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2-REGULARITY AND 2-NORMALITY CONDITIONS FOR SYSTEMS WITH IMPULSIVE CONTROLS

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Abstract: In this paper a controlled system with impulsive controls in the neighborhood of an abnormal point is investigated. The set of pairs (u, μ) is considered as a class of admissible controls, where u is a measurable essentially bounded function and μ is a finite-dimensional Borel measure, such that for any Borel set B, $\mu(B)$ is a subset of the given convex closed pointed cone.

In this article the concepts of 2-regularity and 2-normality for the abstract mapping Φ, operating from the given Banach space into a finite-dimensional space, are introduced. The concepts of 2-regularity and 2-normality play a great role in the course of derivation of the first and the second order necessary conditions for the optimal control problem, consisting of the minimization of a certain functional on the set of the admissible processes. These concepts are also important for obtaining the sufficient conditions for the local controllability of the nonlinear systems.

The convenient criterion for 2-regularity along the prescribed direction and necessary conditions for 2-normality of systems, linear in control, are introduced in this article as well.

Keywords: Impulsive control, 2-regularity condition, 2-normality condition.

1. PROBLEM DEFINITION

Consider a controllable dynamic system

$$dx(t) = f(x(t),u(t),t)dt + G(t)d\mu(t),\ t \in [t_1,t_2], \quad (1)$$

$$x(t_1) = x_1,\ \ x(t_2) = x_2, \quad (2)$$

$$W(x_1,x_2) = 0,\ \ \mu \in \mathbb{K}. \quad (3)$$

Here $t \in [t_1,t_2]$ is time, $t_1 < t_2$ are given, x is a phase variable, which accepts value in the n-dimensional arithmetical space $\mathbb{R}^n$, $u = (u_1,u_2,...,u_m) \in \mathbb{R}^m$ is a control, f, G,W are respectively n-dimensional, $n \times k$-dimensional, and w-dimensional vector-functions (n, m, w are natural numbers). The function W is assumed to be twice continuously differentiable. The function f is assumed to be piece-wise continuously differentiable, that is, the interval $[t_1,t_2]$ is presentable in terms of a finite number of intervals $[\tau_i,\tau_{i+1}]$ so, that the restriction of f at $\mathbb{R}^n \times \mathbb{R}^m \times [\tau_i,\tau_{i+1}]$ is infinitely differentiable.

The set $\mathbb{K}$ is defined by

$$\mathbb{K} = \left\{ \mu \in C^*([t_1,t_2];\mathbb{R}^k) : \forall \text{ continuous } \varphi : \varphi(t) \in K^0\ \forall t, \int_B \varphi(t)\, d\mu \geq 0\ \forall \text{Borel } B \subset [t_1,t_2] \right\},$$

where $K \subseteq \mathbb{R}^k$ is a given convex closed pointed cone, and K^0 is its dual. In other words, μ is a k-dimensional Borel measure such that $\mu(B) \subset K$ for all Borel subsets B.

An admissible control is any pair $(u,\mu) : \mu \in \mathbb{K},\ u \in L_\infty^m[t_1,t_2]$.

The triple $(x(t),u(t),\mu(t))$ $t \in [t_1,t_2]$, is called an admissible process, if $(u(\cdot),\mu(\cdot))$ is an admissible control, and $x(\cdot)$ is a corresponding solution of equation (1), satisfying the endpoint constraints

$$x(t) = x(t_1) + \int_{t_1}^{t} f(x(\tau),u(\tau),\tau)d\tau + \int_{[t_1,t]} G(\tau)\mu(\tau)\ \ \forall \tau \in [t_1,t_2].$$

For deriving the sufficient conditions for local controllability of the system (1)-(3), and also in the course of derivation of the first and the second order necessary conditions for the optimal control problem, consisting in the minimization of a certain functional on the set of admissible processes (1)-(3) concepts of regularity, 2-regularity and 2-normality in considered point $(\hat{x}_1,\hat{u}(\cdot),\hat{\mu}(\cdot))$ play a great role.

Before giving strict definitions of these concepts for the system (1)-(3), we shall explain the essence of these definitions for the abstract mapping Φ, operating from the given Banach space Z to $\mathbb{R}^k$.

Let $\hat{z}$ be a given point from Z, and let the mapping Φ be twice continuously differentiable in a neighborhood of $\hat{z}$.

Definition 1. *Mapping* Φ *is called regular (normal) at the point* $\hat{z}$ *, if*

$$\operatorname{im}\Phi'(\hat{z}) = \mathbb{R}^k \tag{4}$$

where im *is the image of the linear operator* Φ'.

It is known, that if the mapping Φ is regular at the point $\hat{z}$, then the implicit function theorem holds. Besides, for the minimization problem:

$$\varphi(z) \to \min, \ \ \Phi(z) = 0\,, \tag{5}$$

where φ is a given smooth function, the Lagrange principle is valid (for $\lambda_0 = 1$) well as the as necessary conditions of the second order. If the point $\hat{z}$ is abnormal, that is $\operatorname{im}\Phi'(\hat{z}) \neq \mathbb{R}^k$, then the statement of the classical theorem of implicit function, does not hold. Similarly, for the minimization problem (5) the Lagrange principle is not informative ($\lambda_0 = 0$), and the classical second-order necessary conditions can be false.

Thus there is a problem of finding conditions more delicate than the condition (4), which would guarantee local resolvability of the equation $\Phi(z) = y$ for any z close to the point $\hat{y} = \Phi(\hat{z})$, and also imply substantial necessary conditions of the first and the second order for the problem (5). Conditions of that type are 2-regularity (obtained in [1]) and 2-normality [2]. Let us define these conditions.

Let

$$T(\hat{z}) = \{h \in Z : \Phi'(\hat{z})h = 0, \Phi''(\hat{z})[h,h] \in \operatorname{im}\Phi'(\hat{z})\},$$

and let $h \in T(\hat{z})$. Define a linear operator $G(\hat{z},h) : Z \times \operatorname{Ker}\Phi'(\hat{z}) \to \mathbb{R}^k$ according to:

$$G(\hat{z},h)[\xi_1,\xi_2] = \Phi'(\hat{z})\xi_1 + \Phi''(\hat{z})[h,\xi_2].$$

Definition 2. *The mapping* Φ *is called 2-regular at the point* $\hat{z}$ *in the direction* h *, if*

$$\operatorname{im} G(\hat{z},h) = \mathbb{R}^k. \tag{6}$$

As it is known [1], the existence of vector $h \in T(\hat{z})$, along which the mapping Φ is 2-regular at the point $\hat{z}$, guarantees resolvability of the equation $\Phi(z) = y$ for any y close enough to $\hat{y} = \Phi(\hat{z})$. Besides that, in the problem (5) for any such h some necessary conditions of the first and the second order are valid also in an abnormal case (that is when $\operatorname{im}\Phi'(\hat{z}) \neq \mathbb{R}^k$).

