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NONLINEAR RECEDING HORIZON CONTROL OF PRODUCTION INVENTORY SYSTEMS WITH DETERIORATING ITEMS

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Abstract: This paper is concerned with the receding horizon control of the production rate of a deteriorating production system with a nonlinear inventory-level-dependent demand. Both continuous and periodic review policies are discussed and numerical illustrations are provided.

Keywords: Production systems, continuous-review, periodic-review, deterioration, receding horizon control.

1. INTRODUCTION

Receding Horizon Control (RHD) or Model Predictive Control (MPC), is by now a well established control method (see for instance [6, 7]). It is a scheme that, at each instant of time, implements the first component of an optimal control vector minimizing some performance criterion. It has emerged as a successful control strategy, especially Linear Model Predictive Control (LMPC), i.e., predictive control for linear systems, which is widely used both in academic and industrial fields.

Motivated by the success of LMPC, Nonlinear Model Predictive Control (NMPC) has gained significant interest over the past decade. Various NMPC strategies that lead to stability of the closed-loop have been developed in recent years and key questions such as the efficient solution of the occurring open-loop control problem have been extensively studied.

Recently, Hedjar et al. [3] investigated the predictive control of a linear periodic-review production inventory system with deteriorating items. In this paper, we generalize their model to the case of a nonlinear system. We consider here not only the periodic-review but continuous-review policy as well. The nonlinearity of the system is obtained by assuming that the demand rate depends on the stock on hand. Researchers as Levin et al. [5] and Silver and Peterson [9] have observed that the presence of more quantities of the same product tends to attract more customers. In other words, the consumption rate may be influenced by the stock levels. This phenomenon is termed as a stock-dependent consumption rate. The optimization of production systems with stock-dependent demand rate is widely available [2, 10], but literature on the optimal control of such systems is rather sparse [1].

The next section is concerned with the formulation of the models (continuous and periodic) and the derivation of the optimal solutions. For the continuous model we use an approximation approach to transform the given problem into an unconstrained quadratic minimization problem in terms of the control variable only. Similarly, the periodic model is also reformulated into an unconstrained nonlinear (not necessarily quadratic) minimization problem. Some illustrative numerical examples are presented in each case. The last section concludes the paper.

2. MODELS FORMULATION AND ANALYSIS

Given a planning horizon of length $H > 0$, we consider a manufacturing firm producing a single item at rate P. The inventory level is denoted by I and the demand rate, function of the stock on hand, is denoted by $D(I)$. We assume that the stocked items deteriorate at rate $\mu > 0$.

2.1. Continuous-Review Model

In the continuous-review model, the dynamics of the inventory level are governed by the following state equation

$$\frac{d}{dt}I(t) = P(t) - D(I(t)) - \theta I(t), \quad \forall t \in [0, H]\,. \tag{2.1}$$

As in [8], we assume that the firm has set an inventory goal level $\hat{I}$ and a production goal rate $\hat{P}$ and penalties are incurred for deviating from these goals. Given an instant of time t in $[0;H]$ and given a prediction horizon of length $T > 0$ ($T << H$), then the objective is to determine the production rate that minimizes the performance index

$$J := J(I,P) = \frac{1}{2} p\Delta I^2(t+T) + \frac{1}{2}\int_t^{t+T} [q\Delta I^2(\tau) + r\Delta P^2(\tau)]d\tau \ , \qquad (2.2)$$

subject to (2.1). Here p, q and r are positive costs and the shift operator Δ is such that

$$\Delta I(t) = I(t) - \hat{I} \text{ and } \Delta P(t) = P(t) - \hat{P}, \quad \forall t \in [0,T].$$

We adopt the widely used quadratic objective function of Holt, Modigliani, Muth, and Simon (HMMS) [4]. The interpretation of this objective function is that penalties are incurred when the inventory level and production rate deviate from their respective goals.

