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# OPTIMALITY AND DUALITY FOR A CLASS OF NONDIFFERENTIABLE MINIMAX FRACTIONAL PROGRAMMING PROBLEMS

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**Abstract:** Necessary and sufficient optimality conditions are established for a class of nondifferentiable minimax fractional programming problems with square root terms. Subsequently, we apply the optimality conditions to formulate a parametric dual problem and we prove some duality results.

**Keywords:** Fractional programming, generalized invexity, optimality conditions, duality.

## 1. INTRODUCTION

Let us consider the following continuous differentiable mappings:

$$f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R},\quad h:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R},\quad g:\mathbb{R}^n\to\mathbb{R}^p,\quad \Psi:\mathbb{R}_+\to\mathbb{R},$$

where $\frac{d\Psi(x)}{dx} \overset{def}{=} \Psi'(x) > 0$, and $g = (g_1, \cdots, g_p)$. We denote

$$\mathscr{P} = \left\{ x \in \mathbb{R}^n \mid g_j(x) \le 0,\ j = 1, 2, \cdots, p \right\} \tag{1.1}$$

and consider the compact subset $Y \subseteq \mathbb{R}^m$. Let $B_r, r = \overline{1,\beta},$ and $D_q, q = \overline{1,\delta},$ be $n \times n$ positive semi definite matrices such that for each $(x, y) \in \mathscr{P} \times Y,$ we have:

$$f(x, y) + \sum_{r=1}^{\beta} \sqrt{x^T B_r x} \ge 0, \quad h(x, y) - \sum_{q=1}^{\delta} \sqrt{x^T D_q x} > 0.$$

In this paper we consider the following non differentiable minimax fractional programming problem:

$$\inf_{x \in \mathscr{P}} \sup_{y \in Y} \Psi\left[ \left( f(x, y) + \sum_{r=1}^{\beta} \sqrt{x^T B_r x} \right) \Big/ \left( h(x, y) - \sum_{q=1}^{\delta} \sqrt{x^T D_q x} \right) \right]. \tag{P}$$

For $\beta = \delta = 1,$ and $\Psi \equiv 1,$ this problem was studied by Lai et al. [3], and further, if $B_1 = D_1 = 0,$ (P) is a differentiable minimax fractional programming problem which has been studied by Chandra and Kumar [2], Liu and Wu [5]. Many authors investigated the optimality conditions and duality theorems for minimax (fractional) programming problems. For details, one can consult [1, 4, 7].

In an earlier work, under conditions of convexity, Schmittendorf [6] established necessary and sufficient optimality conditions for the problem:

$$\inf_{x \in \mathscr{P}} \sup_{y \in Y} \phi(x, y), \tag{P1}$$

where $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous differentiable mapping. Later, Yadev and Mukherjee [9] employed the optimality conditions of Schmittendorf [6] to construct two dual problems and derived duality theorems for (convex) differentiable fractional minimax programming. In [2], Chandra and Kumar constructed two modified dual problems for which they proved duality theorems for (convex) differentiable fractional minimax programming. Liu and Wu [5] relaxed the convexity assumption in the sufficient optimality of [2] and employed the optimality conditions so as to construct one parametric dual and two other dual models of parametric-free problems. Several authors considered the optimality and duality theorems for nondifferentiable non convex minimax fractional programming problems, one can consult [4, 7].

We present necessary and sufficient optimality conditions for problem (P) and we apply the optimality conditions so as to construct one parametric dual problem for which we state weak duality, strong duality, and strictly converse duality theorems.