Let $\mathbb{F}_2(\hat{z})$ be a cone, consisting of $\lambda \in \mathbb{R}^k$, $\lambda \neq 0$, such that $\Phi'(\hat{z})^*\lambda = 0$ and there exists a subspace $\Pi = \Pi(\lambda)$ inside of Z :

$$\Pi \subseteq \operatorname{Ker}\Phi'(\hat{z}), \quad \operatorname{codim}\Pi \leq k;$$

$$\frac{\partial^2}{\partial z^2}\langle \lambda, \Phi(\hat{z})\rangle [z,z] \geq 0 \quad \forall z \in \Pi.$$

We should note that the cone $\mathbb{F}_2(\hat{z})$ can be empty. For example it is obviously empty if the mapping Φ is normal at the point $\hat{z}$, since from (4) it follows that $\Phi'(\hat{z})^*\lambda \neq 0 \ \forall \lambda \neq 0$. Besides that, after joining zero to $\mathbb{F}_2(\hat{z})$ it becomes closed, but not necessarily convex.

Definition 3. *The mapping Φ is called 2-normal at the point $\hat{z}$, if the cone $\operatorname{conv}\mathbb{F}_2(\hat{z})$ is pointed, i.e. does not contain nonzero subspaces (the case $\mathbb{F}_2(\hat{z}) \neq \varnothing$ is not excluded, since an empty cone is pointed by the definition).*

The goal of the present paper consists in deriving the conditions of 2-regularity and 2-normality for the considered dynamic system (1)-(3). To this end we shall present the system (1)-(3) in an abstract form.

Let us fix a point $(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot)) \in \mathbb{R}^n \times L_\infty^m[t_1, t_2] \times \mathbb{K}$, so that, $(\hat{x}, \hat{u}, \hat{\mu})$ is an admissible process, and $\hat{x}(t_1) = \hat{x}_1$.

For any $(x_1, u(\cdot), \mu(\cdot)) \in \mathbb{R}^n \times L_\infty^m[t_1, t_2] \times \mathbb{K}$, close enough to $(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot))$, by virtue of the theorems of existence and continuous dependence of the solution of the Cauchy problem on initial conditions and right part there is a unique decision $x(\cdot)$ of the Cauchy problem.

$$dx(t) = f(x(t), u(t), t)dt + G(t)d\mu(t), \quad x(t_1) = x_1, \ t \in [t_1, t_2]. \tag{7}$$

For the specified $(x_1, u(\cdot), \mu(\cdot))$, let us define the mapping $\Phi : \mathbb{R}^n \times L_\infty^m[t_1, t_2] \times C^* \to \mathbb{R}^w$ according to the formula

$$\Phi(x_1, u(\cdot), \mu(\cdot)) = W(x_1, x(t_2; x_1, u(\cdot), \mu(\cdot))).$$

Here $x(t; x_1, u(\cdot), \mu(\cdot))$, $t \in [t_1, t_2]$ is a solution of the Cauchy problem (7).

To interpret the concepts of 2-regularity and 2-normality for the system (1)-(3), it is necessary to derive formulas for calculation of derivatives of the mapping Φ with respect to $(u(\cdot), \mu(\cdot))$.

For the given $\xi(\cdot) \in L_\infty^m[t_1, t_2]$, $v \in \mathbb{K}$ let us denote by $\delta_1 x_{\xi v}(\cdot)$ the solution of the system

$$d(\delta_1 x_{\xi v})(t) = \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t)\delta_1 x_{\xi v}(t)dt + \frac{\partial f}{\partial u}(\hat{x}(t), \hat{u}(t), t)\xi(t)dt + G(t)dv(t) \tag{8}$$

with the initial condition

$$\delta_1 x_{\xi v}(t_1) = 0. \tag{9}$$

Lemma 1. *The derivative operator $\dfrac{\partial \Phi}{\partial(u, \mu)}(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot)) : L_\infty^m[t_1, t_2] \times C^* \to \mathbb{R}^w$ satisfies the formula*

$$\frac{\partial \Phi}{\partial(u, \mu)}(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot))(\xi, v) = \frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)\delta_1 x_{\xi v}(t_2),$$

where $\delta_1 x_{\xi v}(\cdot)$ is the solution of(8)-(9), $\xi(\cdot) \in L_\infty^m[t_1, t_2]$, $v \in \mathbb{K}$.

Proof: Using the theorem of continuous dependence of the solution on parameters in the space $R^n \times L_\infty^m[t_1,t_2] \times C^*$, in certain neighborhood $U(\hat{x}_1)\times V(\hat{u})\times O(\hat{\mu})$ of the point $(\hat{x}_1,\hat{u},\hat{\mu})$ is an operator

$$F:(x_1,u,\mu)\in U(\hat{x}_1)\times V(\hat{u})\times O(\hat{\mu})\mapsto x(t_2)\in \mathbb{R}^n,$$

which is defined associates with the triple (x_1,u,μ) the value of the corresponding solution $x(t)$ of the equation (1) at point t_2 $(x(t_2))$, and this operator is continuous at the point $(\hat{x}_1,\hat{u},\hat{\mu})$ (this also implies, that it will be continuous at all points of the given neighborhood).