Note that in order to have $\Delta P \to 0$, one has to choose the desired production $\hat{P} = D(\hat{I}) + \theta\hat{I}$. Let $F(t) = q\Delta I^2(t) + r\Delta P^2(t)$. The interval [$t$; t + T] is divided into $2m$ subintervals of equal width $h = T/2m$. Using the composite Simpson's rule for $2m$ intervals, the objective function can be approximated as

$$J \approx \frac{1}{2} p\Delta I^2(t+T) + \frac{h}{3}\left[F(t) + 2\sum_{i=1}^{m-1} F(t+2ih) + 4\sum_{i=1}^{m} F(t+(2i-1)h) + F(t+2mh) \right]. \quad (2.3)$$

By the first-order Taylor approximation, we have for $i = 1,...,2m$.

$$\begin{aligned}\Delta I(t+ih) &= \Delta I(t) + ih\frac{d}{dt}I(t) + ihO(ih) \\ &= (1 - ih\theta)\Delta I(t) + ih[D(\hat{I}) - D(I(t))] + ih\Delta P(t) + ihO(ih),\end{aligned}$$

where $O(h) \to 0$ 0 as $h \to 0$. Hence, the objective function can be further approximated as

$$J \approx \frac{h}{3}\left[M(I(t),h) + U(t)^T \mathbf{R} U(t) + G(t)^T U(t) \right], \qquad (2.4)$$

where

$$U(t) = [\Delta P(t), \Delta P(t+h), ..., \Delta P(t+(2m-1)h)]^T$$

$$\mathbf{R} = \text{diag}(\gamma, 4r, 2r, 4r, 2r, ..., 4r, 2r, 4r) \quad \text{with} \quad \gamma = r + \frac{\mu}{2}$$

$$G(t) = [\lambda\Delta I(t) + \mu(D(\hat{I}) - D(I(t))), 0, ..., 0]^T,$$

with

$$\mu = 8hm^2(3p + 2hqm), \quad \lambda = 12m(p + qhm) - \mu\theta,$$

and where $M(I(t),h)$ is independent of $U(t)$. It is easy to see that the minimum of J is reached at

$$U(t) = -\frac{1}{2}\mathbf{R}^{-1}G(t), \text{ whenever } r > 0.$$

In receding horizon, we obtain $\Delta P(t)$ as

$$\Delta P(t) = [1,0,...,0]U(t) = -\frac{\lambda \Delta I(t) + \mu(D(\hat{I}) - D(I(t)))}{2\gamma}.$$

If $r = 0$, then relation (2.4) becomes

$$J \approx \frac{h}{3}\left\{ M(I(t),h) + \frac{\mu}{2}\Delta P^2(t) + [\lambda \Delta I(t) + \mu(D(\hat{I}) - D(I(t)))]\,\Delta P(t) \right\}, \tag{2.5}$$

and so the minimum of J is reached at

$$\Delta P(t) = -\left[\frac{12m(p+hqm)}{\mu} - \theta\right]\Delta I(t) + D(I(t)) - D(\hat{I}).$$

Thus, state equation (2.1) yields

$$\frac{d}{dt}\Delta I(t) = -\frac{12m(p+hqm)}{\mu}\Delta I(t) \quad \text{and so} \quad \Delta I(t) = \Delta I(0)e^{-\frac{12m(p+hqm)}{\mu}t}.$$

For illustration purposes, we use in the following example the demand rate that is widely used in the literature (see for instance [2, 10]), $D(I(t)) = \alpha I(t)^{\beta}$, $\forall t \in [0, H,]$, with $\alpha > 0$ and $0 < \beta < 1$.

Example 2.1. Let $\beta = 0.1$, $\alpha = 3$, $\theta = 0.01$, $r = 10$, $p = 5$, $q = 1$, $m = 5$, $T = 1$, $I_0 = 5$, and $\hat{I} = 10$. Then, h = 0.01 and $\hat{P} = \alpha\hat{I}^{\beta} + \theta\hat{I} = 19.0287$. Figure 1 shows the variations of the optimal inventory level I^*, the optimal production rate P^*, and the corresponding optimal cost J^*. As expected I^* converges to $\hat{I}$, P^* converges to $\hat{P}$, and J^* converges to 0.