## 2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, we denote by $\mathbb{R}^n$ the $n$-dimensional Euclidean space and by $\mathbb{R}^n_+$ its nonnegative orthant. Let us consider the set $\mathcal{P}$ defined by (1.1), and for each $x \in \mathcal{P}$, we define

$$J(x) = \left\{ j \in \{1,2,\cdots,p\} \mid g_j(x) = 0 \right\},$$

$$Y(x) = \left\{ y \in Y \;\middle|\; \Psi\left( \frac{f(x,y) + \sum_{r=1}^{\beta} \sqrt{x^T B_r x}}{h(x,y) - \sum_{q=1}^{\delta} \sqrt{x^T D_q x}} \right) = \sup_{z \in Y} \Psi\left( \frac{f(x,z) + \sum_{r=1}^{\beta} \sqrt{x^T B_r x}}{h(x,z) - \sum_{q=1}^{\delta} \sqrt{x^T D_q x}} \right) \right\},$$

$$K(x) = \left\{ (s,t,\overline{y}) \in \mathbb{N} \times \mathbb{R}^s_+ \times \mathbb{R}^{ms} \;\middle|\; \begin{array}{l} 1 \le s \le n+1, \ \sum_{i=1}^{s} t_i = 1, \\ \text{and } \overline{y} = (\overline{y}_1, \cdots, \overline{y}_s) \in \mathbb{R}^{ms} \\ \text{with } \overline{y}_i \in Y(x), \ i = \overline{1,s} \end{array} \right\}.$$

Since $f$ and $h$ are continuous differentiable functions and $Y$ is a compact set in $\mathbb{R}^m$, it follows that for each $x_0 \in \mathcal{P}$, we have $Y(x_0) \neq \varnothing$. We denote for any $\overline{y}_i \in Y(x_0)$,

$$k_0 = \left( f(x_0, \overline{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_0^T B_r x_0} \right) \Big/ \left( h(x_0, \overline{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_0^T D_q x_0} \right). \tag{2.1}$$

Let $A$ be an $m \times n$ matrix and let $M$, $M_i$, $i = 1, \cdots, k$, be $n \times n$ symmetric positive semi definite matrices.

**Lemma 2.1** *[8] We have*

$$Ax \ge 0 \;\Rightarrow\; c^T x + \sum_{i=1}^{k} \sqrt{x^T M_i x} \ge 0,$$

*if and only if there exist* $y \in \mathbb{R}^m_+$ *and* $v_i \in \mathbb{R}^n$, $i = \overline{1,k}$, *such that*

$$Av_i \ge 0, \ \ v_i^T M_i v_i \le 1, \ i = \overline{1,k}, \ \ A^T y = c + \sum_{i=1}^{k} M_i v_i.$$

**Lemma 2.2** *[6] Let* $x_0$ *be a solution of the minimax problem (P1) and the vectors* $\nabla g_j(x_0)$, $j \in J(x_0)$ *are linearly independent. Then there exist a positive integer s,*

$1 \le s \le n+1,$ *real numbers* $t_i \ge 0,$ $i = \overline{1,s},$ $\mu_j \ge 0,$ $j = \overline{1,p},$ *and vectors* $\overline{y}_i \in Y(x_0),$ $i = \overline{1,s},$ *such that*

$$\sum_{i=1}^{s} t_i \nabla_x \psi(x_0, \overline{y}_i) + \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) = 0; \quad \mu_j g_j(x_0) = 0, \ j = \overline{1,p}; \quad \sum_{i=1}^{s} t_i \ne 0.$$

Let us consider for the next definitions the differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$, the real number $\rho \in \mathbb{R}$, and the following functions: $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$

**Definition 2.1** *The differentiable function* $\varphi$ *is* $(\eta, \rho, \theta)$*-invex at* $x_0 \in C$ *if the following hold:* $\varphi(x) - \varphi(x_0) \ge \eta(x, x_0)^T \nabla \varphi(x_0) + \rho\theta(x, x_0), \quad \forall x \in C.$

If $-\varphi$ is $(\eta, \rho, \theta)$-invex at $x_0 \in C,$ then $\varphi$ is called $(\eta, \rho, \theta)$-incave at $x_0 \in C.$ If the inequality holds strictly, then $\varphi$ is called to be strictly $(\eta, \rho, \theta)$-invex.

