AN EPQ MODEL UNDER CASH DISCOUNT AND PERMISSIBLE DELAY IN PAYMENTS DERIVED WITHOUT DERIVATIVES

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Abstract: The main purpose of this paper is to investigate the case where the retailer's unit selling price and the purchasing price per unit are not necessarily equal within the economic production quantity (EPQ) framework under cash discount and permissible delay in payments. We establish the retailer's inventory system as a cost minimization problem to determine the retailer's optimal inventory cycle time, optimal order quantity and optimal payment time. This paper provides an algebraic approach to determine the optimal cycle time, optimal order quantity and optimal payment time. This approach provides one theorem to efficiently determine the optimal solution. Some previously published results of other researchers are deduced as special cases. Finally, numerical examples are given to illustrate the result and the managerial insights are also obtained.

Keywords: Economic production quantity, EPQ, permissible delay in payments, trade credit, cash discount, algebraic method.

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1. INTRODUCTION

The traditional economic order quantity (EOQ) model assumes that retailer's capitals are adequate and must pay for the items as soon as the items are received. In practice, supplier will offer retailer a delay period, which is the trade credit period, in paying for the amount of purchase. Before end of the trade credit period, retailer can sell goods and accumulate revenue and earn interest. A higher interest is charged if the payment is not settled by end of the trade credit period. In the real world, supplier often makes use of this policy to promote commodities. Many researchers discussed this topic that investigates inventory problems under varying conditions.

Goyal (1985) established a single-item inventory model under permissible delay in payments. Chung (1998) developed an alternative approach to determine the economic order quantity under condition of permissible delay in payments. Aggarwal and Jaggi (1995) considered the inventory model with an exponential deterioration rate under the condition of trade credit. This line of research was extended to the varying rate of deterioration Chang et al.(2002), and with inflation Liao et al.(2000) and Sarker et al.(2000a), allowable shortage Jamal et al.(1997) and Chang and Dye(2001) and linear demand Chang et al.(2001) Buyer's inventory policy was investigated Chen and Chuang(1999) under trade credit by the concept of discounted cash flow. Hwang and Shinn (1997) modeled an inventory system for retailer's pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payment. Jamal et al. (2000) and Sarker et al. (2000b) addressed the optimal payment time under permissible delay in payment with deterioration. Teng (2002) assumed that the selling price not equal to the purchasing price to modify the Goyal's model (1985). Chung et al. (2002) and Chung and Huang (2003b) discussed this issue under the selling price not equal to the purchasing price and different payment rule. Shinn and Hwang (2003) determined the retailer's optimal price and order size simultaneously under the condition of order-size-dependent delay in payments. They assumed that the length of the credit period is a function of the retailer's order size, and also the demand rate is a function of the selling price. Chung and Huang (2003a) extended this problem within the EPQ framework and developed an efficient procedure to determine the retailer's optimal ordering policy. Huang (2003) extended this issue under two levels of trade credit and developed an efficient solution procedure to determine the optimal lot-sizing policy of the retailer. Arcelus et al. (2003) modeled the retailer's profit-maximizing retail promotion strategy, when confronted with a vendor's trade promotion offer of credit and/or price discount on the purchase of regular or perishable merchandise. Abad and Jaggi (2003) developed a joint approach to determine for the seller the optimal unit price and the length of the credit period when end demand is price sensitive. Salameh et al. (2003) extended this issue to continuous review inventory model. Chang et al. (2003) and Chung and Liao (2004) investigated the problem of determining the economic order quantity for exponentially deteriorating items under permissible delay in payments depending on the ordering quantity. Huang (2004a) investigated that the unit selling price and the unit purchasing price are not necessarily equal and the retailer uses different payment rule to pay the payment under supplier's trade credit policy. Huang (2004b) extended this issue to the EPQ framework. Recently, Chung et al. (2005) investigated retailer's lot-sizing policy under permissible delay in payments depending on the ordering quantity. Chang and Dye (2005) investigated an inventory model for

deteriorating items with time varying demand and deterioration rates when the credit period depends on the retailer's ordering quantity.

Therefore, it makes economic sense for the retailer to delay the settlement of the replenishment account up to the last moment of the permissible period allowed by the supplier. From the viewpoint of the supplier, the supplier hopes that the payment is paid from retailer as soon as possible. It can avoid the possibility of resulting in bad debt. So, in most business transactions, the supplier will not only offer the fixed period to settle the account but also may allow a cash discount to encourage the retailer to pay for his/her purchasing cost as soon as possible. The retailer can obtain the cash discount when the payment is paid within cash discount period offered by the supplier. Otherwise, the retailer will pay full payment within the trade credit period. In general, the cash discount period is shorter than the trade credit period. Many articles related to the inventory policy under cash discount and payment delay can be found in Chang (2002), Ouyang et al. (2002) , Huang and Chung (2003) and Ouyang et al. (2005).