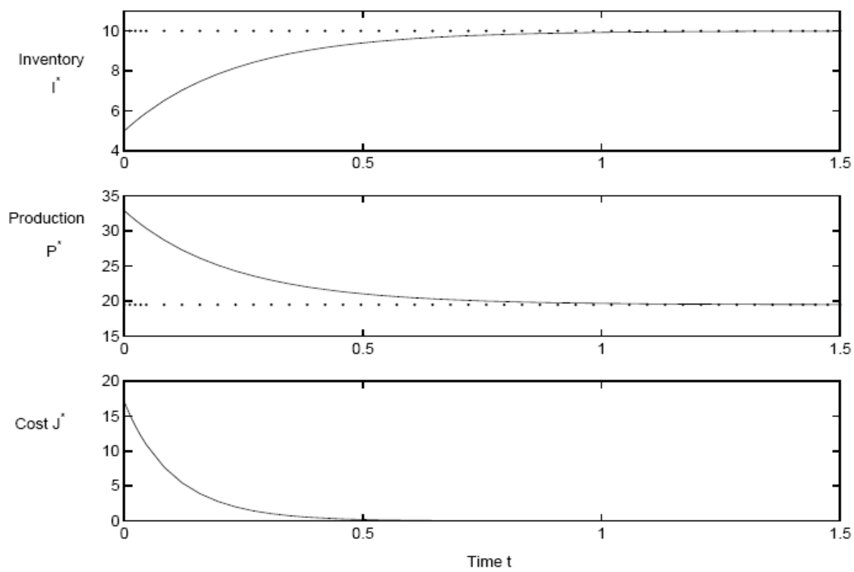

Figure 1: Variations of I^*, P^*, and J^* as functions of time t.

To assess the effect of the cost parameters (p; r; q) on the solutions, a sensitivity analysis has been conducted. We were mainly interested in the optimal initial production rate $P_0 := P^*(0)$ and the rise time (RT), which is the time required for the stock to achieve the target. We noticed that as q increases the rise time decreases and P_0 increases. Also, as r increases the rise time increases and P_0 decreases. We observed that the cost p has not effect RT and P_0. We could have performed many other sensitivity analysis in a similar way, but we refrained from doing so for brevity.

2.2. PERIODIC-REVIEW MODEL

In the periodic-review model the inventory is reviewed every h units of time. Without loss of generality we may assume that $N_H := H/h$ is integer. In this case, the dynamics of the inventory level are given by the following difference equation:

$$\frac{I(k+1)-I(k)}{h} = P(k) - D(I(k)) - \theta I(k), \quad \forall k \in \{0,...,N_H\}\ . \tag{2.6}$$

Introduce the shifted variables $\Delta I(k)$, $\Delta u(k)$, and $\Delta D(I(k))$ as

$$\Delta I(k) = I(k) - \hat{I}(k),\ \Delta P(k) = P(k) - \hat{P}(k) \quad \text{and} \quad \Delta D(z) = D(z+\hat{I}) - D(\hat{I})\ .$$

Note that the inventory goal level $\hat{I}(k)$ must satisfy the equation

$$\hat{I}(k+1) = \hat{I}(k) + h[\hat{P}(k) - D(\hat{I}(k)) - \theta\hat{I}(k)], \quad \forall k \in \{0,...,N_H - 1\}, \tag{2.7}$$

and therefore equations (2.6) and (2.7) lead to

$$\Delta I(k+1) = (1-\theta h)\Delta I(k) - h\Delta D(k) + h\Delta P(k). \tag{2.8}$$

The above discrete equation represents the model of the periodic-review production inventory systems with items deterioration.

Now we are in a position to formulate the finite horizon optimal control problem. Given an instant k, a prediction horizon of length N, the current state of the inventory $I(k)$, and the target inventory $\hat{I}(k)$, we want to find the vector P of N production rates:

$$\mathbf{P} = (\Delta P(k), \Delta P(k+1), ..., \Delta P(k+N-1))^T, \tag{2.9}$$

that minimizes the performance index:

$$\mathbf{V}(\Delta I(k), \mathbf{P}) = p\Delta I(k+N)^2 + \sum_{i=k}^{k+N-1} [q\Delta I(i)^2 + r\Delta P(i)^2], \tag{2.10}$$

subject to the equation (2.8) and the equality constraints:

$$\Delta P(k+i) = 0 \quad \text{if} \quad i \geq N \tag{2.11}$$

where $p\Delta I(k+N)^2$ is the terminal cost function. To minimize the predicted cost function (2.10), define

$$\alpha_{\Delta P(k+i)}(z) := \begin{cases} (1-\theta h)(z+h\Delta P(k+i)) - h\Delta D(z+h\Delta P(k+i)), & 1 \le i \le N, \\ (1-\theta h)z - h\Delta D(z), & i = 0 \end{cases}$$

then state equation (2.8) leads to the *i-step ahead predictor*

$$\Delta I(k+1) = \alpha_{\Delta P(k+i-1)} \circ \cdots \circ \alpha_{\Delta P(k)}(\Delta I(k)) + h\Delta P(k+i), \quad 1 \le i \le N, \tag{2.12}$$