Let $u\in V(\hat{u}),\ \mu\in O(\hat{\mu}),$ and let x be a solution of the equation (1). Let $\xi = u-\hat{u},\ v=\mu-\hat{\mu},\ \overline{x}=x-\hat{x},$ so that $u=\hat{u}+\xi,\ \mu=\hat{\mu}+v,\ x=\hat{x}+\overline{x}.$ Then we have

$$\hat{x}(t)+\overline{x}(t)=\hat{x}_1+\int_{t_1}^{t} f(\hat{x}(\tau)+\overline{x}(\tau),\hat{u}(\tau)+\xi(\tau),\tau)\,d\tau+\int_{[t_1,t]} G(\tau)\,d(\hat{\mu}+v)(\tau),$$

$$\hat{x}(t)=\hat{x}_1+\int_{t_1}^{t} f(\hat{x}(\tau),\hat{u}(\tau),\tau)\,d\tau+\int_{[t_1,t]} G(\tau)\,d\hat{\mu}(\tau).$$

Subtracting the second equation from the first one and factorizing f up to linear terms, we get

$$\overline{x}(t)=\int_{t_1}^{t}(f_x(\hat{x}(\tau),\hat{u}(\tau),\tau)\overline{x}(\tau)+f_u(\hat{x}(\tau),\hat{u}(\tau),\tau)\xi(\tau)+\chi(\tau,\overline{x}(\tau),\xi(\tau)))d\tau+ \\ +\int_{[t_1,t]} G(\tau)dv(\tau), \tag{10}$$

where $\|\chi\|_C = o(\|\overline{x}\|_C+\|\xi\|_{L_\infty})$ when $\|\overline{x}\|_C\to 0,\ \|\xi\|_{L_\infty}\to 0.$

Let us consider the function $\delta_1 x_{\xi v}(t)$, which is the solution of (8)-(9), that is

$$\delta_1 x_{\xi v}(t)=\int_{t_1}^{t}(f_x(\hat{x}(\tau),\hat{u}(\tau),\tau)\delta_1 x_{\xi v}(\tau)+f_u(\hat{x}(\tau),\hat{u}(\tau),\tau)\xi(\tau))d\tau+ \\ +\int_{[t_1,t]} G(\tau)dv(\tau). \tag{11}$$

Let us estimate the difference $r(t)=\overline{x}(t)-\delta_1 x_{\xi v}(t).$ From (10)-(11) we get

$$r(t)=\int_{t_1}^{t}(f_x(\hat{x}(\tau),\hat{u}(\tau),\tau)r(\tau))+\chi(\tau,\overline{x}(\tau),\xi(\tau))d\tau.$$

From Gronwall's inequality [4] it follows, that then

$$\|r\|_C\le const\,\|\chi(t,\overline{x},\xi)\|_{L_1},$$

and so

$$\| r \|_C \le o(\| \bar{x} \|_C + \| \xi \|_{L_\infty}).$$

On the other hand $\bar{x} = r + \delta_1 x_{\xi v}$, so that $\| r \|_C \le o(\| r \|_C) + o(\| \delta_1 x_{\xi v} \|_C + \| \xi \|_{L_\infty})$. From (8)-(9) and again from Gronwall's inequality it follows, that $\| \delta_1 x_{\xi v} \|_C \le const(\| \xi \|_{L_1} + V_{t_1}^{t_2}[v])$, so that $\| r \|_C \ (1 - o(1)) \le \ o(\| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v])$, and hence $\| r \|_C \le o(\| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v])$, which means, that $\delta_1 x_{\xi v}(t)$ is the main linear part of quantity $\bar{x}(t)$ (that is increment of a phase variable *x*), generated by increment $\xi = u - \hat{u}$ X $v = \mu - \hat{\mu}$. Thus $\dfrac{\partial F}{\partial(u,\mu)}(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot))(\xi, v) = \ = \delta_1 x_{\xi v}(t_2)$. This implies the statement of the lemma.

For the given $\xi(\cdot), \eta(\cdot) \in L_\infty^m[t_1, t_2], v, \theta \in \mathbb{K}$ let us denote by $\delta_2 x_{\xi\eta v\theta}(\cdot)$ the solution of the system of the equations in variations

$$\begin{aligned} & d(\delta_2 x_{\xi\eta v\theta})(t) \\ & = \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t)\delta_2 x_{\xi\eta v\theta}(t)dt + \frac{\partial^2 f}{\partial x^2}(\hat{x}(t), \hat{u}(t), t)[\delta_1 x_{\xi v}(t), \delta_1 x_{\eta\theta}(t)] \\ & + \frac{\partial^2 f}{\partial x \partial u}(\hat{x}(t), \hat{u}(t), t)[\delta_1 x_{\xi v}(t), \eta(t)] + \frac{\partial^2 f}{\partial x \partial u}(\hat{x}(t), \hat{u}(t), t)[\delta_1 x_{\eta\theta}(t), \xi(t)] \\ & + \frac{\partial^2 f}{\partial u^2}(\hat{x}(t), \hat{u}(t), t)[\xi(t), \eta(t)] \end{aligned} \tag{12}$$

with the initial condition

$$\delta_2 x_{\xi\eta v\theta}(t_1) = 0 . \tag{13}$$

Lemma 2. *The operator* $\dfrac{\partial^2 \Phi}{\partial(u,\mu)^2}(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot)) : L_\infty^m[t_1, t_2] \times C^* \times L_\infty^m[t_1, t_2] \times C^* \to \mathbb{R}^w$ *satisfies the formula*

$$\begin{aligned} & \frac{\partial^2 \Phi}{\partial(u,\mu)^2}(\hat{x}_1, \hat{u}(\cdot), \mu(\cdot))[(\xi, v), (\eta, \theta)] = \\ & \frac{\partial^2 W}{\partial x_2^2}(\hat{x}_1, \hat{x}_2)[\delta_1 x_{\eta\theta}(t_2), \delta_1 x_{\xi v}(t_2)] + \frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)\delta_2 x_{\xi\eta v\theta}(t_2), \end{aligned}$$

where $\delta_2 x_{\xi\eta v\theta}(\cdot)$ is the solution of (12)-(13), $\xi(\cdot), \eta(\cdot) \in L_\infty^m[t_1, t_2], v, \theta \in \mathbb{K}$.

Proof: Similarly as in the proof of Lemma 1, we shall consider the function $\delta_2 x_{\xi\xi vv}(t)$, which is the solution of (12)-(13) at $\eta = \xi$, $\theta = v$. Then, factorizing $f(\hat{x}(t) + \bar{x}, \hat{u}(t) + \xi, t)$ $- f(\hat{x}(t), \hat{u}(t), t)$ up to the terms of the second order, we get

$$\begin{aligned}\bar{x}(t) = \int_{t_1}^{t} (& f_x(\hat{x}(\tau),\hat{u}(\tau),\tau)\bar{x}(\tau) + f_u(\hat{x}(\tau),\hat{u}(\tau),\tau)\xi(\tau) + \\ & + \frac{1}{2} f_{xx}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\bar{x}(\tau),\bar{x}(\tau)] + f_{xu}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\bar{x}(\tau),\xi(\tau)] + \\ & + \frac{1}{2} f_{uu}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\xi(\tau),\xi(\tau)] + \varsigma(\tau,\bar{x}(\tau),\xi(\tau)))d\tau + \\ & + \int_{[t_1,t]} G(\tau)dv(\tau),\end{aligned} \tag{14}$$