The main purpose is that we want to investigate the case where the retailer's unit selling price and the purchasing price per unit are not necessarily equal within the EPQ framework under cash discount and permissible delay in payments. That is, this paper incorporates all Chung and Huang (2003a), Huang and Chung (2003) and Teng (2002) under above conditions. In addition, we try to provide an algebraic approach to determine the optimal cycle time. In previous most published papers those have been derived using differential calculus to find the optimal solution and to prove optimality condition with second-order derivatives. In recent papers, Grubbström and Erdem (1999) and Cárdenas-Barrón (2001) showed that the formulae for the EOQ and EPQ with backlogging can be derived without differential calculus. Yang and Wee (2002) developed algebraically the optimal replenishment policy of the integrated vendor-buyer inventory system without using differential calculus. Wu and Ouyang (2003) modify Yang and Wee (2002) to allow shortages using algebraic method. In this paper, we provide one theorem to efficiently determine the optimal cycle time, optimal order quantity and optimal payment time. Finally, numerical examples are given to illustrate the result and the managerial insights are also obtained.

2. MODEL FORMULATION

For convenience, most notation and assumptions similar to all Chung and Huang (2003a), Huang and Chung (2003) and Teng (2002) will be used in this paper.

2.1. Notation:

A = cost of placing one order
c = unit purchasing price
D = demand rate per year
h = unit stock holding cost per year excluding interest charges
I_e = interest which can be earned per \$ per year
I_k = interest charges per \$ investment in inventory per year
M_1 = the period of cash discount in years
M_2 = the period of trade credit in years, $M_1 < M_2$

P = replenishment rate per year, $P > D$

$$\rho = 1 - \frac{D}{P} > 0$$

r = cash discount rate, $0 \leq r < 1$

s = unit selling price

T = the cycle time in years (decision variable)

$TVC_1(T)$ = the annual total relevant cost when payment is paid at time M_1 and $T > 0$

$$= \begin{cases} TVC_{11}(T) & \text{if } M_1 \leq PM_1/D \leq T \\ TVC_{12}(T) & \text{if } M_1 \leq T \leq PM_1/D \\ TVC_{13}(T) & \text{if } 0 < T \leq M_1 \end{cases}$$

$TVC_2(T)$ = the annual total relevant cost when payment is paid at time M_2 and $T > 0$

$$= \begin{cases} TVC_{21}(T) & \text{if } M_2 \leq PM_2/D \leq T \\ TVC_{22}(T) & \text{if } M_2 \leq T \leq PM_2/D \\ TVC_{23}(T) & \text{if } 0 < T \leq M_2 \end{cases}$$

$TVC(T)$ = the annual total relevant cost when $T > 0$

$$= \begin{cases} TVC_1(T) & \text{if the payment is paid at time } M_1 \\ TVC_2(T) & \text{if the payment is paid at time } M_2 \end{cases}$$

T^* = the optimal cycle time of $TVC(T)$.

2.2. Assumptions

(1) Demand rate, D, is known and constant.
(2) Replenishment rate, P, is known and constant.
(3) Shortages are not allowed.
(4) Time horizon is infinite.
(5) $s \geq c$.
(6) Supplier offers a cash discount after settlement of an order if payment is paid within M_1, otherwise the full payment is paid within M_2. The account is settled when the payment is paid.
(7) During the time the account is not settled, generated sales revenue is deposited in an interest-bearing account. At the end of the period, the retailer pays off all units sold and keeps his/her profits, and starts paying for the interest charges on the items in stock.

2.3. Mathematical model

The total annual relevant cost consists of the following elements.

(1) Annual ordering cost = $\dfrac{A}{T}$.

(2) Annual stock holding cost (excluding interest charges)

$$= \frac{hT(P-D)\frac{DT}{P}}{2T} = \frac{DTh}{2}(1-\frac{D}{P}) = \frac{DTh\rho}{2} \ .$$

Since the supplier offers a cash discount if payment is paid within M_1, there are two payment policies for the retailer. First, the payment is paid at time M_1 to get the cash discount, **Case 1**. Second, the payment is paid at time M_2 not to get the cash discount, **Case 2**. So purchasing cost, interest payable and interest earned, we shall discuss these two cases as follows.

(3) Annual purchasing cost:
Case 1: Payment is paid at time M_1, the annual purchasing cost= $c(1-r)D$.
Case 2: Payment is paid at time M_2, the annual purchasing cost= cD.

