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THE TRUNCATED HYPER-POISSON QUEUES: $H_k / M^{a,b} / C / N$ WITH BALKING, RENEGING AND GENERAL BULK SERVICE RULE

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Abstract: The aim of this paper is to derive the analytical solution of the queue: $H_k / M^{a,b} / C / N$ with balking and reneging in which (i) units arrive according to a hyper-Poisson distribution with k independent branches, (ii) the queue discipline is FIFO; and (iii) the units are served in batches according to a general bulk service rule. The steady-state probabilities, recurrence relations connecting various probabilities introduced are found and the expected number of units in the queue is derived in an explicit form. Also, some special cases are obtained.

Keywords: Hyper-Poisson queues, balking, reneging, general bulk service rule.

1. INTRODUCTION

Morse [7] has discussed the case of hyper-Poisson arrivals with two branches, Gupta [4] considered more general case of hyper-Poisson arrivals with k branches and finite waiting space, the service pattern being single exponential, and White et al. [12] solved the system: $H_2 / M / 2 / 2$ numerically. Sim and Templeton [11] discussed the system: $M / M^{a,b} / C$ and Medhi [6] also discussed the system: $M / M^{a,b} / C$. Goyal [3] treated the time-dependent solution of the queues with hyper exponential arrivals and bulk exponential service by using the generating function, Easton and Chaudhry [2] treated the Erlangian queues: $E_k / M^{a,b} / I$, and Sathiya Moorthi and Ganesan [8]

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discussed the queues: $H_k / M^{a,b} / C$. The present paper treats the analytical solution of the queues: $H_k / M^{a,b} / C / N$ with balking and reneging. Finally, some previously published works are shown to be special cases of the model herein.

2. DESCRIPTION OF THE SYSTEM

The arrival channel consists of k independent branches. A unit arriving for service joins the r^{th} branch a fraction σ_r of the time on the average, so that $\sum_{r=1}^{k} \sigma_r = 1$. The unit in the r^{th} branch joins the system (queue or service) with rate λ_r per unit time. The maximum number of units allowed in the queue is N. The units are served in batches according to a general bulk service rule (See, Chaudhry and Templeton [1] and Medhi [5]) such that the server starts service only when a minimum number of units, say "a", are present in the queue, the maximum service capacity being, say "b", $1 \le a \le x \le b < N$. This means that if there are x units waiting at the completion of a service, the following rule for service is followed:

a) For $0 \le x < a$, do not served,
b) For $a \le x \le b$, serve a batch of x units,
c) For $x > b$, serve a batch of b unit; the remaining $(x-b)$ units continue to wait in the queue. The service times of batches of size x $(a \le x \le b)$ are assumed to have independent identical exponential distribution with mean $1/\mu$.

Consider the balk concept with probability:

$\beta =$ prob. {a unit joins the queue},

where $0 \le \beta < 1$ if $0 \le n \le N$; $i = C$ and $\beta = 1$ if $0 \le q \le a-1$, $0 \le i \le C-1$.

It is assumed that the units may renege according to an exponential distribution, $f(t) = \alpha\, e^{-\alpha t}$, $t > 0$, with parameter α . The probability of reneging in a short period of time Δt is given by $r_n = n\alpha\Delta t$, for $1 \le n \le N$, $i = C$ and $r_n = 0$, for $0 \le q \le a-1$, $0 \le i \le C-1$.

3. THE STEADY-STATE EQUATIONS AND THEIR SOLUTION

Define $p_{C,n,r}$ the equilibrium probability that there are n units waiting in the queue, the unit in the arrival channel being in the r^{th} branch and C servers are busy, where $0 \le n \le N$ and $1 \le r \le k$, and $p_{i,q,r}$ the equilibrium probability that there are q units waiting in the queue, the unit in the arrival channel being in the r^{th} branch and i servers are busy, where $0 \le q \le a-1$ and $0 \le i \le C-1$.

Then, the steady-state probability difference equations are:

$$\lambda_r\, p_{0,0,r} = \mu\, p_{1,0,r} \,,\ 1 \le r \le k \tag{1}$$

$$\lambda_r\, p_{0,q,r} = \sigma_r \sum_{s=1}^{k} \lambda_s\, p_{0,q-1,s} + \mu\, p_{1,q,r}\ ,\ 1 \le q \le a-1,\ a \ge 2\ ,\ 1 \le r \le k \tag{2}$$

$$(\lambda_r + i\,\mu)\ p_{i,q,r} = \sigma_r \sum_{s=1}^{k} \lambda_s\, p_{i,q-1,s} + (i+1)\,\mu\, p_{i+1,q,r}\ ,1 \le q \le a-1,\ a \ge 2\ ,$$
$$1 \le i \le C-2,\ C \ge 3\ ,\ 1 \le r \le k \tag{3}$$

$$[\lambda_r + (C-1)\,\mu]\, p_{C-1,q,r} = \sigma_r \sum_{s=1}^{k} \lambda_s\, p_{C-1,q-1,s} + (q\,\alpha + C\,\mu)\, p_{C,q,r}\ ,$$
$$1 \le q \le a-1,\ a \ge 2\ ,1 \le r \le k \tag{4}$$