**Definition 2.2** *The differentiable function* $\varphi$ *is* $(\eta, \rho, \theta)$*-pseudo-invex at* $x_0 \in C$ *if the following hold:* $\eta(x, x_0)^T \nabla \varphi(x_0) \ge -\rho\theta(x, x_0) \ \Rightarrow \ \varphi(x) \ge \varphi(x_0), \quad \forall x \in C,$

If $-\varphi$ is $(\eta, \rho, \theta)$-pseudo-invex at $x_0 \in C,$ then $\varphi$ is called $(\eta, \rho, \theta)$-pseudo-incave at $x_0 \in C.$

**Definition 2.3** *The differentiable function* $\varphi$ *is strictly* $(\eta, \rho, \theta)$*-pseudo-invex at* $x_0 \in C$ *if the following hold:* $\eta(x, x_0)^T \nabla \varphi(x_0) \ge -\rho\theta(x, x_0) \ \Rightarrow \ \varphi(x) > \varphi(x_0), \quad \forall x \in C, \ x \ne x_0.$

**Definition 2.4** *The differentiable function* $\varphi$ *is* $(\eta, \rho, \theta)$*-quasi-invex at* $x_0 \in C$ *if the following hold:* $\varphi(x) \le \varphi(x_0) \ \Rightarrow \ \eta(x, x_0)^T \nabla \varphi(x_0) \le -\rho\theta(x, x_0), \quad \forall x \in C.$

If $-\varphi$ is $(\eta, \rho, \theta)$-quasi-invex at $x_0 \in C,$ then $\varphi$ is called $(\eta, \rho, \theta)$-quasi-incave at $x_0 \in C.$

## 3. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

For any $x \in \mathcal{P},$ let us denote the following index sets:

$$\mathcal{B}(x) = \left\{ r \in \{1, 2, \cdots, \beta\} \mid x^T B_r x > 0 \right\},$$
$$\overline{\mathcal{B}}(x) = \{1, 2, \cdots, \beta\} \setminus \mathcal{B}(x) = \left\{ r \mid x^T B_r x = 0 \right\},$$
$$\mathcal{D}(x) = \left\{ q \in \{1, 2, \cdots, \delta\} \mid x^T D_q x > 0 \right\},$$
$$\overline{\mathcal{D}}(x) = \{1, 2, \cdots, \delta\} \setminus \mathcal{D}(x) = \left\{ q \mid x^T D_q x = 0 \right\}.$$

Using Lemma 2.2, we may prove the following necessary optimality conditions for problem (P).

**Theorem 3.1** *(Necessary Condition) If* $x_0$ *is an optimal solution of problem (P) for which* $\overline{\mathcal{B}}(x_0) = \emptyset,$ $\overline{\mathcal{D}}(x_0) = \emptyset,$ *and* $\nabla g_j(x_0),$ $j \in J(x_0)$ *are linearly independent, then* *there exist* $(s,\bar{t},\bar{y}) \in K(x_0)$, $k_0 \in \mathbb{R}_+$, $w_r \in \mathbb{R}^n, r = \overline{1,\beta}$, $v_q \in \mathbb{R}^n, q = \overline{1,\delta}$, *and* $\bar{\mu} \in \mathbb{R}_+^p$ *such that*

$$\sum_{i=1}^{s} \bar{t}_i \Psi'(k_0) \left[ \nabla f(x_0,\bar{y}_i) + \sum_{r=1}^{\beta} B_r w_r - k_0 \left( \nabla h(x_0,\bar{y}_i) - \sum_{q=1}^{\delta} D_q v_q \right) \right] + \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) = 0, \tag{3.1}$$

$$f(x_0,\bar{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_0^T B_r x_0} - k_0 \left( h(x_0,\bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_0^T D_q x_0} \right) = 0, \quad \forall\, i = \overline{1,s}, \tag{3.2}$$

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(x_0) = 0, \tag{3.3}$$

$$\bar{t}_i \geq 0, \quad \sum_{i=1}^{s} \bar{t}_i = 1, \tag{3.4}$$

$$\begin{aligned} & w_r^T B_r w_r \leq 1,\ x_0^T B_r w_r = \sqrt{x_0^T B_r x_0},\ r = \overline{1,\beta}; \\ & v_q^T D_q v_q \leq 1,\ x_0^T D_q v_q = \sqrt{x_0^T D_q x_0},\ q = \overline{1,\delta}. \end{aligned} \tag{3.5}$$