(4) Annual cost of interest charges for the items kept in stock:
Case 1: Payment is paid at time M_1

Case 1.1: $M_1 \leq \frac{PM_1}{D} \leq T$.

In this case, the retailer pays the payment at M_1 to get cash discount and the account is settled. Hence, the retailer must pay the cost of interest charges for unsold items behind M_1. Therefore, the annual interest payable

$$= cI_k(1-r)[\frac{DT^2\rho}{2} - \frac{(P-D)M_1^2}{2}]/T = cI_k(1-r)\rho(\frac{DT^2}{2} - \frac{PM_1^2}{2})/T \ .$$

Case 1.2: $M_1 \leq T \leq \frac{PM_1}{D}$.

Same discussion as above case 1.1, the annual interest payable

$$= cI_k(1-r)[\frac{D(T-M_1)^2}{2}]/T \ .$$

Case 1.3: $T \leq M_1$.

In this case, all items have sold when the payment is paid at time M_1. Therefore, there is no interest charges are paid for the items.

Case 2: Payment is paid at time M_2

Case 2.1: $M_2 \leq \frac{PM_2}{D} \leq T$.

In this case, the retailer cannot get the cash discount since the retailer pays the payment at M_2, then the account is settled. Hence, the retailer must pay the cost of interest charges for unsold items behind M_2. Therefore, the annual interest payable

$$= cI_k[\frac{DT^2\rho}{2} - \frac{(P-D)M_2^2}{2}]/T = cI_k\rho(\frac{DT^2}{2} - \frac{PM_2^2}{2})/T \ .$$

Case 2.2: $M_2 \leq T \leq \dfrac{PM_2}{D}$.

Same discussion as above case 2.1, the annual interest payable

$$= cI_k[\frac{D(T-M_2)^2}{2}]/T .$$

Case 2.3: $T \leq M_2$.

In this case, all items have sold when the payment is paid at time M_2. Therefore, there is no interest charges are paid for the items.

(5) Annual interest earned:

Case 1: Payment is paid at time M_1

During the time the account is not settled, generated sales revenue is deposited in an interest-bearing account. Hence, the retailer can earn the interest from sales revenue during $(0, M_1]$.

Case 1.1: $M_1 \leq \dfrac{PM_1}{D} \leq T$.

Annual interest earned $= sI_e(\dfrac{DM_1^2}{2})/T$.

Case 1.2: $M_1 \leq T \leq \dfrac{PM_1}{D}$.

Annual interest earned $= sI_e(\dfrac{DM_1^2}{2})/T$.

Case 1.3: $T \leq M_1$.

Annual interest earned $= sI_e[\dfrac{DT^2}{2} + DT(M_1 - T)]/T$.

Case 2: Payment is paid at time M_2

During the time the account is not settled, generated sales revenue is deposited in an interest-bearing account. Hence, the retailer can earn the interest from sales revenue during $(0, M_2]$.

Case 2.1: $M_2 \leq \dfrac{PM_2}{D} \leq T$.

Annual interest earned $= sI_e(\dfrac{DM_2^2}{2})/T$.

Case 2.2: $M_2 \leq T \leq \dfrac{PM_2}{D}$.

Annual interest earned $= sI_e(\dfrac{DM_2^2}{2})/T$.

Case 2.3: $T \leq M_2$.

Annual interest earned $= sI_e[\dfrac{DT^2}{2} + DT(M_2 - T)]/T$.

The annual total relevant cost for the retailer can be expressed as:

$TVC(T)$ = ordering cost + stock-holding cost + purchasing cost + interest payable − interest earned.

We show that the annual total relevant cost is given by

Case 1: Payment is paid at time M_1

$$TVC_1(T) = \begin{cases} TVC_{11}(T) & \text{if } M_1 \le PM_1/D \le T & (1.1) \\ TVC_{12}(T) & \text{if } M_1 \le T \le PM_1/D & (1.2) \\ TVC_{13}(T) & \text{if } 0 < T \le M_1 & (1.3) \end{cases}$$

where:

$$TVC_{11}(T) =$$
$$= \frac{A}{T} + \frac{DTh\rho}{2} + c(1-r)D + cI_k(1-r)\rho\left(\frac{DT^2}{2} - \frac{PM_1^2}{2}\right)/T - sI_e\left(\frac{DM_1^2}{2}\right)/T, \quad (2)$$

$$TVC_{12}(T) =$$
$$= \frac{A}{T} + \frac{DTh\rho}{2} + c(1-r)D + cI_k(1-r)\left[\frac{D(T-M_1)^2}{2}\right]/T - sI_e\left(\frac{DM_1^2}{2}\right)/T \quad (3)$$

and

$$TVC_{13}(T) = \frac{A}{T} + \frac{DTh\rho}{2} + c(1-r)D - sI_e\left[\frac{DT^2}{2} + DT(M_1 - T)\right]/T \quad (4)$$

Then, we find $TVC_{11}(PM_1/D) = TVC_{12}(PM_1/D)$ and $TVC_{12}(M_1) = TVC_{13}(M_1)$. Hence $TVC_1(T)$ is continuous and well-defined. All $TVC_{11}(T)$, $TVC_{12}(T)$, $TVC_{13}(T)$ and $TVC_1(T)$ are defined on $T > 0$.

