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ON DUALITY FOR NONSMOOTH LIPSCHITZ OPTIMIZATION PROBLEMS

Vasile PREDA
University of Bucharest, Bucharest
preda@fmi.unibuc.ro

Miruna BELDIMAN
Institute of Mathematical Statistics and Applied Mathematics, Romanian Academy, Bucharest

Anton BĂTĂTORESCU
University of Bucharest, Bucharest

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Abstract: We present some duality theorems for a non-smooth Lipschitz vector optimization problem. Under generalized invexity assumptions on the functions the duality theorems do not require constraint qualifications.

Keywords: Nonsmooth Lipschitz vector optimization, Fritz John type necessary optimization conditions, duality theorems.

1. INTRODUCTION

We shall introduce some definitions used in this article and formulate a vector optimization problem together with its Mond-Weir dual.

The real *n*-dimensional vector space will be denoted by $\mathbb{R}^n$ and we will use the following conventions for any two vectors $x, y \in \mathbb{R}^n$:

$x < y \quad \Leftrightarrow \quad x_i < y_i,\ \forall i \in \{1,...,n\}$,

$x \leq y \quad \Leftrightarrow \quad x_i \leqq y_i,\ \forall i \in \{1,...,n\}$, and $x \neq y$,

$x \nless y$ is the negation of $x < y$.

Throughout this paper we will denote to a real Banach space by X, the topological dual of X by X^*, and the value of a function ξ in X^* at d by $\langle \xi, d \rangle$.

We will consider this function for the definitions that follow:

$\varphi : X \to \mathbb{R}$

Definition 1.1 *(Clarke [1]) The function φ is locally Lipschitz if for any $x \in X$ there exist a neighborhood $N(x)$ of x and a constant $K_x > 0$ such that for any $y, z \in N(x)$ we have*

$$|\varphi(y) - \varphi(z)| \leqq K_x \|y - z\|.$$

Definition 1.2 *(Clarke [1]) The generalized directional derivative of a local Lipschitz function φ at x in the direction d is denoted by*

$$\varphi^o(x;d) = \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{\varphi(y + td) - \varphi(y)}{t}.$$

Definition 1.3 *The Clarke generalized subgradient of a locally Lipschitz function φ at x is denoted by*

$$\partial^c \varphi(x) = \left\{ \xi \in X^* \mid \varphi^o(x;d) \geqq \langle \xi, d \rangle,\ \forall d \in X \right\}.$$

Definition 1.4 *(see also Giorgi and Guerraggio [2]) Let us consider:*

$\eta : X \times X \to X, \quad \rho \in \mathbb{R}, \quad d : X \times X \to \mathbb{R}_+.$

We say that:

φ is (η, ρ) -pseudoinvex if for $\forall x, y \in X$,

$$\varphi^o(x; \eta(y,x)) \geqq \rho d(y,x) \quad \Rightarrow \quad \varphi(y) \geqq \varphi(x),$$

or, equivalently, for $\forall x, y \in X, \forall \xi \in \partial^c \varphi(x)$,

$$\varphi(y) < \varphi(x) \quad \Rightarrow \quad \langle \xi, \eta(y,x) \rangle < \rho d(y,x).$$

φ is (η, ρ) -quasiinvex if for $\forall x, y \in X$,

$$\varphi(y) \leqq \varphi(x) \quad \Rightarrow \quad \varphi^o(x; \eta(y,x)) \leqq \rho d(y,x)$$

or, equivalently, for $\forall x, y \in X, \forall \xi \in \partial^c \varphi(x)$,

$$\varphi(y) \leqq \varphi(x) \quad \Rightarrow \quad \langle \xi, \eta(y,x) \rangle \leqq \rho d(y,x)$$

φ is strictly (η, ρ) -pseudoinvex if for $\forall x, y \in X$, with $x \neq y$,

$$\varphi^o(x; \eta(y,x)) \geqq \rho d(y,x) \quad \Rightarrow \quad \varphi(y) > \varphi(x),$$

or, equivalently, for $\forall x, y \in X$, with $x \neq y$, and $\forall \xi \in \partial^c \varphi(x)$,

$$\varphi(y) \leqq \varphi(x) \quad \Rightarrow \quad \langle \xi, \eta(y,x) \rangle < \rho d(y,x).$$