$$(\lambda_r + i\mu) p_{i,0,r} = \sigma_r \sum_{s=1}^{k} \lambda_s p_{i-1,a-1,s} + (i+1)\,\mu\, p_{i+1,0,r}\ ,1 \le i \le C-1, C \ge 2\ ,1 \le r \le k \tag{5}$$

$$(\beta\,\lambda_r + C\,\mu)\, p_{C,0,r} = \sigma_r \sum_{s=1}^{k} \lambda_s\, p_{C-1,a-1,s} + \sum_{j=a}^{b} (j\,\alpha + C\mu)\ p_{C,j,r}\ ,\ 1 \le r \le k \tag{6}$$

$$(\beta\,\lambda_r + C\,\mu + n\,\alpha)\, p_{C,n,r} = \sigma_r \sum_{s=1}^{k} \beta\,\lambda_s\, p_{C,n-1,s} + [(n+b)\,\alpha + C\,\mu]\, p_{C,n+b,r}$$
$$1 \le n \le N-b,\ b < N,\ \ 1 \le r \le k \tag{7}$$

$$(\beta\,\lambda_r + C\,\mu + n\,\alpha)\, p_{C,n,r} = \sigma_r \sum_{s=1}^{k} \beta\,\lambda_s\, p_{C,n-1,s}\ , N-b+1 \le n \le N-1, b \ge 2,$$
$$1 \le r \le k \tag{8}$$

$$(\beta\,\lambda_r + C\,\mu + N\,\alpha)\, p_{C,N,r} = \sigma_r \sum_{s=1}^{k} \beta\,\lambda_s\, p_{C,N-1,s} + \sigma_r \sum_{s=1}^{k} \beta\,\lambda_s\ p_{C,N,s}\ ,\ 1 \le r \le k\ . \tag{9}$$

Note that equations (2) and (4) do not hold for $a = 1$, equation (3) does not hold for $C = 2$, equation (5) does not hold for $C = 1$ and also equation (8) does not hold for $b = 1$.

Summing up equation (9) over *r*, we have

$$\sum_{s=1}^{k} \beta\,\lambda_s\ p_{C,N-1,s} = (C\,\mu + N\,\alpha) \sum_{s=1}^{k} p_{C,N,s}\ . \tag{10}$$

Also, from (8) and (10), we obtain

$$\sum_{s=1}^{k} \beta\,\lambda_s\ p_{C,n-1,s} = \sum_{j=n}^{N} \sum_{s=1}^{k} (C\mu + j\alpha)\, p_{C,j,s}\ ,\ N-b+1 \le n \le N\ ,\ b \ge 1, \tag{11}$$

while (7) and (11) give

$$\sum_{s=1}^{k} \beta\,\lambda_s\ p_{C,n-1,s} = \sum_{j=n}^{n+b-1} \sum_{s=1}^{k} (C\,\mu + j\,\alpha)\ p_{C,j,s}\ ,\ 1 \le n \le N-b,\ b < N\ . \tag{12}$$

Now, from (6) and (12),

$$\sum_{s=1}^{k} \lambda_s \ p_{C-1,a-1,s} = \sum_{j=0}^{a-1} \ \sum_{s=1}^{k} (C\mu + j\alpha) \ p_{C,j,s} \ . \tag{13}$$

Also, from (4) and (13), we obtain

$$\sum_{s=1}^{k} \lambda_s \ p_{C-1,q-1,s} = \sum_{j=0}^{q-1} \ \sum_{s=1}^{k} (C\mu + j\alpha) \ p_{C,j,s} \ + (C-1)\mu \sum_{j=q}^{a-1} \ \sum_{s=1}^{k} p_{C-1,j,s} \ ,$$
$$1 \le q \le a-1, \ a \ge 2 \ , \tag{14}$$

thus (14), (5) and (3) give

$$\sum_{s=1}^{k} \lambda_s \ p_{i,0,s} = (i+1)\mu \sum_{s=1}^{k} p_{i+1,0,s} \ + i\mu \sum_{j=1}^{a-1} \ \sum_{s=1}^{k} p_{i,j,s} \ , \ 1 \le i \le C-1, \ C \ge 2 \tag{15}$$

$$\sum_{s=1}^{k} \lambda_s \ p_{i,q-1,s} = (i+1)\mu \sum_{j=0}^{q-1} \ \sum_{s=1}^{k} p_{i+1,j,s} \ + i\mu \sum_{j=q}^{a-1} \ \sum_{s=1}^{k} p_{i,j,s} \ ,$$
$$1 \le i \le C-1, 1 \le q \le a-1, \ \ a \ge 2, \ C \ge 2 \ . \tag{16}$$