Case 2: Payment is paid at time M_2

$$TVC_2(T) = \begin{cases} TVC_{21}(T) & \text{if } M_2 \le PM_2/D \le T & (5.1) \\ TVC_{22}(T) & \text{if } M_2 \le T \le PM_2/D & (5.2) \\ TVC_{23}(T) & \text{if } 0 < T \le M_2 & (5.3) \end{cases}$$

where:

$$TVC_{21}(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cD + cI_k\rho\left(\frac{DT^2}{2} - \frac{PM_2^2}{2}\right)/T - sI_e\left(\frac{DM_2^2}{2}\right)/T, \quad (6)$$

$$TVC_{22}(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cD + cI_k\left[\frac{D(T-M_2)^2}{2}\right]/T - sI_e\left(\frac{DM_2^2}{2}\right)/T \quad (7)$$

and

$$TVC_{23}(T) = \frac{A}{T} + \frac{DТh\rho}{2} + cD - sI_e\left[\frac{DT^2}{2} + DT(M_2 - T)\right]/T.$$ (8)

Then, we find $TVC_{21}(PM_2/D) = TVC_{22}(PM_2/D)$ and $TVC_{22}(M_2) = TVC_{23}(M_2)$. Hence $TVC_2(T)$ is continuous and well-defined. All $TVC_{21}(T)$, $TVC_{22}(T)$, $TVC_{23}(T)$ and $TVC_2(T)$ are defined on $T > 0$.

2.4. Find optimal solution using algebraic method

We can rewrite

$$TVC_{11}(T) =$$ (9)

$$= \frac{D\rho[h + c(1-r)I_k]}{2T}\left\{T - \sqrt{\frac{2A + DM_1^2[c(1-r)I_k - sI_e] - PM_1^2 c(1-r)I_k}{D\rho[h + c(1-r)I_k]}}\right\}^2$$

$$+ \left\{\sqrt{D\rho[h + c(1-r)I_k]\{2A + DM_1^2[c(1-r)I_k - sI_e] - PM_1^2 c(1-r)I_k\}} + cD(1-r)\right\}$$

Equation (9) represents that the minimum of $TVC_{11}(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_{11}^* is

$$T_{11}^* = \sqrt{\frac{2A + DM_1^2[c(1-r)I_k - sI_e] - PM_1^2 c(1-r)I_k}{D\rho[h + c(1-r)I_k]}} \quad \text{if}$$

$$2A + DM_1^2[c(1-r)I_k - sI_e] - PM_1^2 c(1-r)I_k > 0.$$ (10)

Therefore, Equation (9) has a minimum value for the optimal value of T_{11}^* reducing $TVC_{11}(T)$ to

$$TVC_{11}(T_{11}^*) =$$
$$= \sqrt{D\rho[h + c(1-r)I_k]\{2A + DM_1^2[c(1-r)I_k - sI_e] - PM_1^2 c(1-r)I_k\}}$$
$$+ cD(1-r).$$ (11)

Similarly, we can derive

$$TVC_{12}(T) = \frac{D[h\rho + c(1-r)I_k]}{2T}\left\{T - \sqrt{\frac{2A + DM_1^2[c(1-r)I_k - sI_e]}{D[h\rho + c(1-r)I_k]}}\right\}^2$$

$$+ \left\{\sqrt{D[h\rho + c(1-r)I_k]\{2A + DM_1^2[c(1-r)I_k - sI_e]\}} + cD(1-r)(1-M_1I_k)\right\}.$$ (12)

Equation (12) represents that the minimum of $TVC_{12}(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value T_{12}^* is

$$T_{12}* = \sqrt{\frac{2A + DM_1^2[c(1-r)I_k - sI_e]}{D[h\rho + c(1-r)I_k]}} \text{ if } 2A + DM_1^2[c(1-r)I_k - sI_e] > 0. \quad (13)$$

Therefore, Equation (12) has a minimum value for the optimal value of $T_{12}*$ reducing $TVC_{12}(T)$ to

$$TVC_{12}(T_{12}*) =$$
$$= \sqrt{D[h\rho + c(1-r)I_k]\{2A + DM_1^2[c(1-r)I_k - sI_e]\}} + cD(1-r)(1 - M_1 I_k). \quad (14)$$

