\$50/unit				c=\$100/unit				c=\$150/unit			
	Δ_1	Δ_4	Δ_5	T*	Δ_1	Δ_4	Δ_5	T*	Δ_1	Δ_4	Δ_5	T*
5	<0	<0	<0	T_1*=0.109818	<0	<0	<0	T_1*=0.100062	>0	<0	<0	T_2*=0.09347
10	>0	<0	<0	T_2*=0.09269	>0	<0	<0	T_2*=0.08757	>0	<0	<0	T_2*=0.08435
15	>0	<0	<0	T_2*=0.08124	>0	<0	<0	T_2*=0.07912	>0	<0	<0	T_2*=0.07767
W=150 units, (W/D =0.075 year)												
k(\$/unit/year)	c=\$50/unit				c=\$100/unit				c=\$150/unit			
	Δ_1	Δ_2	Δ_3	T*	Δ_1	Δ_2	Δ_3	T*	Δ_1	Δ_2	Δ_3	T*
5	<0	<0	<0	T_1*=0.113402	<0	<0	<0	T_1*=0.10253	>0	<0	<0	T_2*=0.09611
10	<0	<0	<0	T_1*=0.103889	>0	<0	<0	T_2*=0.09705	>0	<0	<0	T_2*=0.09228
15	>0	<0	<0	T_2*=0.09798	>0	<0	<0	T_2*=0.09306	>0	<0	<0	T_2*=0.08963
W=250 units, (W/D =0.125 year)												
k(\$/unit/year)	c=\$50/unit				c=\$100/unit				c=\$150/unit			
	Δ_6	Δ_7	Δ_3	T*	Δ_6	Δ_7	Δ_3	T*	Δ_6	Δ_7	Δ_3	T*
5	>0	<0	<0	T_6*=0.119324	>0	<0	<0	T_6*=0.105145	>0	>0	<0	T_3*=0.09818
10	>0	<0	<0	T_6*=0.119324	>0	<0	<0	T_6*=0.105145	>0	>0	<0	T_3*=0.09818
15	>0	<0	<0	T_6*=0.119324	>0	<0	<0	T_6*=0.105145	>0	>0	<0	T_3*=0.09818

6. CONCLUSIONS

This paper adopts the algebraic method to modify Huang's model [13] by considering two warehouses. Using this approach presented in this paper, we can find the optimal cycle time without using differential calculus. This should also mean that this algebraic approach is more accessible in order to ease the learning of basic inventory theories for younger students who lack the knowledge of differential calculus. Furthermore, we develop three easy-to-use theorems to help the retailer to accurately and quickly determine the optimal lot-sizing policy. Then we deduce Huang's model [13] and Goyal's model [11] as special cases of this paper. Finally, a numerical example is given to illustrate all theorems developed in this paper and we obtain a lot of managerial insights from this numerical example.

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