$\|\varsigma\|_C = o((\|\bar{x}\|_C + \|\xi\|_{L_\infty})^2)$ when $\|\bar{x}\|_C \to 0,\ \|\xi\|_{L_\infty} \to 0.$

Let us estimate the difference $R(t) = \bar{x}(t) - \delta_1 x_{\xi v}(t) - \frac{1}{2}\delta_2 x_{\xi\xi vv}(t),$ where $\delta_2 x_{\xi\xi vv}(t)$ is the solution of (12)-(13) under $\eta = \xi,\ \theta = v,$ that is

$$\begin{aligned}\delta_2 x_{\xi\xi vv}(t) = \int_{t_1}^{t} (& f_x(\hat{x}(\tau),\hat{u}(\tau),\tau)\delta_2 x_{\xi\xi vv}(\tau) + f_{xx}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\delta_1 x_{\xi v}(\tau),\delta_1 x_{\xi v}(\tau)] + \\ & + 2 f_{xu}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\delta_1 x_{\xi v}(\tau),\xi(\tau)] + f_{uu}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\xi(\tau),\xi(\tau)]d\tau.\end{aligned} \tag{15}$$

From (11), (14) and (15) we have

$$\begin{aligned}R(t) = \int_{t_1}^{t} (& f_x(\hat{x}(\tau),\hat{u}(\tau),\tau)R(\tau) + \frac{1}{2} f_{xx}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\bar{x}(\tau),\bar{x}(\tau)] - \\ & - \frac{1}{2} f_{xx}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\delta_1 x_{\xi v}(\tau),\delta_1 x_{\xi v}(\tau)] + \\ & + f_{xu}(\hat{x}(\tau),\hat{u}(\tau),\tau)[\bar{x}(\tau) - \delta_1 x_{\xi v}(\tau),\xi(\tau)] + \\ & + \varsigma(\tau,\bar{x}(\tau),\xi(\tau)))d\tau.\end{aligned}$$

From Gronwall's inequality it follows that then

$$\|R\|_C \le C_1 \|\varsigma(t,\bar{x},\xi)\|_{L_1} + C_2 \|\bar{x} - \delta_1 x_{\xi v}\|_C^2 + C_3 \|\bar{x} - \delta_1 x_{\xi v}\|_C \|\xi\|_{L_\infty},$$

where $C_1,\ C_2,\ C_3$ are some constants.

Using the inequality derived in the proof of Lemma 1 $\|\bar{x} - \delta_1 x_\xi\|_C \le$ $\le o(\|\xi\|_{L_\infty} + V_{t_1}^{t_2}[v]),$ we get $\|R\|_C \le o((\|\bar{x}\|_C + \|\xi\|_{L_\infty})^2) + o((\|\xi\|_{L_\infty} + V_{t_1}^{t_2}[v])^2).$

But $\bar{x} = R + \delta_1 x_{\xi v} + \frac{1}{2}\delta_2 x_{\xi\xi vv},$ so

$$\|R\|_C \le o((\|R\|_C + \|\delta_1 x_{\xi v}\|_C + \|\delta_2 x_{\xi\xi vv}\|_C + \|\xi\|_{L_\infty} + V_{t_1}^{t_2}[v])^2).$$

From (11), (15) and from Gronwall's inequality it follows that

$$\|\delta_2 x_{\xi\xi vv}\|_C \le C_4 \|\delta_1 x_{\xi v}\|_C^2 + C_5 \|\delta_1 x_{\xi v}\|_C \|\xi\|_{L_\infty} + C_6 \|\xi\|_{L_\infty}^2,$$

$$\| \delta_1 x_{\xi v} \|_C \le C_7 (\| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v]),$$

where C_4, C_5, C_6, C_7 are some constants. Then

$$\| R \|_C \le o\Big(\big(\| R \|_C + (\| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v])^2 + \| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v] \big)^2 \Big).$$

Hence

$$\| R \|_C \ (1 - o(| R \|_C) \le o\Big((\| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v])^2 \Big) + o + (\| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v]),$$

and it follows

$$\| R \|_C \le o\Big((\| \xi \|_{L_\infty} + V_{t_1}^{t_2}[v])^2 \Big).$$

Thus

$\dfrac{\partial^2 F}{\partial (u,\mu)^2}(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot))[(\xi, v), (\eta, \theta)] = \delta_2 x_{\xi\eta v\theta}(t_2),$ where $\delta_2 x_{\xi\eta v\theta}(\cdot)$ is a solution of the system of the equations in variations (12) with the initial condition (13). The statement of the lemma follows from here.

2. 2-REGULARITY CONDITION

Definition 4. *Let* $h \in L_\infty^m[t_1, t_2]$, $g \in \mathbb{K}$. *For the problem (1)-(3) at the point* $(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot))$ *the 2-regularity condition in the direction* (h, g) *is satisfied if*

$$\forall y \in \mathbb{R} \ \ \exists \xi_1, \xi_2 \in L_\infty^m[t_1, t_2], \ v_1, v_2 \in \mathbb{K}:$$

$$\frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)\delta_1 x_{\xi_1 v_1}(t_2) + \frac{\partial^2 W}{\partial x_2^2}(\hat{x}_1, \hat{x}_2)[\delta_1 x_{hg}(t_2), \delta_1 x_{\xi_2 v_2}(t_2)] + \frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)\delta_2 x_{h\xi_2 g v_2}(t_2) = y,$$

$$\frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)\delta_1 x_{\xi_2 v_2}(t_2) = 0.$$

From Lemmas 1 and 2 it obviously follows that 2-regularity in terms of definition 4 means 2-regularity of the introduced mapping Φ at point $\hat{w} = (\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot))$ in the direction $\overline{h} = (h, g)$. The following lemma gives the criterion for 2-regularity of the system (1)-(3).