which can be written in the following form

$$\mathbf{X} = \mathbf{G}(\mathbf{P}, \Delta I(k)), \tag{2.13}$$

where **X** is an *N* component column vector and **G** is a map from $\mathbb{R}^N \times \mathbb{R}$ to $\mathbb{R}^N$, given by

$$\mathbf{X} = \begin{pmatrix} \Delta I(k+1) \\ \Delta I(k+2) \\ \vdots \\ \Delta I(k+N_h) \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} G_1(\mathbf{P}, \Delta I(k)) \\ G_2(\mathbf{P}, \Delta I(k)) \\ \vdots \\ G_N(\mathbf{P}, \Delta I(k)) \end{pmatrix},$$

with

$$G_i(\mathbf{P}, \Delta I(k)) = \alpha_{\Delta P(k+i-1)} \circ \cdots \circ \alpha_{\Delta P(k)}(\Delta I(k)) + h\Delta P(k+i), \quad 1 \le i \le N.$$

Thus, the performance index (2.10) can be expressed as:

$$\mathbf{V}(\Delta I(k), \mathbf{P}) = q\Delta I(k)^2 + \mathbf{X}^T \mathbf{Q} \mathbf{X} + \mathbf{P}^T \mathbf{R} \mathbf{P}, \tag{2.14}$$

where $\mathbf{Q} = diag(q, q, \ldots, q, p)$, $\mathbf{R} = diag(r, r, \ldots, r)$, and $\dim(\mathbf{Q}) = \dim(\mathbf{R}) = N \times N$.

The performance index (2.14) with the predicted model (2.13) becomes:

$$\mathbf{V}(\Delta I(k), \mathbf{P}) = q\Delta I(k)^2 + [\mathbf{G}(\mathbf{P}, \Delta I(k))]^T \mathbf{Q} [\mathbf{G}(\mathbf{P}, \Delta I(k))] + \mathbf{P}^T \mathbf{R} \mathbf{P}. \tag{2.15}$$

The necessary condition for $\mathbf{P}^*$ to be an optimal predicted control that minimizes the cost function (2.15) is $\nabla_P \mathbf{V}(\Delta I(k), \mathbf{P}^*) = 0$, which is equivalent to

$$q\sum_{i=1}^{N-1} G_i(\mathbf{P}^*, \Delta I(k)) \frac{\partial G_i(\mathbf{P}^*, \Delta I(k))}{\partial \Delta P(k+j)} + pG_n(\mathbf{P}^*, \Delta I(k)) \frac{\partial G_N(\mathbf{P}^*, \Delta I(k))}{\partial \Delta P(k+j)} + r\Delta P(k+j) = 0, \qquad j = 1, \ldots, N. \tag{2.15}$$

It is not hard to show that the necessary optimality condition (2.16) is also sufficient if the functions G_i are convex.

Due to the nonlinearity of the functions G_i, the resolution of Equations (2.16) gets harder and harder for large values of *N*.

Example 2.2: Assume N = 3 and put for simplicity $x_i := \Delta P(k+i-1),\ i=1,2,3.$ In this case, $G_i := G_i(\mathbf{P},\Delta I(k))$ are given by

$$G_1 = (1-\theta h)\Delta I(k) - h\Delta D(\Delta I(k)) + hx_1$$
$$G_2 = (1-\theta h)G_1 - h\Delta D(G_1) + hx_2$$
$$G_3 = (1-\theta h)G_2 - h\Delta D(G_2) + hx_3.$$

Obviously, G_1 is convex in **P**. We check the convexity of G_2 and G_3 in **P**. The Hessian matrices of G_2 and G_3 are given by

$$\nabla^2 G_2 = \begin{pmatrix} -h^3\Delta D''(G_1) & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\nabla^2 G_3 = h^3 \begin{pmatrix} \Delta D''(G_1)[h\Delta D'(G_2)-\bar{h}] - h^{-1}[\bar{h} - h\Delta D'(G_1)]^2 \Delta D''(G_2) & (h\Delta D'(G_1)-\bar{h})\Delta D''(G_2) & 0 \\ (h\Delta D'(G_1)-\bar{h})\Delta D''(G_2) & -\Delta D''(G_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\bar{h} = 1-\theta h.$ Assuming D concave yields $\Delta D''(G_1) \le 0$ and then $\nabla^2 G_2$ is positive semi-definite which ensures the convexity of G_2 in **P**. In order to obtain the convexity of G_3 it suffices to make the following additional assumption on D and h:

$$h < \frac{1}{\theta + D'(I_0)}. \tag{2.17}$$

Simple computations show that the Hessian matrix $\nabla^2 G_3$ is semi-definite positive and therefore the function G_3 is convex in **P** provided (2.17) is satisfied. Thus (2.16) yields the following necessary and sufficient optimality condition:

$$qhG_1 + qG_2 h(\bar{h} - h\Delta D'(G_1)) + pG_3 h(\bar{h}) - h\Delta D'(G_1))\,(\bar{h} - h\Delta D'(G_2)) + rx_1 = 0$$
$$qhG_2 + pG_3 h(\bar{h} - h\Delta D'(G_2)) + rx_2 = 0$$
$$phG_3 + rx_3 = 0$$

which can be solved numerically. To illustrate the results obtained, we perform a simulation with the same nonlinear form of the demand rate $D(I) = \alpha I^{\beta}$. We also set $\beta = 0.81, \alpha = 3, \theta = 0.01, r = 10, p = 5, q = 1, h = 0.4, I_0 = 5,$ and $\hat{I} = 10$. Then, $\hat{P} = \alpha \hat{I}^{\beta} + \theta \hat{I} =$ $= 19.0287.$ Figure 2 shows the variations of the optimal inventory level I^*, the optimal production rate P^*, and the corresponding optimal cost J^*. As expected I^* converges to $\hat{I}$, P^* converges to $\hat{P}$, and J^* converges to 0. As in the continuous-review case, a sensitivity analysis has been done to evaluate the effect of the cost parameters (p; r; q) on the optimal initial production rate P_0 and the rise time (RT). We noticed the same effect as in the continuous-review case.

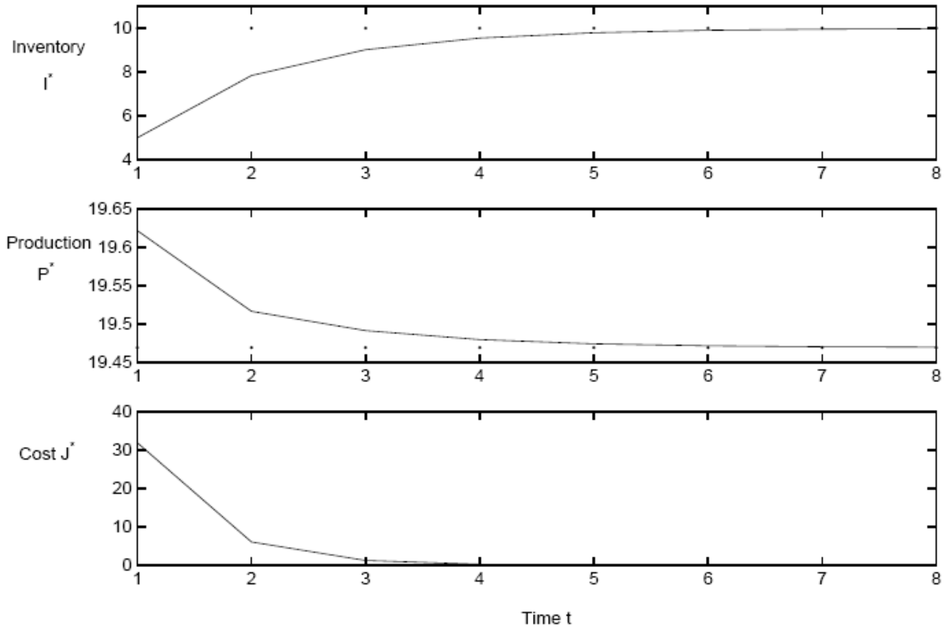

Figure 2: Variations of I^*, P^*, and J^* as functions of time *t*.

3. CONCLUSION

We presented in this paper the predictive control of both continuous-review and periodic-review production inventory systems with deteriorating items. The model in the continuous case consists of a predictive cost function and a differential equation while in the discrete case, it consists of a predictive cost function and a recurrent difference equation. These two models are reformulated as unconstrained minimization problems in terms of the control variable. The optimal production rates are the solutions of the obtained unconstrained minimization problems. Numerical simulations show the effectiveness of the proposed approach. The multi-variable and stochastic cases are being investigated by the authors.

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