Similarly,

$$TVC_{13}(T) = \frac{D(h\rho + sI_e)}{2T}\left\{T - \sqrt{\frac{2A}{D(h\rho + sI_e)}}\right\}^2$$
$$+ \left\{\sqrt{2AD(h\rho + sI_e)} + D[c(1-r) - sI_e M_1]\right\}. \quad (15)$$

Equation (15) represents that the minimum of $TVC_{13}(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value $T_{13}*$ is

$$T_{13}* = \sqrt{\frac{2A}{D(h\rho + sI_e)}} \ . \quad (16)$$

Therefore, Equation (15) has a minimum value for the optimal value of $T_{13}*$ reducing $TVC_{13}(T)$ to

$$TVC_{13}(T_{13}*) = \sqrt{2AD(h\rho + sI_e)} + D[c(1-r) - sI_e M_1]. \quad (17)$$

Similarly,

$$TVC_{21}(T) = \frac{D\rho(h + cI_k)}{2T}\left\{T - \sqrt{\frac{2A + DM_2^2(cI_k - sI_e) - PM_2^2 cI_k}{D\rho(h + cI_k)}}\right\}^2$$
$$+ \left\{\sqrt{D\rho(h + cI_k)[2A + DM_2^2(cI_k - sI_e) - PM_2^2 cI_k]} + cD\right\}. \quad (18)$$

Equation (18) represents that the minimum of $TVC_{21}(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value $T_{21}*$ is

$$T_{21}* = \sqrt{\frac{2A + DM_2^2(cI_k - sI_e) - PM_2^2 cI_k}{D\rho(h + cI_k)}} \text{ if }$$

$$2A + DM_2^2(cI_k - sI_e) - PM_2^2 cI_k > 0. \quad (19)$$

Therefore, Equation (18) has a minimum value for the optimal value of $T_{21}*$ reducing $TVC_{21}(T)$ to

$$TVC_{21}(T_{21}*)= \sqrt{D\rho(h+cI_k)[2A+DM_2{}^2(cI_k-sI_e)-PM_2{}^2cI_k]}+cD \ . \qquad (20)$$

Similarly,

$$TVC_{22}(T) = \frac{D(h\rho+cI_k)}{2T}\left\{T-\sqrt{\frac{2A+DM_2{}^2(cI_k-sI_e)}{D(h\rho+cI_k)}}\right\}^2$$
$$+\left\{\sqrt{D(h\rho+cI_k)[2A+DM_2{}^2(cI_k-sI_e)]}+cD(1-M_2I_k)\right\}. \qquad (21)$$

Equation (21) represents that the minimum of $TVC_{22}(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value $T_{22}*$ is

$$T_{22}* = \sqrt{\frac{2A+DM_2{}^2(cI_k-sI_e)}{D(h\rho+cI_k)}} \quad \text{if} \quad 2A+DM_2{}^2(cI_k-sI_e)>0. \qquad (22)$$

Therefore, Equation (21) has a minimum value for the optimal value of $T_{22}*$ reducing $TVC_{22}(T)$ to

$$TVC_{22}(T_{22}*) = \sqrt{D(h\rho+cI_k)[2A+DM_2{}^2(cI_k-sI_e)]}+cD(1-M_2I_k) \ . \qquad (23)$$

Similarly,

$$TVC_{23}(T) = \frac{D(h\rho+sI_e)}{2T}\left\{T-\sqrt{\frac{2A}{D(h\rho+sI_e)}}\right\}^2$$
$$+\left\{\sqrt{2AD(h\rho+sI_e)}+D(c-sI_eM_2)\right\}. \qquad (24)$$

Equation (24) represents that the minimum of $TVC_{23}(T)$ is obtained when the quadratic non-negative term, depending on T, is made equal to zero. Therefore, the optimum value $T_{23}*$ is

$$T_{23}* = \sqrt{\frac{2A}{D(h\rho+sI_e)}} \ . \qquad (25)$$

Therefore, Equation (24) has a minimum value for the optimal value of $T_{23}*$ reducing $TVC_{23}(T)$ to

$$TVC_{23}(T_{23}*) = \sqrt{2AD(h\rho+sI_e)}+D(c-sI_eM_2) \ . \qquad (26)$$

3. DETERMINATION OF THE OPTIMAL CYCLE TIME $T*$

The main purpose of this section is to develop a solution procedure to determine the optimal cycle time $T*$.