Lemma 3. *For the problem (1)-(3) at the point* $(\hat{x}_1, \hat{u}(\cdot), \hat{\mu}(\cdot))$ *2-regularity condition in the direction* (h, g) *holds true if and only if, there is no* $r \in \mathbb{R}^w$, $r \neq 0$, $q \in \mathbb{R}^w$, *such that for functions* ψ_1, ψ_2, *which are solutions of the Cauchy problem*

$$\dot{\psi}_1 = -\frac{\partial f^*}{\partial x}(\hat{x}(t), \hat{u}(t), t)\psi_1, \tag{16}$$

$$\dot{\psi}_2 = -\frac{\partial f^*}{\partial x}(\hat{x}(t),\hat{u}(t),t)\psi_2 - E_1^*(t)\psi_1, \tag{17}$$

$$\psi_1(t_2) = -\frac{\partial W^*}{\partial x_2}(\hat{x}_1,\hat{x}_2)r, \tag{18}$$

$$\psi_2 = -\delta_1 x_{hg}(t_2)^* \frac{\partial^2 W^*}{\partial x_2^2}(\hat{x}_1,\hat{x}_2)r - \frac{\partial W^*}{\partial x_2}(\hat{x}_1,\hat{x}_2)q, \tag{19}$$

the following take a place:

$$\frac{\partial f^*}{\partial u}(\hat{x}(t),\hat{u}(t),t)\psi_1(t) = 0 \quad \text{Lebesque-a.e.}, \tag{20}$$

$$\frac{\partial f^*}{\partial u}(\hat{x}(t),\hat{u}(t),t)\psi_2(t) + E_2^*(t)\psi_1(t) = 0 \quad \text{Lebesque-a.e.}, \tag{21}$$

$$\langle \psi_1(t), G(t)\upsilon \rangle \le 0 \quad \forall \upsilon \in K,\ \forall t \in [t_1,t_2], \tag{22}$$

$$\langle \psi_2(t), G(t)\upsilon \rangle \le 0 \quad \forall \upsilon \in K,\ \forall t \in [t_1,t_2], \tag{23}$$

$$\langle \psi_1(t), G(t)\hat{\upsilon}(t) \rangle = 0 \quad \hat{\mu} - \text{a.e.} \tag{24}$$

$$\langle \psi_2(t), G(t)\hat{\upsilon}(t) \rangle = 0 \quad \hat{\mu} - \text{a.e.} \tag{25}$$

where $\hat{\upsilon}(t) = \frac{d\hat{\mu}}{d|\hat{\mu}|}(t)$ *is the Radon Nicodym derivative.*

Here

$$E_1(t) = \frac{\partial^2 f}{\partial x^2}(\hat{x}(t),\hat{u}(t),t)\delta_1 x_{hg}(t) + \frac{\partial^2 f}{\partial x \partial u}(\hat{x}(t),\hat{u}(t),t)h(t),$$
$$E_2(t) = \frac{\partial^2 f}{\partial x \partial u}(\hat{x}(t),\hat{u}(t),t)\delta_1 x_{hg}(t) + \frac{\partial^2 f}{\partial u^2}(\hat{x}(t),\hat{u}(t),t)h(t).$$

Proof: By virtue of the theorem of separability for convex sets, the 2-regularuty condition is violated if and only if

$$\exists r \in \mathbb{R}^w,\ r \ne 0:\ \Big\langle \frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_1 x_{\xi_1 v_1}(t_2) +$$
$$+\frac{\partial^2 W}{\partial x_2^2}(\hat{x}_1,\hat{x}_2)[\delta_1 x_{hg}(t_2), \delta_1 x_{\xi_2 v_2}(t_2)] + \frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_2 x_{h\xi_2 g v_2}(t_2), r \Big\rangle = 0 \tag{26}$$
$$\forall \xi_1 \in L_\infty^m[t_1,t_2],\ v_1, \in \mathbb{K};\ \forall \xi_2 \in L_\infty^m[t_1,t_2],\ \forall v_2 \in \mathbb{K}: \frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_1 x_{\xi_2 v_2}(t_2) = 0.$$

Let us interpret condition (26). For this purpose we shall consider the linear optimal control problem

$$\left\langle \frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_1 x_{\xi_1 v_1}(t_2)+\frac{\partial^2 W}{\partial x_2^2}(\hat{x}_1,\hat{x}_2)[\delta_1 x_{hg}(t_2),\delta_1 x_{\xi_2 v_2}(t_2)]+ \right.$$
$$\left. +\frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_2 x_{h\xi_2 g v_2}(t_2), r \right\rangle \to \min, \tag{27}$$

$$d(\delta_1 x_{\xi_1 v_1})(t)=\frac{\partial f}{\partial x}(\hat{x}(t),\hat{u}(t),t)\delta_1 x_{\xi_1 v_1}(t)dt+\frac{\partial f}{\partial u}(\hat{x}(t),\hat{u}(t),t)\xi_1(t)dt+G(t)dv_1(t), \tag{28}$$

$$d(\delta_1 x_{\xi_2 v_2})(t)=\frac{\partial f}{\partial x}(\hat{x}(t),\hat{u}(t),t)\delta_1 x_{\xi_2 v_2}(t)dt+\frac{\partial f}{\partial u}(\hat{x}(t),\hat{u}(t),t)\xi_2(t)dt+G(t)dv_2(t), \tag{29}$$

$$d(\delta_2 x_{h\xi_2 g v_2})(t)=\frac{\partial f}{\partial x}(\hat{x}(t),\hat{u}(t),t)\delta_2 x_{h\xi_2 g v_2}(t)\,dt+E_1(t)\delta_1 x_{\xi_2 v_2}(t)\,dt+E_2(t)\xi_2(t), \tag{30}$$

$$\delta_1 x_{\xi_1 v_1}(t_1)=0, \tag{31}$$

$$\delta_1 x_{\xi_2 v_2}(t_1)=0, \tag{32}$$

$$\frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_1 x_{\xi_2 v_2}(t_2)=0, \tag{33}$$

$$\delta_2 x_{h\xi_2 g v_2}(t_1)=0. \tag{34}$$

In this problem control variables are $(\xi_1(\cdot),v_1(\cdot),\xi_2(\cdot),v_2(\cdot))$, while phase variables - are $(\delta_1 x_{\xi_1 v_1}(\cdot),\delta_1 x_{\xi_2 v_2}(\cdot),\delta_2 x_{h\xi_2 g v_2}(\cdot))$.