For the rest of our presentation we will consider the following locally Lipschitz functions:

$$f_i : X \to \mathbb{R}, \quad i \in \{1,...,p\},$$
$$g_j : X \to \mathbb{R}, \quad j \in \{1,...,m\}.$$

We can define the vector optimization problem (VP):

$$\min f(x) := (f_1(x), f_2(x),..., f_p(x)), \tag{VP}$$

subject to: $g_j(x) \leqq 0, \quad j \in \{1,...,m\}$,

and its Mond-Weir vector dual problem (VD):

$$\max f(v), \qquad \text{VD)}$$

$$\text{subject to: } 0 \in \sum_{i=1}^{p} \mu_i \partial^c f_i(v) + \sum_{j=1}^{m} \lambda_j \partial^c g_j(v), \tag{0.1}$$

$$\lambda_j g_j(v) \geqq 0, \quad j \in \{1,...,m\}, \tag{0.2}$$

$$(\mu_1,...,\mu_p, \lambda_1,...,\lambda_m) \geq 0. \tag{0.3}$$

Definition 1.5 *A (VP)-feasible point* $\overline{x} \in X$ *is said to be a weakly efficient solution for (VP) if there doesn't exist any other (VP)-feasible point* $y \in X$ *such that* $f(y) < f(\overline{x})$.

In a similar manner, a weakly efficient solution for (VD) is defined.

2. DUALITY THEOREMS

In this section we will establish the weak and the strong duality relations between the problems (VP) and (VD). Usually, see references [3-5, 7], the dual problem is formulated by using the Kuhn-Tucker type necessary optimality conditions:

$$0 \in \sum_{i=1}^{p} \mu_i \partial^c f_i(v) + \sum_{j=1}^{m} \lambda_j \partial^c g_j(v),$$
$$\lambda_j g_j(v) = 0, \quad j \in \{1,...,m\},$$
$$(\mu_1,...,\mu_p) \geq 0, \quad (\lambda_1,...,\lambda_m) \geq 0.$$

Since the equality conditions $\lambda_j g_j(v) = 0$, and $(\mu_1,...,\mu_p) \geq 0$ are not present in the statement of the problem (VD), we do not require any constraint qualification for our duality results by using Fritz-John type necessary optimality conditions and (strict) pseudoinvexity assumptions on the functions.

Theorem 2.1 *(Weak Duality) Suppose that the functions* f_i *are* (η, ρ_i) *-pseudoinvex,* $i \in \{1,...,p\}$, *and* g_j *are strictly* (η, ρ'_j) *-pseudoinvex,* $j \in \{1,...,m\}$. *Then, for any feasible solution x of (VP) and any feasible solution* (v, μ, λ) *of (VD), such that* $\sum_{i=1}^{p} \mu_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho'_j \leqq 0$, *we have* $f(x) \not< f(v)$, *where* $\mu = (\mu_1,...,\mu_p) \in \mathbb{R}^p$ *and* $\lambda = (\lambda_1,...,\lambda_m) \in \mathbb{R}^m$.

Proof: Let us suppose that, on the contrary, there exists a (VP)-feasible solution x and a (VD)-feasible solution (v, μ, λ) such that

$$f_i(x) < f_i(v), \quad \text{for all } i \in \{1,...,p\}. \tag{0.4}$$

We will prove that the strict inequalities (0.4) contradict the inclusion (0.1). Since the functions f_i are (η, ρ_i) -pseudoinvex, we have for any $\xi_i \in \partial^c f_i(v)$, $i \in \{1,...,p\}$,

$$\langle \xi_i, \eta(x,v) \rangle < \rho_i d(x,v). \tag{0.5}$$

We shall consider these two cases:

Case 1: $\lambda = 0$. From (0.3) and (0.5) we get

$$\sum_{i=1}^{p} \langle \mu_i \xi_i, \eta(x,v) \rangle < \sum_{i=1}^{p} \mu_i \rho_i d(x,v) \leqq 0$$

for any $\xi_i \in \partial^c f_i(v)$. This contradicts the inclusion (0.1).