From equation (10) the optimal value of T for the case of $T \geq PM_1/D$ is $T_{11}* \geq PM_1/D$. We can substitute equation (10) into $T_{11}* \geq PM_1/D$ to obtain the optimal value of T

if and only if $-2A+\dfrac{M_1{}^2}{D}\left[cI_k(1-r)(P^2-D^2)+sI_eD^2+hP(P-D)\right] \leq 0.$

Similar disscussion, we can obain following results:

$$M_1 \leq T_{12}* \leq PM_1/D$$

if and only if $-2A + \dfrac{M_1^2}{D}\left[cI_k(1-r)(P^2 - D^2) + sI_eD^2 + hP(P-D)\right] \geq 0$ and

if and only if $-2A + DM_1^2(h\rho + sI_e) \leq 0$.

$T_{13}* \leq M_1$ if and only if $-2A + DM_1^2(h\rho + sI_e) \geq 0$.

$T_{21}* \geq PM_2/D$ if and only if $-2A + \dfrac{M_2^2}{D}\left[cI_k(P^2 - D^2) + sI_eD^2 + hP(P-D)\right] \leq 0$.

$$M_2 \leq T_{22}* \leq PM_2/D$$

if and only if $-2A + \dfrac{M_2^2}{D}\left[cI_k(P^2 - D^2) + sI_eD^2 + hP(P-D)\right] \geq 0$ and

if and only if $-2A + DM_2^2(h\rho + sI_e) \leq 0$.

$T_{23}* \leq M_2$ if and only if $-2A + DM_2^2(h\rho + sI_e) \geq 0$.

Let

$$\Delta_1 = -2A + \frac{M_1^2}{D}\left[cI_k(1-r)(P^2 - D^2) + sI_eD^2 + hP(P-D)\right], \tag{27}$$

$$\Delta_2 = -2A + DM_1^2(h\rho + sI_e), \tag{28}$$

$$\Delta_3 = -2A + \frac{M_2^2}{D}\left[cI_k(P^2 - D^2) + sI_eD^2 + hP(P-D)\right] \tag{29}$$

and

$$\Delta_4 = -2A + DM_2^2(h\rho + sI_e). \tag{30}$$

From equations (27)-(30), we can obtain $\Delta_3 > \Delta_1 > \Delta_2$ and $\Delta_3 > \Delta_4 > \Delta_2$ since $M_2 > M_1$. Summarized above arguments, we can obtain following results.

Theorem 1:

    (A)   If $\Delta_2 \geq 0$, then $TVC(T*) = \min\{TVC_1(T_{13}*), TVC_2(T_{23}*)\}$. Hence $T*$ is $T_{13}*$ or $T_{23}*$ associated with the least cost.

    (B)   If $\Delta_1 \geq 0$, $\Delta_2 < 0$ and $\Delta_4 \geq 0$, then $TVC(T*) = \min\{TVC_1(T_{12}*), TVC_2(T_{23}*)\}$. Hence $T*$ is $T_{12}*$ or $T_{23}*$ associated with the least cost.

    (C)   If $\Delta_1 \geq 0$, $\Delta_2 < 0$ and $\Delta_4 < 0$, then $TVC(T*) = \min\{TVC_1(T_{12}*), TVC_2(T_{22}*)\}$. Hence $T*$ is $T_{12}*$ or $T_{22}*$ associated with the least cost.

(D)   If $\Delta_1 < 0$ and $\Delta_4 \geq 0$, then $TVC(T^*) = \min\{TVC_1(T_{11}^*),\ TVC_2(T_{23}^*)\}$. Hence T^* is T_{11}^* or T_{23}^* associated with the least cost.

(E)   If $\Delta_1 < 0$, $\Delta_3 > 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_1(T_{11}^*),\ TVC_2(T_{22}^*)\}$. Hence T^* is T_{11}^* or T_{22}^* associated with the least cost.

(F)   If $\Delta_3 \leq 0$, then $TVC(T^*) = \min\{TVC_1(T_{11}^*),\ TVC_2(T_{21}^*)\}$. Hence T^* is T_{11}^* or T_{21}^* associated with the least cost.

Theorem 1 immediately determines the optimal cycle time T^* after computing the numbers Δ_1, Δ_2, Δ_3 and Δ_4. Theorem 1 is really very simple.

4. SPECIAL CASES

In this section, some previously published models are deduced as special cases.