Pontryagin's function (Hamiltonian) $\bar{H}$ and the minor Lagrangian $\bar{l}$ of the problem (27)-(34) are given by

$$\bar{H}(x,\xi_1,\xi_2,v_1,v_2,t,\psi_1,\psi_2,\psi_3)=$$
$$=\left\langle \frac{\partial f}{\partial x}(\hat{x},\hat{u},t)\delta_1 x_{\xi_1 v_1}+\dot{\psi}_1+\frac{\partial f}{\partial u}(\hat{x},\hat{u},t)\xi_1,\psi_1 \right\rangle$$
$$+\left\langle \frac{\partial f}{\partial x}(\hat{x},\hat{u},t)\delta_1 x_{\xi_2 v_2}+\frac{\partial f}{\partial u}(\hat{x},\hat{u},t)\xi_2,\psi_2 \right\rangle+$$
$$+\left\langle \frac{\partial f}{\partial x}(\hat{x},\hat{u},t)\delta_2 x_{h\xi_2 g v_2}+E_1(t)\delta_1 x_{\xi_2 v_2}+E_2(t)\xi_2,\psi_3 \right\rangle,$$

$$\bar{l}(\delta_1 x_{\xi_1 v_1}(t_2),\delta_1 x_{\xi_2 v_2}(t_2),\delta_2 x_{h\xi_2 g v_2}(t_2),q)=\left\langle \frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_1 x_{\xi_1 v_1}(t_2)+ \right.$$
$$\left. +\frac{\partial^2 W}{\partial x_2^2}(\hat{x}_1,\hat{x}_2)[\delta_1 x_{hg}(t_2),\delta_1 x_{\xi_2 v_2}(t_2)]+\frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_2 x_{h\xi_2 g v_2}(t_2),r \right\rangle+$$
$$+\left\langle \frac{\partial W}{\partial x_2}(\hat{x}_1,\hat{x}_2)\delta_1 x_{\xi_2 v_2}(t_2),q \right\rangle.$$

Here $q \in \mathbb{R}^w$, and ψ_1, ψ_2, ψ_3 are *n*-dimensional column vectors.

According to the Pontryagin's maximum principle there exist a vector q and solutions ψ_1, ψ_2, ψ_3 of the Cauchy problem

$$\dot{\psi}_1 = -\frac{\partial f^*}{\partial x}(\hat{x}(t), \hat{u}(t), t)\psi_1,$$

$$\dot{\psi}_2 = -\frac{\partial f^*}{\partial x}(\hat{x}(t), \hat{u}(t), t)\psi_2 - E_1^*(t)\psi_1,$$

$$\dot{\psi}_3 = -\frac{\partial f^*}{\partial x}(\hat{x}(t), \hat{u}(t), t)\psi_3,$$

$$\psi_1(t_2) = -\frac{\partial W^*}{\partial x_2}(\hat{\sigma}, \hat{x}_2)r,$$

$$\psi_2 = -\delta_1 x_{hg}(t_2)^* \frac{\partial^2 W^*}{\partial x_2^2}(\hat{\sigma}, \hat{x}_2)r - \frac{\partial W^*}{\partial x_2}(\hat{\sigma}, \hat{x}_2)q,$$

$$\psi_3(t_2) = -\frac{\partial W^*}{\partial x_2}(\hat{\sigma}, \hat{x}_2)r,$$

Such that

$$\frac{\partial f^*}{\partial u}(\hat{x}(t), \hat{u}(t), t)\psi_1(t) = 0 \quad \text{Lebesque-a.e.},$$

$$\frac{\partial f^*}{\partial u}(\hat{x}(t), \hat{u}(t), t)\psi_2(t) + E_2^*(t)\psi_3(t) = 0 \quad \text{Lebesque-a.e.},$$

$$\langle \psi_1(t), G(t)\upsilon \rangle \le 0 \quad \forall \upsilon \in K, \ \forall t \in [t_1, t_2],$$

$$\langle \psi_2(t), G(t)\upsilon \rangle \le 0 \quad \forall \upsilon \in K, \ \forall t \in [t_1, t_2],$$

$$\langle \psi_1(t), G(t)\hat{\upsilon}(t) \rangle = 0 \quad \hat{\mu} - \text{a.e.}$$

$$\langle \psi_2(t), G(t)\hat{\upsilon}(t) \rangle = 0 \quad \hat{\mu} - \text{a.e.}$$

where $\hat{\upsilon}(t) = \frac{d\hat{\mu}}{d|\hat{\mu}|}(t)$ is the Radon Nicodym derivative.

The lemma's statement immediately follows from the last relations.

3. 2-NORMALITY CONDITION

Let us define on sets $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^n$ and $\mathbb{R}^{2n} \times \mathbb{R}^k$ the Hamiltonian function H and the Lagrangian l by:

$$H(x,u,t,\psi) = \langle \psi, f(x,u,t) \rangle,$$
$$l(x_1, x_2, \lambda) = \langle \lambda, W(x_1, x_2) \rangle.$$

Here $\lambda \in \mathbb{R}^w$, and ψ is the n-dimenzional vector-column. Let $(\hat{x}, \hat{u}, \hat{\mu})$ be the given admissible process.

Definition 5. *The process $(\hat{x}, \hat{u}, \hat{\mu})$ satisfies Eiler-Lagrange equation, if the vector $\lambda \neq 0$ exists such that for a vector-function ψ, which is the solution of the Cauchy problem*

$$\dot{\psi} = -\partial H(\hat{x}(t), \hat{u}(t), t, \psi(t)) / \partial x, \quad \psi(t_1) = \partial l(\hat{x}_1, \hat{x}_2, \lambda) / \partial \sigma, \tag{35}$$

the following holds:

$$\psi(t_2) = -\partial l(\hat{x}_1, \hat{x}_2, \lambda) / \partial x_2, \quad \partial H(\hat{x}(t), \hat{u}(t), t, \psi(t)) / \partial u = 0 \quad \text{Lebesque-a.e.},$$
$$\langle \psi(t), G(t)\upsilon \rangle \leq 0 \quad \forall\, \upsilon \in K, \ \forall t \in [t_1, t_2],$$
$$\langle \psi(t), G(t)\hat{\upsilon}(t) \rangle = 0 \quad \hat{\mu} \ \text{– a.e.}$$

Here $\hat{\upsilon}(t) = \frac{d\hat{\mu}}{d|\hat{\mu}|}(t)$ is the Radon Nicodym derivative, $\hat{x}_1 = \hat{x}(t_1)$, $\hat{x}_2 = \hat{x}(t_2)$.

Let us denote by $\Lambda(\hat{x}, \hat{u}, \hat{\mu})$ the set of vectors λ which correspond to the given extremal $(\hat{x}, \hat{u}, \hat{\mu})$ by virtue of Eiler-Lagrange equations.