From (15) and (5) after summing over *r,* we get

$$\sum_{s=1}^{k} \lambda_s \ p_{i-1,a-1,s} = i\mu \sum_{j=0}^{a-1} \ \sum_{s=1}^{k} p_{i,j,s} \ \ , \ 1 \le i \le C-1, \ C \ge 2 \ . \tag{17}$$

From (5), (16) and (2) after summing over *r*, we have

$$\sum_{s=1}^{k} \lambda_s \ p_{0,q-1,s} = \mu \sum_{j=0}^{q-1} \ \sum_{s=1}^{k} p_{1,j,s} \ \ , \ 1 \le q \le a \ . \tag{18}$$

Using (10) in (9), we obtain

$$-(\beta\lambda_r + C\mu + N\alpha)\, p_{C,N,r} + \sigma_r \sum_{s=1}^{k} (\beta\lambda_s + C\mu + N\alpha)\, p_{C,N,s} = 0 \ , \ 1 \le r \le k \ . \tag{19}$$

It is easy to verify that the determinant formed by the coefficients of $p_{C,N,r}$, $1 \le r \le k$, is zero and therefore we can solve the system of linear equations (19) for any $k-1$ probabilities involved in terms of the k^{th}. Let us solve for $p_{C,N,s}$, $1 \le s \le k-1$, in terms of $p_{C,N,k}$. Leaving out the k^{th} equation, then the system of linear equations (19) takes the form:

$$(\sigma_r - 1)\ (\beta\lambda_r + C\mu + N\alpha)\, p_{C,N,r} + \sigma_r \sum_{s \ne r}^{k-1} (\beta\lambda_s + C\mu + N\alpha)\, p_{C,N,s}$$
$$= -\sigma_r\ (\beta\lambda_k + C\mu + N\alpha)\, p_{C,N,k} \ , \ 1 \le r \le k-1 \ . \tag{20}$$

We have the matrix representation of (20) as

$$\mathbf{A}\mathrm{R}_N = -(\beta\lambda_k + C\mu + N\alpha) \ p_{C,N,k} \ \mathsf{S}_1 \ , \tag{21}$$

where **A** is the $(k-1)\times(k-1)$ matrix given by

$$\mathbf{A}=\left[a_{ij}\right]$$

such that

$$a_{ij}=\sigma_i\,(\beta\lambda_j+C\mu+N\alpha)\,,\quad i\neq j$$

$$a_{ii}=(\sigma_i-1)\,(\beta\lambda_i+C\mu+N\alpha)\ ,$$

$$\mathrm{R}_N^T=\left[p_{C,N,1},\,p_{C,N,2},\ \cdots\ ,\ p_{C,N,k-1}\right]\ ,$$

and

$$\mathrm{S}_1^T=\left[\sigma_1,\sigma_2,\ \ldots\ ,\sigma_{k-1}\right].$$

Now, the inverse matrix of A is given by

$$\mathbf{A}^{-1}=[a_{ij}^{/}]$$

where

$$a_{ij}^{/}=\frac{-\sigma_i}{\sigma_k\ (\beta\lambda_i+C\mu+N\alpha)}\ ,\ \ i\neq j$$

$$a_{ij}^{/}=\frac{-(\sigma_i+\sigma_k)}{\sigma_k\ (\beta\lambda_i+C\mu+N\alpha)}\ .$$

Using this value of A^{-1} in (21), we have

$$p_{C,N,r}=\frac{(\beta\lambda_k+C\mu+N\alpha)\ \sigma_r}{(\beta\lambda_r+C\mu+N\alpha)\ \sigma_k}\ \ p_{C,N,k}\ ,\ 1\leq r\leq k-1\,. \tag{22}$$

Similarly, from (11) and (8), we obtain

$$\left\{\beta\lambda_r+(1-\sigma_r)\ (C\mu+n\alpha)\right\}p_{C,n,r}-(C\mu+n\alpha)\ \sigma_r\sum_{s\neq r}^{k}\ p_{C,n,s}$$

$$=\sigma_r\sum_{j=n+1}^{N}\sum_{s=1}^{k}(C\mu+j\alpha)\,p_{C,j,s}\ ,\ \ N-b+1\leq n\leq N-1, b\geq 2,\ \ 1\leq r\leq k\ ,$$

which can be written the family of linear system in the matrix form as

$$\mathbf{B}_n\mathbf{P}_n=\phi_n\ \mathrm{S}_2\ ,\ \ N-b+1\leq n\leq N-1,\ \ b\geq 2 \tag{23}$$

where

$$\mathbf{B}_n=[b_{ij}(n)]$$

such that

$$b_{ij}(n) = -\sigma_i\,(C\mu + n\alpha)\,, \qquad i \neq j$$
$$b_{ii}(n) = \beta\lambda_i + (C\mu + n\alpha)\,(1-\sigma_i)\,,$$
$$\phi_n = \sum_{j=n+1}^{N}\sum_{s=1}^{k}\,(C\mu + j\alpha)\,p_{C,j,s}\,,$$
$$\mathbf{P}_n^T = \left[p_{C,n,1},\, p_{C,n,2},\, \dots\,,\, p_{C,n,k}\right],$$