Case 2: $\lambda \neq 0$. Let $M = \{ j \in \{1,...,m\} \mid \lambda_j > 0 \}$. From (0.2) we have

$$g_j(v) \geqq 0, \quad \text{for all } j \in M.$$

Since $g_j(x) \leqq 0$, it follows

$$g_j(x) \leqq g_j(v), \quad \text{for all } j \in M.$$

Relation (0.4) implies $x \neq v$, and from the strict (η, ρ'_j) -pseudoinvexity of g_j we have

$$\langle \bar{\xi}_j, \eta(x,v) \rangle < \rho'_j d(x,v)$$

for all $j \in M$ and any $\bar{\xi}_j \in \partial^c g_j(v)$. Since $\lambda_j = 0$ for all $j \notin M$, we have

$$\sum_{j=1}^{m} \langle \lambda_j \bar{\xi}_j, \eta(x,v) \rangle < \sum_{j=1}^{m} \lambda_j \rho'_j d(x,v) \tag{0.6}$$

for any $\bar{\xi}_j \in \partial^c g_j(v)$, $j \in \{1,...,m\}$.

On the other hand, the inequality (0.5) implies that

$$\sum_{i=1}^{p} \langle \mu_i \xi_i, \eta(x,v) \rangle \leqq \sum_{i=1}^{p} \mu_i \rho_i d(x,v) \tag{0.7}$$

for any $\xi_i \in \partial^c f_i(v)$. Combining inequalities (0.6) and (0.7), we obtain

$$\left\langle \sum_{i=1}^{p} \mu_i \xi_i + \sum_{j=1}^{m} \lambda_j \overline{\xi}_j, \eta(x,v) \right\rangle < \left(\sum_{i=1}^{p} \mu_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho'_j \right) d(x,v) \leqq 0$$

for any $\xi_i \in \partial^c f_i(v)$ and $\overline{\xi}_j \in \partial^c g_j(v)$. This contradicts the inclusion (0.1).

Theorem 2.2 *(Strong Duality) Let $\overline{x}$ be a weakly efficient solution for (VP). Then, there exist $\overline{\mu} \in \mathbb{R}^p$ and $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{x}, \overline{\mu}, \overline{\lambda})$ is a feasible solution for (VD) and the objective values of problems (VP) and (VD) are equal. Moreover, if all functions f_i are (η, ρ_i)-pseudoinvex, g_j are strictly (η, ρ'_j)-pseudoinvex and $\sum_{i=1}^{p} \overline{\mu}_i \rho_i + \sum_{j=1}^{m} \overline{\lambda}_j \rho'_j \leqq 0$, then $(\overline{x}, \overline{\mu}, \overline{\lambda})$ is a weakly efficient solution for (VD).*

Proof: Let $\overline{x}$ be a weakly efficient solution for (VP) and let us define

$$h(x) = \max_{1 \le i \le p} \left[f_i(x) - f_i(\overline{x}) \right].$$

Following the Minami's approach [6], we can easily check that $\overline{x}$ is an optimal solution of the following scalar optimization problem:

$$\min \left\{ h(x) \mid \text{subject to } g(x) \leqq 0 \right\}.$$

From Theorem 6.1.1 in [1] we get that there exist $\mu^* \in \mathbb{R}$ and $\lambda^* \in \mathbb{R}^m$ such that

$$0 \in \mu^* \partial^c h(\overline{x}) + \sum_{j=1}^{m} \lambda_j^* \partial^c g_j(\overline{x}),$$
$$\lambda_j^* g_j(\overline{x}) = 0, \quad j \in \{1,...,m\},$$
$$(\mu^*, \lambda_1^*, ..., \lambda_m^*) \ge 0.$$