For the formulation of the second order conditions for the process $(\hat{x}, \hat{u}, \hat{\mu})$ we shall consider the following system of the equations

$$d(\delta x)(t) = \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t), t)\delta x(t)\, dt + \frac{\partial f}{\partial u}(\hat{x}(t), \hat{u}(t), t)\delta u(t)\, dt + G(t) d(\delta\mu)(t). \tag{36}$$

Here $\delta u \in L_\infty^m[t_1, t_2]$, $\delta\mu \in \mathbb{T}_{\mathbb{K}}(\hat{\mu})$, and the solution of the equation in variations should satisfy the conditions:

$$\frac{\partial W}{\partial x_1}(\hat{x}_1, \hat{x}_2)\delta x(t_1) = 0, \quad \frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)\delta x(t_2) = 0. \tag{37}$$

Let $\lambda \in \Lambda(\hat{x}, \hat{u})$. On the space $X = R^n \times L_\infty^m[t_1, t_2] \times C^*$ of points $(\varsigma, \delta u, \delta\mu)$ we shall define the quadratic form Ω_λ by the formula

$$\Omega_\lambda(\varsigma, \delta u, \delta\mu) = \frac{\partial^2 l}{\partial (x_1, x_2)^2}(\hat{x}_1, \hat{x}_2, \lambda)[(\delta x(t_1), \delta x(t_2)), (\delta x(t_1), \delta x(t_2))] -$$
$$- \int_{t_1}^{t_2} \frac{\partial^2 H}{\partial (x,u)^2}(\hat{x}, \hat{u}, t, \psi)[(\delta x(t), \delta u(t)), (\delta x(t), \delta u(t))]\, dt.$$

Hereinafter δx is the solution of the system of the equations in variations (36) with the initial condition $\delta x(t_1) = \varsigma$ corresponding to $(\delta u, \delta\mu)$.

Let χ denote the linear subspace of X, which consists of those $(\varsigma, \delta u, \delta\mu)$, such that $\varsigma = \delta x$.

Let r be a natural number and let $\Lambda_r = \Lambda_r(\hat{x}, \hat{u}, \hat{\mu})$ denote the set of those $\lambda \in \Lambda(\hat{x}, \hat{u}, \hat{\mu})$, for which the index of narrowing the form Ω_λ on the subspace χ does not exceed r.

Definition 6. *Admissible process $(\hat{x}, \hat{u}, \hat{\mu})$ is called 2-normal if the cone $conv\Lambda_k(\hat{x}, \hat{u}, \hat{\mu})$ is pointed.*

From Legandre's condition [4] it follows

Lemma 4. *For some r let the cone $\Lambda_r(\hat{x}, \hat{u}, \hat{\mu})$ be non empty. Then there exists $\lambda \in \Lambda_r(\hat{x}, \hat{u}, \hat{\mu})$, such that*

$$\frac{\partial^2 H}{\partial u^2}(\hat{x}(t), \hat{u}(t), t, \psi(t)) \le 0 \quad \text{Lebesque-a.e.}$$

Let us apply 2-normality concept to linear controllable systems.

Assume that $f(x,u,t) = a_0(x,t) + \sum_{i=1}^{m} u_i a_i(x,t)$, where a_0, a_i are given piecewise smooth vector functions. Then

$$dx(t) = a_0(x,t)dt + \sum_{i=1}^{m} u_i a_i(x,t)dt + G(t)d\mu(t), \tag{38}$$

$$x(t_1) = x_1, \quad x(t_2) = x_2, \tag{39}$$

$$W(x_1, x_2) = 0. \tag{40}$$

$$H(x,u,t,\psi) = \langle \psi, a_0(x,t) \rangle + \sum_{i=1}^{m} u_i \langle \psi, a_i(x,t) \rangle. \tag{41}$$

Consider the process $(\hat{x}, \hat{u})$ $(\hat{x}(t_1) = \hat{x}_1, \hat{x}(t_2) = \hat{x}_2)$. Without loss of generality we shall assume that $\hat{u}(t) \equiv 0$. The corresponding system of the equations in variations is the following:

$$d(\delta x)(t) = A(t)\delta x(t)dt + B(t)\delta u(t)dt + G(t)d(\delta\mu)(t),$$

$$\frac{\partial W}{\partial x_1}(\hat{x}_1, \hat{x}_2)\delta x(t_1) = 0, \quad \frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)\delta x(t_2) = 0. \tag{42}$$

where $\delta u \in L_\infty^m[t_1, t_2]$, $\delta\mu \in \mathbb{T}_{\mathbb{K}}(\hat{\mu})$, A and B are defined by formulas:

$$A(t) = \frac{\partial a_0(\hat{x}(t), t)}{\partial x}, \quad B(t) = (b^1(t), \dots, b^m(t)), \quad b^i = a_i(\hat{x}(t), t).$$

For any $\lambda \in \Lambda_r(\hat{x},\hat{u})$ let

$$D_\lambda(t) = -\frac{\partial^2 H(\hat{x}(t),\hat{u}(t),t,\psi_{\lambda(t)})}{\partial x^2},$$

$$C_\lambda(t) = -\frac{\partial^2 H(\hat{x}(t),\hat{u}(t),t,\psi_{\lambda(t)})}{\partial x \partial u}.$$

Then

$$\Omega_\lambda(\varsigma, \delta u) = \left\langle \frac{\partial^2 l}{\partial(x_1,x_2)^2}(\hat{x}_1,\hat{x}_2,\lambda)(\delta x(t_1),\delta x(t_2)),(\delta x(t_1),\delta x(t_2)) \right\rangle +$$
$$+\int_{t_1}^{t_2} \left(\langle D_\lambda \delta x, \delta x \rangle + 2\langle C_\lambda^* \delta x, \delta u \rangle \right) dt,$$

where δx is the solution of the system of the equations in variations (42) with the initial condition $\delta x(t_1) = \varsigma$.

Assume that matrixes $A(t), B(t), C(t), D(t)$ and all their derivatives can have on $[t_1, t_2]$ jumps only in finite number p of points $\tau_1,...,\tau_p$.