and

$$\mathrm{S}_2^T = \left[\sigma_1, \sigma_2,\, \dots\,, \sigma_k\right].$$

Now, $\mathbf{B}_n^{-1}$ is given by

$$\mathbf{B}_n^{-1} = \left[b'_{ij}(n)\right]$$

such that

$$b'_{ij}(n) = \frac{\sigma_i\,(C\mu + n\alpha)}{\{\beta\lambda_i + C\mu + n\alpha\}\,\{\beta\lambda_j + C\mu + n\alpha\}\,D_n}\,, \qquad i \neq j$$

$$b'_{ii}(n) = \frac{1}{\beta\lambda_i + C\mu + n\alpha} + \frac{\sigma_i\,(C\mu + n\alpha)}{\{\beta\lambda_i + C\mu + n\alpha\}^2\,D_n}\,,$$

and

$$D_n = 1 - (C\mu + n\alpha)\sum_{s=1}^{k}\frac{\sigma_s}{\beta\lambda_s + C\mu + n\alpha}\,.$$

Using this value of $\mathbf{B}_n^{-1}$ in (23), we have

$$p_{C,n,r} = \phi_n\left[\,b'_{rr}(n)\,\sigma_r + \sum_{s\neq r}^{k} b'_{rs}(n)\,\sigma_s\right] = \frac{\sigma_r\;\phi_n}{[\beta\lambda_r + C\mu + n\alpha]\,D_n}\,,$$
$$N - b + 1 \leq n \leq N-1, b \geq 2,\ 1 \leq r \leq k\,. \tag{24}$$

From (7) and (12), we find

$$\left\{\beta\lambda_r + (1-\sigma_r)\,(C\mu + n\alpha)\right\}p_{C,n,r} - \sigma_r(C\mu + n\alpha)\sum_{s\neq r}^{k}\,p_{C,n,s}$$
$$= \sigma_r\left\{\sum_{j=n+1}^{n+b-1}\sum_{s=1}^{k}(C\mu + j\alpha)\,p_{C,j,s}\right\} + [\,C\mu + (n+b)\alpha]\,p_{C,n+b,r}\,, 1 \leq n \leq N-b\,,$$
$$1 \leq r \leq k\,.$$

Also, which can be written the family of linear system in the matrix form as

$$\mathbf{B}_n\,\mathbf{P}_n = \psi_n\;\mathrm{S}_2 + [C\mu + (n+b)\alpha]\,\mathrm{P}_{n+b}\,, \qquad 1 \leq n \leq N-b\,,$$

where

$$\psi_n = \sum_{j=n+1}^{n+b-1} \sum_{s=1}^{k} (C\mu + j\alpha)\; p_{C,j,s}\;.$$

Therefore

$$p_{C,n,r} = \frac{\sigma_r\;\psi_n}{[\beta\lambda_r + C\mu + n\alpha]\;D_n} + \frac{C\mu + (n+b)\alpha}{[\beta\lambda_r + C\mu + n\alpha]} \Big\{ p_{C,n+b,r}$$
$$+ \sum_{s=1}^{k} \frac{\sigma_r\,(C\mu + n\alpha)\; p_{C,n+b,s}}{[\beta\lambda_s + C\mu + n\alpha]\;D_n} \Big\},\; 1 \le n \le N-b, 1 \le b < N,\; 1 \le r \le k\,. \quad (25)$$

From (13) and (6), we obtain

$$(\beta\lambda_r + C\mu)\,p_{C,0,r} = \sigma_r \left\{ \sum_{j=0}^{a-1} \sum_{s=1}^{k} (C\mu + j\alpha)\,p_{C,j,s} \right\} + \sum_{j=a}^{b} (C\mu + j\alpha)\; p_{C,j,r}\,,$$

$1 \le r \le k$

or

$$\{\beta\lambda_r + C\mu(1-\sigma_r)\}\,p_{C,0,r} - \sigma_r\; C\mu \sum_{s\ne r}^{k} \; p_{C,0,s} = \eta\;\sigma_r \; + \sum_{j=a}^{b} (C\mu + j\alpha)\; p_{C,j,r}\,,$$

which can be written the family of linear system in the matrix form as

$$\mathbf{B}_0 \mathbf{P}_0 = \eta\; \mathsf{S}_2 + \mathsf{Q}$$

where

$$\mathbf{B}_0 = \left[\; b_{ij}(0) \;\right]$$

such that

$$b_{ij}(0) = -C\mu\;\sigma_i\,,\;\; i \ne j$$

$$b_{ii}(0) = \beta\;\lambda_i + C\mu\;(1-\sigma_i)\,,$$

$$\eta = \sum_{j=1}^{a-1} \sum_{s=1}^{k} (C\mu + j\alpha)\,p_{C,j,s}\,,$$

and

$$\mathsf{Q}^T = \left[\sum_{j=a}^{b} (C\mu + j\alpha)\; p_{C,j,1},\; \sum_{j=a}^{b} (C\mu + j\alpha)\; p_{C,j,2},\; ...,\; \sum_{j=a}^{b} (C\mu + j\alpha)\; p_{C,j,k} \right].$$