(I) Huang and Chung's model (2003)

When $P \to \infty$ and $s = c$, let

$$\overline{TVC}_{11}(T) = \frac{A}{T} + \frac{DTh}{2} + c(1-r)D + \frac{c(1-r)I_k D(T-M_1)^2}{2T} - \frac{cI_e DM_1^2}{2T},$$

$$\overline{TVC}_{12}(T) = \frac{A}{T} + \frac{DTh}{2} + c(1-r)D - DcI_e(M_1 - \frac{T}{2}),$$

$$\overline{TVC}_{21}(T) = \frac{A}{T} + \frac{DTh}{2} + cD + \frac{cI_k D(T-M_2)^2}{2T} - \frac{cI_e DM_2^2}{2T}$$

and

$$\overline{TVC}_{22}(T) = \frac{A}{T} + \frac{DTh}{2} + cD - DcI_e(M_2 - \frac{T}{2}).$$

Equations (1.1-1.3) and (5.1-5.3) will be reduced as follows:

$$TVC_1(T) = \begin{cases} \overline{TVC}_{11}(T) & \text{if } M_1 \leq T \\ \overline{TVC}_{12}(T) & \text{if } 0 < T \leq M_1 \end{cases}$$

(31.1)

(31.2)

and

$$TVC_2(T) = \begin{cases} \overline{TVC}_{21}(T) & \text{if } M_2 \leq T \\ \overline{TVC}_{22}(T) & \text{if } 0 < T \leq M_2 \end{cases}$$

(32.1)

(32.2)

Equations (31.1-31.2) and (32.1-32.2) will be consistent with equations 1(a, b) and 4(a, b) in Huang and Chung (2003), respectively. Hence, Huang and Chung (2003) will be a special case of this paper.

(II) Chung and Huang's model (2003a)

When $r=M_1=0$, $M_2=M$ and $s=c$, let

$$TVC_4(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cI_k\rho(\frac{DT^2}{2} - \frac{PM^2}{2})/T - cI_e(\frac{DM^2}{2})/T,$$

$$TVC_5(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cI_k[\frac{D(T-M)^2}{2}]/T - cI_e(\frac{DM^2}{2})/T$$

and

$$TVC_6(T) = \frac{A}{T} + \frac{DTh\rho}{2} - cI_e[\frac{DT^2}{2} + DT(M-T)]/T,$$

Equations (1.1-1.3) and (5.1-5.3) will be reduced as follows:

$$TVC(T) = \begin{cases} TVC_4(T) & \text{if} & T \geq \dfrac{PM}{D} & (33.1) \\[2mm] TVC_5(T) & \text{if} & M \leq T \leq \dfrac{PM}{D} & (33.2) \\[2mm] TVC_6(T) & \text{if} & 0 < T \leq M & (33.3) \end{cases}$$

Equations (33.1-33.3) will be consistent with equations 6(a, b, c) in Chung and Huang (2003a), respectively. Hence, Chung and Huang (2003a) will be a special case of this paper.

(III) Teng's model (2002)

When $P \to \infty$, $r=M_1=0$ and $M_2=M$, let

$$TVC_7(T) = \frac{A}{T} + \frac{DTh}{2} + cI_k[\frac{D(T-M)^2}{2}]/T - sI_e(\frac{DM^2}{2})/T$$

and

$$TVC_8(T) = \frac{A}{T} + \frac{DTh}{2} - sI_e[\frac{DT^2}{2} + DT(M-T)]/T.$$

Equations (1.1-1.3) and (5.1-5.3) will be reduced as follows:

$$TVC(T) = \begin{cases} TVC_7(T) & \text{if } M \leq T & (34.1) \\[2mm] TVC_8(T) & \text{if } 0 < T \leq M & (34.2) \end{cases}$$

Equations (34.1-34.2) will be consistent with equations (1) and (2) in Teng (2002), respectively. Hence, Teng (2002) will be a special case of this paper.

(IV) Goyal's model (1985)

When $P \to \infty$, $r = M_1 = 0$, $M_2 = M$ and $s = c$, let

$$TVC_9(T) = \frac{A}{T} + \frac{DTh}{2} + cI_k[\frac{D(T-M)^2}{2}]/T - cI_e(\frac{DM^2}{2})/T ,$$

and

$$TVC_{10}(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e[\frac{DT^2}{2} + DT(M-T)]/T .$$

Equations (1.1-1.3) and (5.1-5.3) will be reduced as follows:

$$TVC(T) = \begin{cases} TVC_9(T) & \text{if} \quad M \leq T \qquad\qquad (35.1) \\ TVC_{10}(T) & \text{if} \quad 0 < T \leq M \qquad (35.2) \end{cases}$$

　　　Equations (35.1-35.2) will be consistent with equations (1) and (4) in Goyal (1985), respectively. Hence, Goyal (1985) will be a special case of this paper.

5. NUMERICAL EXAMPLES

To illustrate the result obtained in this paper, let us apply the proposed method to efficiently solve the following numerical examples. The optimal cycle time, optimal order quantity and optimal payment time are summarized in Table 1 and Table 2, respectively.