Lemma 5. *Suppose that for the some r the cone* $\Lambda_r(\hat{x},\hat{u},\hat{\mu})$ *is not empty. Then* $\lambda \in \Lambda_r(\hat{x},\hat{u},\hat{\mu})$, *such exists, that conditions are satisfied*

1) for the solution ψ *of adjoint equation (35), corresponding to the vector* λ, *the following leads*

$$\frac{\partial^2 H}{\partial u^2}(\hat{x}(t),\hat{u}(t),t,\psi(t)) = 0 \quad \forall t \notin \{\tau_1,...,\tau_p\}, \tag{43}$$

$$\frac{\partial}{\partial u}\frac{\partial}{\partial t}\frac{\partial H}{\partial u}(\hat{x}(t),\hat{u}(t),t,\psi(t)) = 0 \quad \forall t \notin \{\tau_1,...,\tau_p\}, \tag{44}$$

$$\frac{\partial}{\partial u}\frac{\partial^2}{\partial t^2}\frac{\partial H}{\partial u}(\hat{x}(t),\hat{u}(t),t,\psi(t)) \ge 0 \quad \forall t \notin \{\tau_1,...,\tau_p\}, \tag{45}$$

2)

$$\operatorname{ind} w(\varsigma, \delta u, \upsilon) \le r$$

3)

$$\sum_{i=1}^{q} \operatorname{ind} \Delta(C_\lambda^* B)_{\tau_i} + \operatorname{ind} C_\lambda^*(t_2) B(t_2) \le r. \tag{46}$$

Here

$$w(\varsigma, \delta u, \upsilon) = \left\langle \frac{\partial^2 l}{\partial(x_1,x_2)^2}(\hat{x}_1,\hat{x}_2,\lambda)(\delta x(t_1),\delta x(t_2)),(\delta x(t_1),\delta x(t_2)) \right\rangle +$$
$$+\int_{t_1}^{t_2} \left(\langle D_\lambda \xi, \xi \rangle + 2\langle P\xi, \upsilon \rangle + \langle Q\upsilon, \upsilon \rangle \right) dt,$$

$P = B^* D_\lambda - C_\lambda^* - C_\lambda^* A, \quad Q = B^* D_\lambda B - (C_\lambda^* AB + B^* AC_\lambda) + (1/2) C_\lambda^* \dot{B} - \dot{C}_\lambda^* B,$
$\Delta M_\tau = M(\tau + 0) - M(\tau - 0)$ *is the jump of the matrix* M *at point* τ *, and* $\operatorname{ind}\Psi$ *denotes the index of the quadratic form of certain symmetric matrix* Ψ *. Let us note that, as can be seen from the proof given below, all matrices in (46) are symmetric.*

Proof: Under the hypothesis of the theorem there exists such $\lambda \in \Lambda_r(\hat{x}, \hat{u})$, that $\operatorname{ind}\Omega_\lambda |_\chi \le r$.

Let us convert the form Ω_λ according to Hooch by virtue of system (42). For convenience, in the sequel we shall omit the subscript λ is matrix functions $C, D,$ and quadratic form $\tilde{\Omega}$.

Let us introduce new variables $\dot{\upsilon} = \delta u,\ \upsilon(t_1) = 0,\ \xi = x - B\upsilon.$ Functions ξ and υ belong to spaces $\tilde{W}^n_{\infty,1}$ and $\tilde{W}^m_\infty$ respectively and satisfy correlations

$$\dot{\xi} = A(t)\xi + (AB - \dot{B})\upsilon,$$
$$\frac{\partial W}{\partial x_1}(\hat{x}_1, \hat{x}_2)\xi(t_1) = 0, \quad \frac{\partial W}{\partial x_2}(\hat{x}_1, \hat{x}_2)(\xi(t_2) + B(t_2)\upsilon(t_2)) = 0.$$

Here $\tilde{W}^n_{\infty,1}$ is the space of *n*-dimensional functions, which have piecewise-Lipschitzian first-order derivative, and $\tilde{W}^m_\infty$ is the space of *m*-dimensional piecewise- Lipschitzian functions.

Then

$$\Omega(\varsigma, \delta u, \delta\mu) = \tilde{\Omega}(\varsigma, \delta u, \upsilon) = w(\varsigma, \delta u, \upsilon) + \sum_{i=1}^{q} \left\langle \Delta(C^* B)_{\tau_i}\, \upsilon(\tau_i), \upsilon(\tau_i) \right\rangle +$$
$$+2\sum_{i=1}^{q} \left\langle \xi(\tau_i), \Delta C_{\tau_i} \upsilon(\tau_i) \right\rangle + 2\left\langle \xi(t_2), \Delta C(t_2)\upsilon(t_2) \right\rangle + \left\langle C^*(t_2) B(t_2) \upsilon(t_2), \upsilon(t_2) \right\rangle.$$

Here

$$w(\varsigma,\, \delta u,\, \upsilon) = \left\langle \frac{\partial^2 l}{\partial (x_1, x_2)^2}(\hat{x}_1, \hat{x}_2, \lambda)(\delta x(t_1), \delta x(t_2)), (\delta x(t_1), \delta x(t_2)) \right\rangle + \\ + \int_{t_1}^{t_2} \left(\langle D\xi, \xi \rangle + 2\langle P\xi, \upsilon \rangle + \langle Q\upsilon, \upsilon \rangle + \langle V\upsilon, \delta u \rangle \right) dt, \tag{47}$$
$$P = B^* D - \dot{C}^* - C^* A, \quad V = C^* B - B^* C,$$
$$Q = B^* DB - (C^* AB + B^* AC) + (1/2) C^* \dot{B} - \dot{C}^* B.$$

Let us note, that Hooch's conversion does not change the index of the quadratic form.

By the virtue of necessary conditions of finiteness of index of the form *w* on χ the following holds

$$V(t) = 0 \ \dot{\forall} t; \quad Q(t) = Q(t)^*, \ Q(t) \ge 0 \ \dot{\forall} t. \tag{48}$$

The first of these conditions is called Hooch's condition, and the second one is a generalization of Legendre's conditions.

From conditions (48) follows, that all the matrices entering (46) are symmetric.

In [3] the following inequality is proved:

$$\operatorname{ind}\Omega \geq \operatorname{ind} w(\varsigma, \delta u, \upsilon) + \sum_{i=1}^{q} \operatorname{ind}\Delta(C^*B)_{\tau_i} + \operatorname{ind} C^*(t_2)B(t_2).$$

Since $\operatorname{ind}\Omega \leq r$, last inequality implies $\sum_{i=1}^{q} \operatorname{ind}\Delta(C^*B)_{\tau_i} + \operatorname{ind} C^*(t_2)B(t_2) \leq r$.

Direct differentiation of (41) yields the formulas:

$$V(t) = \frac{\partial}{\partial u}\frac{d}{dt}\frac{\partial H}{\partial u}(\hat{x}(t), \hat{u}(t), t, \psi(t)),$$

$$Q(t) = \frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u}(\hat{x}(t), \hat{u}(t), t, \psi(t)).$$

By the virtue of (47), last two relations imply (44)-(45).

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