Therefore, we have

$$p_{C,0,r} = \frac{\eta\ \sigma_r}{(\beta\lambda_r + C\mu)\ D_0} + \frac{1}{(\beta\lambda_r + C\mu)}\left[\sum_{j=a}^{b}(C\mu + j\alpha)\ p_{C,j,r}\right.$$
$$\left. + \frac{C\mu\sigma_r}{D_0}\sum_{s=1}^{k}\ \sum_{j=a}^{b}\left\{\frac{(C\mu + j\alpha)}{(\beta\lambda_s + C\mu)} p_{C,j,s}\right\}\right],\ 1 \le r \le k\ . \qquad (26)$$

Also, from (14) and (4), we have

$$\{\lambda_r + (C-1)\mu\}\, p_{C-1,q,r} - \sigma_r\,(C-1)\ \mu\sum_{s=1}^{k}\ p_{C-1,q,s} = \sigma_r\, g_q + \ (\mathrm{q}\alpha + \mathrm{C}\mu)\, p_{C,q,r}\ ,$$
$$1 \le q \le a-1,\quad 1 \le r \le k.$$

Then

$$p_{C-1,q,r} = \frac{\sigma_r\ g_q}{\{\lambda_r + (C-1)\mu\}\ D'_{C-1}} + \frac{(q\alpha + C\mu)}{\{\lambda_r + (C-1)\mu\}}\left[p_{C,q,r}\right.$$
$$\left. + \sum_{s=1}^{k}\frac{(C-1)\mu\ \sigma_r\ \ p_{C,q,s}}{\{\lambda_s + (C-1)\mu\}\ D'_{C-1}}\right],\ 1 \le q \le a-1\ ,\ a \ge 2\ ,\ 1 \le r \le k\ , \qquad (27)$$

where

$$g_q = \sum_{j=0}^{q-1}\ \sum_{s=1}^{k}\{C\mu + j\alpha\}\, p_{C,j,s} + \sum_{j=q+1}^{a-1}\ \sum_{s=1}^{k}(C-1)\mu\ \ p_{C-1,j,s}$$

and

$$D'_{C-1} = 1 - \sum_{s=1}^{k}\frac{(C-1)\mu\sigma_s}{\lambda_s + (C-1)\mu}\ .$$

From (16) and (3), we get:

$$\{\lambda_r + i\,\mu(1-\sigma_r)\}\, p_{i,q,r} - i\,\mu\,\sigma_r\sum_{s\ne r}^{k} p_{i,q,s} = \sigma_r\ \theta(i,q) + (i+1)\,\mu\ p_{i+1,q,r}\ ,$$
$$1 \le i \le C-2\ ,\ 1 \le q \le a-1\ ,\ 1 \le r \le k\ ,$$

where

$$\theta(i,q) = (i+1)\mu\sum_{j=0}^{q-1}\ \sum_{s=1}^{k} p_{i+1,j,s} + i\,\mu\sum_{j=q+1}^{a-1}\ \sum_{s=1}^{k} p_{i,j,s}\ ,\quad 1 \le i \le C-2\ ,$$

which can be written the family of linear system in the matrix form as:

$$\mathbf{A}_i\ \ \mathrm{Q}_{i,q} = \theta(i,q)\mathrm{S}_2 + (i+1)\,\mu\,\mathrm{Q}_{i+1,q}\ ,\ 1 \le i \le C-2\ ,\ 1 \le q \le a-1\ ,$$

where

$$\mathbf{A}_i = \left[a_{rs}(i)\right]$$

such that

$$a_{rs}(i) = -i\,\mu\,\sigma_r\,,\quad r \neq s$$

$$a_{rr}(i) = \lambda_r + i\,\mu(1-\sigma_r)\,,$$

and

$$Q_{i,q}^T = \left[\, p_{i,q,1}\,, p_{i,q,2}\,, \cdots, p_{i,q,k} \,\right].$$

Therefore

$$p_{i,q,r} = \frac{\sigma_r\ \theta(i,q)}{(\lambda_r + i\,\mu)\ D_i'} + \frac{(i+1)\,\mu}{(\lambda_r + i\,\mu)} \left\{ p_{i+1,q,r} + i\mu \sum_{s=1}^{k} \frac{\sigma_r}{(\lambda_r + i\,\mu)\ D_i'}\, p_{i+1,q,s} \right\},$$

$$1 \le i \le C-2\,,\ 1 \le q \le a-1\,,\ \ a \ge 2,\ \ C \ge 3\,,\ 1 \le r \le k \tag{28}$$

where $D_i' = 1 - \sum_{s=1}^{k} \frac{i\,\mu\,\sigma_s}{\lambda_s + i\,\mu}$.