Table 1: The optimal solution with various values of r

A	D	P	c	s	I_k	I_e	h	M_1	M_2	r	Δ_1	Δ_2	Δ_3	Δ_4	Theorem	Optimal cycle time, T^*	Optimal payment time (M_1/M_2)	Optimal order quantity, DT^*
35	1000	1500	10	15	0.15	0.12	5	0.07	0.1	0.1	>0	<0	>0	<0	1-(C)	T_{12}^*=0.14991	M_1	149.9
35	1000	1500	10	15	0.15	0.12	5	0.07	0.1	0.15	>0	<0	>0	<0	1-(C)	T_{12}^*=0.15139	M_1	151.4
35	1000	1500	10	15	0.15	0.12	5	0.07	0.1	0.2	>0	<0	>0	<0	1-(C)	T_{12}^*=0.15295	M_1	153
35	1000	1500	10	15	0.15	0.12	5	0.07	0.1	0.25	<0	<0	>0	<0	1-(E)	T_{11}^*=0.16916	M_1	169.2
35	1000	1500	10	15	0.15	0.12	5	0.07	0.1	0.3	<0	<0	>0	<0	1-(E)	T_{11}^*=0.17048	M_1	170.5
35	1000	1500	10	15	0.15	0.12	5	0.07	0.1	0.35	<0	<0	>0	<0	1-(E)	T_{11}^*=0.17181	M_1	171.8

Table 2: The optimal solution with various values of P and s

Let A=$60/order, D=1000units/year, c=$10/unit, h=$2/unit/year, I_k=$0.15/$/year, I_e=$0.1/$/year, M_1=0.02year and M_2=0.15year.																
P	s=$15/unit						s=$20/unit						s=$25/unit			
	Δ_1	Δ_2	Δ_3	Δ_4	T^*	M_1/M_2	Δ_1	Δ_2	Δ_3	Δ_4	T^*	M_1/M_2	Δ_1	Δ_2	Δ_3	Δ_4	T^*	M_1/M_2
1500	<0	<0	<0	<0	T_{11}^*=0.3202	M_1	>0	<0	>0	<0	T_{22}^*=0.22404	M_2	>0	<0	>0	<0	T_3^*=0.21213	M_2
2000	>0	<0	>0	<0	T_{22}^*=0.21909	M_2	>0	<0	>0	<0	T_{22}^*=0.20857	M_2	>0	<0	>0	<0	T_{22}^*=0.19748	M_2
2500	>0	<0	>0	<0	T_{22}^*=0.21082	M_2	>0	<0	>0	<0	T_{22}^*=0.20069	M_2	>0	<0	>0	<0	T_{22}^*=0.19003	M_2

To study the effect of cash discount rate, r, on the optimal cycle time and optimal order quantity for the retailer derived by the proposed method, we solve the example on Table 1 with various values of r. The following inferences can be made based on Table 1. When r is increasing, the optimal cycle time and optimal order quantity for the retailer are increasing. So, the retailer will order more quantity to take more the benefits of cash discount as a larger cash discount rate. Of course, the retailer will pay the payment within M_1 to get the cash discount.

To study the effects of replenishment rate per year, P, and unit selling price, s, on the optimal cycle time and optimal order quantity for the retailer derived by the proposed method, we solve the example on Table 2 with various values of P and s. The following inferences can be made based on Table 2. When P is increasing, the optimal cycle time and optimal order quantity for the retailer are decreasing. So, the retailer will shorten the ordering time interval since the replenishment speed is faster. When s is increasing, the optimal cycle time and optimal order quantity for the retailer are decreasing. This result implies that the retailer will order less quantity to take the benefits of the payment delay more frequently.

6. CONCLUSIONS

This paper is to the case where the retailer's unit selling price and the purchasing price per unit are not necessarily equal within the economic production quantity framework under cash discount and permissible delay in payments, reflecting the real-life situations. Then, we provide a very efficient solution procedure to determine the optimal cycle time T^*. Theorem 1 helps the retailer accurately and quickly determining the optimal inventory policy under minimizing the annual total relevant cost. In addition, we deduce some previously published results of other researchers as special cases. Finally, numerical examples are given to illustrate the result obtained in this paper. There are some managerial phenomena as follows:

1. The retailer will order more quantity and pay the payment within M_1 to take more the benefits of cash discount as a larger cash discount rate.
2. The retailer will order less quantity to save inventory holding cost when the replenishment speed is faster.
3. The retailer will order less quantity to take the benefits of the delay payments more frequently when the larger the differences between the unit selling price and the unit purchasing price.

A future study will further incorporate the proposed model into more realistic assumptions, such as probabilistic demand, deteriorating items and allowable shortages.

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