By Proposition 2.3.12 in [1] we obtain

$$\partial^c h(\overline{x}) \subset co\left\{ \partial^c f_i(\overline{x}) \mid i \in \{1,...,p\} \right\} =$$
$$= \left\{ \sum_{i=1}^{p} \tau_i \xi_i \;\middle|\; \sum_{i=1}^{p} \tau_i = 1,\ \tau_i \ge 0,\ \xi_i \in \partial^c f_i(\overline{x}) \right\}.$$

Thus, there exist $\overline{\mu} \in \mathbb{R}^p$ and $\overline{\lambda} \in \mathbb{R}^m$ such that

$$0 \in \sum_{i=1}^{p} \bar{\mu}_i \partial^c f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\lambda}_j \partial^c g_j(\bar{x}),$$
$$\bar{\lambda}_j g_j(\bar{x}) \geqq 0, \quad j \in \{1,...,m\},$$
$$(\bar{\mu}_1,...,\bar{\mu}_p,\bar{\lambda}_1,...,\bar{\lambda}_m) \geq 0,$$

i.e., $(\bar{x},\bar{\mu},\bar{\lambda})$ is a feasible solution for (VD) and clearly the values of the objective function of (VP) and (VD) are equal.

If the functions f_i are (η,ρ_i)-pseudoinvex and g_j are strictly (η,ρ'_j)-pseudoinvex, then it follows from Theorem 2.1 that $f(\bar{x}) \nless f(v)$ for any (VD)-feasible solution $(v,\mu,\lambda),$ in particular this is true for $(\bar{x},\bar{\mu},\bar{\lambda}),$ that means that $(\bar{x},\bar{\mu},\bar{\lambda})$ is a weakly efficient solution of (VD).

Example 2.1 *Let us consider the following functions:*

$$f_1(x) = x, f_2(x) = x^2, g(x) = x^2 - 1,$$

where $x \in \mathbb{R}.$ These functions are obviously locally Lipschitz and

$$\partial^c f_1(x) = \{1\}, \partial^c f_2(x) = \{2x\}, \quad \partial^c g(x) = \{2x\}.$$

We consider the vector optimization problem

$$\min\{(f_1(x), f_2(x)) \mid x \in P\} \tag{VP1}$$

where

$$P = \{x \in \mathbb{R} \mid g(x) \leqq 0\} = [-1,1],$$

and its Mond-Weir dual problem

$$\max\{(f_1(v), f_2(v)) \mid (v,\mu_1,\mu_2,\lambda) \in D\} \tag{VD1}$$

where

$$D = \left\{ (v,\mu_1,\mu_2,\lambda) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \mu_1 + \mu_2(2v) + \lambda(2v) = 0 \\ \lambda(v^2-1) \geqq 0 \\ (\mu_1,\mu_2,\lambda) \geq 0 \end{array} \right\}$$

If we take $\eta(x,y) = x - y$ and $\rho = 0,$ then f_1, f_2 are (η,ρ)-pseudoinvex and g is strictly (η,ρ)-pseudoinvex. Let us denote

$$V = \{v \in \mathbb{R} \mid \exists\, (\mu_1,\mu_2,\lambda) \in \mathbb{R}^3, \text{ s.t. } (v,\mu_1,\mu_2,\lambda) \in D\} = (-\infty, 0].$$

It is easy to verify that for any $x \in P$ and any $(v,\mu_1,\mu_2,\lambda) \in D$ we have

$$(f_1(x), f_2(x)) \nless (f_1(v), f_2(v))$$

or, equivalently, for any $x \in [-1,1]$ and any $v \in V$, we have

$$(x, x^2) \nleq (v, v^2) \tag{0.8}$$

which means weak duality between (VP1) and (VD1).

Moreover, $[-1,0]$ is the solution of all weakly efficient solution of (VP1). Since for any $v \in [-1,0] \subset V$ there exists $(\mu_1^v, \mu_2^v, \lambda^v) \in \mathbb{R}^3$, such that $(v, \mu_1^v, \mu_2^v, \lambda^v) \in D$, it follows from (0.8) that $(v, \mu_1^v, \mu_2^v, \lambda^v)$ is a weakly efficient solution of (VD1). Thus, strong duality holds between (VP1) and (VD1).

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