Similarly, from (17) and (5), we obtain:

$$\{\lambda_r + i\,\mu\,(1-\sigma_r)\}\, p_{i,0,r} - i\,\mu\ \sigma_r \sum_{s \neq r}^{k}\ p_{i,0,s} = i\,\mu\ \sigma_r \sum_{j=1}^{a-1} \sum_{s=1}^{k} p_{i,j,s}\ + (i+1)\,\mu\, p_{i+1,0,r}\,,$$

$$1 \le i \le C-1\,,\ 1 \le r \le k\,.$$

This can be written the family of linear system in the matrix form as

$$\mathbf{A}_i\ \ Q_{i,0} = \left(i\,\mu \sum_{j=1}^{a-1} \sum_{s=1}^{k}\ p_{i,j,s} \right) S_2 + (i+1)\,\mu\, Q_{i+1,0}\,, \qquad 1 \le i \le C-1\,.$$

So, we have

$$p_{i,0,r} = \frac{i\,\mu\ \sigma_r}{(\lambda_r + i\,\mu)\ D_i'} \left(\sum_{j=1}^{a-1}\ \sum_{s=1}^{k} p_{i,j,s} \right) + \frac{(i+1)\,\mu}{(\lambda_r + i\,\mu)} \Big[\, p_{i+1,0,r}$$

$$+ \frac{i\,\mu\,\sigma_r}{D_i'} \sum_{s=1}^{k} \frac{1}{(\lambda_s + i\,\mu)}\ p_{i+1,0,s} \Big],\ \ 1 \le i \le C-1,\ C \ge 2\,,\ \ 1 \le r \le k. \tag{29}$$

From (18) and (2), we get

$$p_{0,q,r} = \left(\frac{\sigma_r}{\lambda_r} \right) \left\{ \mu \sum_{j=0}^{q-1}\ \sum_{s=1}^{k} p_{1,j,s} \right\} + \left(\frac{\mu}{\lambda_r} \right)\ p_{1,q,r}\,,\ 1 \le q \le a-1,\ a \ge 2\,,\ 1 \le r \le k. \tag{30}$$

Also, from (1), we obtain:

$$p_{0,0,r} = \frac{\mu}{\lambda_r}\ \ p_{1,0,r}\,,\ \ 1 \le r \le k. \tag{31}$$

Equations (22) and (24)-(31) are the required recurrence relations, that give all the probabilities in terms of $p_{C,N,k}$, which itself may now be determined by using the normalizing condition:

$$\sum_{i=0}^{C-1}\sum_{q=0}^{a-1}\sum_{s=1}^{k} p_{i,q,s} + \sum_{n=0}^{N}\sum_{s=1}^{k} p_{C,n,s} = 1 ,$$

hence all the probabilities are completely known in terms of the queue parameters.

Then, the expected number of units in the queue can be found from the relation:

$$L_q = \sum_{i=0}^{C-1}\sum_{q=1}^{a-1}\sum_{s=1}^{k} q\, p_{i,q,s} + \sum_{n=1}^{N}\sum_{s=1}^{k} n\, p_{C,n,s} . \tag{32}$$

4. NUMERICAL EXAMPLE

In the above system: $H_k / M^{a,b} / C / N$ with balking and reneging, letting k=2, a=2, b=3, C= 3 and N=5 (i.e., the queue: $H_2 / M^{2,3} / 3 / 5$ with balking and reneging), the results are:

$p_{3,5,1} = g\; p_{3,5,2}$, $p_{3,4,1} = f_1\; p_{3,5,2}$, $p_{3,4,2} = f_2\; p_{3,5,2}$, $p_{3,3,1} = \ell_1\; p_{3,5,2}$

$p_{3,3,2} = \ell_2\; p_{3,5,2}$, $p_{3,2,1} = e_1\; p_{3,5,2}$, $p_{3,2,2} = e_2\; p_{3,5,2}$, $p_{3,1,1} = m_1\; p_{3,5,2}$

$p_{3,1,2} = m_2\; p_{3,5,2}$, $p_{3,0,1} = n_1\; p_{3,5,2}$, $p_{3,0,2} = n_2\; p_{3,5,2}$, $p_{2,1,1} = h_1\; p_{3,5,2}$

$p_{2,1,2} = h_2\; p_{3,5,2}$, $p_{2,0,1} = v_1\; p_{3,5,2}$, $p_{2,0,2} = v_2\; p_{3,5,2}$, $p_{1,1,1} = q_1\; p_{3,5,2}$

$p_{1,1,2} = q_2\; p_{3,5,2}$, $p_{1,0,1} = u_1\; p_{3,5,2}$, $p_{1,0,2} = u_2\; p_{3,5,2}$, $p_{0,1,1} = w_1\; p_{3,5,2}$

$p_{0,1,2} = w_2\; p_{3,5,2}$, $p_{0,0,1} = x_1\; p_{3,5,2}$, $p_{0,0,2} = x_2\; p_{3,5,2}$.

Therefore, we obtain:

$$p_{3,5,2} = S^{-1} ,$$

$$S = 1 + g + f_1 + f_2 + \ell_1 + \ell_2 + e_1 + e_2 + m_1 + m_2 + n_1 + n_2$$
$$+ h_1 + h_2 + q_1 + q_2 + u_1 + u_2 + v_1 + v_2 + w_1 + w_2 + x_1 + x_2 ,$$

$$L_q = \{5(1+g) + 4(f_1 + f_2) + 3(\ell_1 + \ell_2) + 2(e_1 + e_2) + m_1 + m_2 +$$
$$+ h_1 + h_2 + q_1 + q_2 + w_1 + w_2\} / S$$

where

$$g = \frac{(\beta\lambda_2 + 3\mu + 5\alpha)\sigma_1}{(\beta\lambda_1 + 3\mu + 5\alpha)\sigma_2},\ f_1 = \frac{(g+1)(3\mu + 5\alpha)\sigma_1}{(\beta\lambda_1 + 3\mu + 4\alpha)D_4},\ f_1 = \frac{(g+1)(3\mu + 5\alpha)\sigma_2}{(\beta\lambda_2 + 3\mu + 4\alpha)D_4}$$

$$\ell_1 = \frac{[(3\mu + 4\alpha)(f_1 + f_2) + (3\mu + 5\alpha)(1+g)]\sigma_1}{(\beta\lambda_1 + 3\mu + 3\alpha)D_3},$$

$$\ell_2 = \frac{[(3\mu+4\alpha)(f_1+f_2)+(3\mu+5\alpha)(1+g)]\sigma_2}{(\beta\lambda_2+3\mu+3\alpha)D_3},$$

$$e_1 = \frac{\sigma_1}{(\beta\lambda_1+3\mu+2\alpha)D_2}\left[(3\mu+3\alpha)(\ell_1+\ell_2)+(3\mu+4\alpha)(f_1+f_2)\right] + \frac{3\mu+5\alpha}{(\beta\lambda_1+3\mu+2\alpha)}\left[g+\frac{\sigma_1(3\mu+2\alpha)}{D_2}\left(\frac{g}{\beta\lambda_1+3\mu+2\alpha}+\frac{1}{\beta\lambda_2+3\mu+2\alpha}\right)\right]$$

$$e_2 = \frac{\sigma_2}{(\beta\lambda_2+3\mu+2\alpha)D_2}\left[(3\mu+3\alpha)(\ell_1+\ell_2)+(3\mu+4\alpha)(f_1+f_2)\right] + \frac{3\mu+5\alpha}{(\beta\lambda_2+3\mu+2\alpha)}\left[g+\frac{\sigma_2(3\mu+2\alpha)}{D_2}\left(\frac{g}{\beta\lambda_1+3\mu+2\alpha}+\frac{1}{\beta\lambda_2+3\mu+2\alpha}\right)\right]$$

$$m_i = \frac{\sigma_i}{(\beta\lambda_i+3\mu+\alpha)D_i}\left[(3\mu+2\alpha)(e_1+e_2)+(3\mu+3\alpha)(\ell_1+\ell_2)\right] + \frac{3\mu+4\alpha}{\beta\lambda_i+3\mu+\alpha}\left[f_i+\frac{\sigma_i(3\mu+\alpha)}{D_1}\left(\frac{f_1}{\beta\lambda_1+3\mu+\alpha}+\frac{f_2}{\beta\lambda_2+3\mu+\alpha}\right)\right], i=1,2$$

$$n_i = \frac{\sigma_i(3\mu+\alpha)(\ell_1+\ell_2)}{(\beta\lambda_i+3\mu)D_0} + \frac{[(3\mu+2\alpha)e_i+(3\mu+3\alpha)\ell_i]}{\beta\lambda_i+3\mu} + \frac{3\mu\sigma_i(3\mu+2\alpha)}{D_0}\cdot \cdot\left(\frac{e_1}{\beta\lambda_1+3\mu}+\frac{e_2}{\beta\lambda_2+3\mu}\right) + \frac{3\mu\sigma_i(3\mu+3\alpha)}{D_0}\left[\frac{\ell_1}{\beta\lambda_1+3\mu}+\frac{\ell_2}{\beta\lambda_2+3\mu}\right], \quad i=1,2$$

$$h_i = \frac{3\mu\sigma_i(n_1+n_2)}{(\lambda_i+2\mu)D_2'} + \frac{3\mu+\alpha}{\lambda_i+2\mu}[m_i+\frac{2\mu\sigma_i}{D_2'}(\frac{m_1}{\lambda_1+2\mu}+\frac{m_2}{\lambda_2+2\mu})], \quad i=1,2$$

$$v_i = \frac{2\mu\sigma_i(h_1+h_2)}{(\lambda_i+2\mu)D_2'} + \frac{3\mu}{\lambda_i+2\mu}[n_i+\frac{2\mu\sigma_i}{D_2'}(\frac{n_1}{\lambda_1+2\mu}+\frac{n_2}{\lambda_2+2\mu})], \quad i=1,2$$

$$q_i = \frac{2\mu\sigma_i(v_1+v_2)}{(\lambda_i+\mu)D_1'} + \frac{2\mu}{\lambda_i+\mu}[h_i+\frac{\mu\sigma_i}{D_1'}(\frac{h_1}{\lambda_1+\mu}+\frac{h_2}{\lambda_2+\mu})], \quad x_i = \frac{\mu}{\lambda_i}u_i, \quad i=1,2$$

$$u_i = \frac{\mu\sigma_i(q_1+q_2)}{(\lambda_i+\mu)D_1'} + \frac{2\mu}{\lambda_i+\mu}[v_i+\frac{\mu\sigma_i}{D_1'}(\frac{v_1}{\lambda_1+\mu}+\frac{v_2}{\lambda_2+\mu})],$$

$$w_i = \frac{\mu\sigma_i(u_1+u_2)}{\lambda_i} + \frac{\mu}{\lambda_i}q_i, \; i=1,2.$$

Now, we introduce the three tables for the expected number in the queue at $\lambda_1 = 2$, $\lambda_2 = 3$, $\sigma_1 = 0.2$ and $\sigma_2 = 0.8$ for the different values of α, β and μ when two of them are fixed.

Table I: Showing the effect of α when $\beta = 0.2$, $\mu = 3$

α	L_q
0.1	0.498574
0.2	0.498572
0.3	0.498570
0.4	0.498569
0.5	0.498568
0.6	0.498566
0.7	0.498565
0.8	0.498564
0.9	0.498562
1	0.498561

Table II: Showing the effect of β when $\alpha = 0.2$, $\mu = 3$

β	L_q
0.1	0.498293
0.2	0.498572
0.3	0.498836
0.4	0.499087
0.5	0.499328
0.6	0.499561
0.7	0.498787
0.8	0.500008
0.9	0.500225
1	0.500438

Table III: Showing the effect of μ when $\alpha = 0.4$, $\beta = 0.2$

μ	L_q
1	0.473718
2	0.494871
3	0.498569
4	0.499482
5	0.499777
10	0.499987
30	0.5
50	0.5

From the above tables, it is clear that L_q is increasing when μ and β are increasing while it is decreasing when α is increasing.

5. SPECIAL CASES

Some queueing systems can be obtained as special cases of this system:

I. If $C=1$, $\beta=1$ and $\alpha=0$ in equations (22), (24)-(26), (30) and (31), respectively, we get the queue $H_k/M^{a,b}/1/N$ without balking and reneging, i.e.

$$p_{N,s}=\frac{(\rho_k+1)\,\sigma_s}{(\rho_s+1)\,\sigma_k}\,p_{N,k}\ ,\ 1\le s\le k,$$

$$p_{n,s}=\frac{\sigma_s}{(\rho_s+1)\,D}\sum_{j=n+1}^{N}\ \sum_{r=1}^{k}p_{j,r}\ ,\ N-b+1\le n\le N-1\,,$$

$$p_{n,s}=\frac{\sigma_s}{(\rho_s+1)\,D}\sum_{j=n+1}^{n+b-1}\ \sum_{r=1}^{k}p_{j,r}+\frac{p_{n+b,s}}{(\rho_s+1)}+\sum_{r=1}^{k}\frac{\sigma_s\ p_{n+b,r}}{(\rho_r+1)\,(\rho_s+1)\,D}\,,$$

$1\le n\le N-b$,

$$p_{0,s}=\frac{\sigma_s}{(\rho_s+1)\,D}\sum_{j=1}^{a-1}\ \sum_{r=1}^{k}p_{j,r}+\sum_{j=a}^{b}\frac{p_{j,s}}{\rho_s+1}+\sum_{r=1}^{k}\left\{\frac{\sigma_s\sum_{j=a}^{b}p_{j,r}}{(\rho_r+1)\,(\rho_s+1)\,D}\right\}$$

$$\rho_s\ \mathrm{Q}_{q,s}=\sigma_s\sum_{j=0}^{q-1}\ \sum_{r=1}^{k}p_{j,r}+p_{q,s}\ ,\ \rho_s\ \mathrm{Q}_{\,0,s}=p_{0,s}\ ,\ 1\le q\le a-1\,,$$

where

$$D=1-\sum_{s=1}^{k}\frac{\sigma_s}{\rho_s+1}\,,\qquad \rho_s=\frac{\lambda_s}{\mu}$$

$p_{1,n,s}=p_{n,s}$ and $p_{0,q,s}=\mathrm{Q}_{\,q,s}$

which coincides with the results of Shawky and Nassar [10].

II. If $a=b=1$ in equations (22), (25), (26), (29) and (31), we get the queue: $H_k/M/C/N$ with balking and reneging, that was studied by Shawky and El-Paoumy [9].

III. If $\sigma_r = \delta_{rs}$, where δ_{rs} is the Kronecker delta function, $\lambda_r = \lambda$, β =1, α = 0 and $N \to \infty$, we have the queue: $M / M^{a,b} / C$ without balking and reneging, which was discussed by Medhi [5], and Sim and Templeton [11].

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