**Proof:** Since all $B_r, r = \overline{1,\beta}$, and $D_q, q = \overline{1,\delta}$, are positive definite and f and h are differentiable functions, it follows that the function

$$\Psi\left[ \left( f(x,y) + \sum_{r=1}^{\beta} \sqrt{x^T B_r x} \right) \Big/ \left( h(x,y) - \sum_{q=1}^{\delta} \sqrt{x^T D_q x} \right) \right]$$

is differentiable with respect to *x* for any given $y \in \mathbb{R}^m$. Using Lemma 2.2, it follows that there exist a positive integer $s, 1 \leq s \leq n+1$, and vectors $t \in \mathbb{R}_+^s$, $\bar{\mu} \in \mathbb{R}_+^p$, $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, so that

$$\sum_{i=1}^{s} t_i \frac{\Psi'(k_0)}{h(x_0,\bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_0^T D_q x_0}} \left[ \nabla f(x_0,\bar{y}_i) + \sum_{r=1}^{\beta} \frac{B_r x_0}{\sqrt{x_0^T B_r x_0}} - k_0 \left( \nabla h(x_0,\bar{y}_i) - \sum_{r=1}^{\beta} \frac{D_q x_0}{\sqrt{x_0^T D_q x_0}} \right) \right] + \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) = 0 \tag{3.6}$$

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(x_0) = 0, \tag{3.7}$$

$$\sum_{i=1}^{s} t_i > 0, \tag{3.8}$$

where $k_0$ is given by (2.1). If we denote

$$w_r = \frac{x_0}{\sqrt{x_0^T B_r x_0}},\ r = \overline{1,\beta}, \quad v_q = \frac{x_0}{\sqrt{x_0^T D_q x_0}},\ q = \overline{1,\delta},$$

$$\bar{t}_i = \frac{t_i^0}{\sum_{i=1}^{s} t_i^0}, \quad \text{where } t_i^0 = \frac{\Psi'(k_0)t_i}{h(x_0,\bar{y}_i) - \sum_{q=1}^{\delta}\sqrt{x_0^T D_q x_0}},$$

we get (3.1) - (3.4). Furthermore, it easily confirms that relation (3.5) also holds, and the theorem is proved.

We notice that, in the above theorem, all matrices $B_r$ and $D_q$ are supposed to be positive definite. If at least one of $\overline{\mathcal{B}}(x_0)$ or $\overline{\mathcal{D}}(x_0)$ is not empty, then the functions involved in the objective function of problem (P) are not differentiable. In this case, the necessary optimality conditions still hold under some additional assumptions. For $x_0 \in \mathcal{P}$ and $(s,\bar{t},\bar{y}) \in K(x_0)$ we define the following vector:

$$\alpha = \sum_{i=1}^{s} \bar{t}_i \Psi'(k_0)\left(\nabla f(x_0,\bar{y}_i) + \sum_{r\in\mathcal{B}(x_0)} \frac{B_r x_0}{\sqrt{x_0^T B_r x_0}} - k_0\left(\nabla h(x_0,\bar{y}_i) - \sum_{r\in\mathcal{D}(x_0)} \frac{D_q x_0}{\sqrt{x_0^T D_q x_0}}\right)\right)$$

Now we define a set *Z* as follows:

$$Z_{\bar{y}}(x_0) = \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{l} z^T \nabla g_j(x_0) \le 0,\ j \in J(x_0), \\ z^T\alpha + \sum_{i=1}^{s} \bar{t}_i \left( \sum_{r\in\overline{\mathcal{B}}(x_0)} \sqrt{z^T B_r z} + \sum_{q\in\overline{\mathcal{D}}(x_0)} \sqrt{z^T((k_0)^2 D_q)z} \right) < 0. \end{array} \right\}$$

Using Lemma 2.1, we establish the following result:

**Theorem 3.2** *Let $x_0$ be an optimal solution of problem (P) and at least one of $\overline{\mathcal{B}}(x_0)$ or $\overline{\mathcal{D}}(x_0)$ is not empty. Let $(s,\bar{t},\bar{y}) \in K(x_0)$ be such that $Z_{\bar{y}}(x_0) = \varnothing$. Then there exist vectors $w_r \in \mathbb{R}^n$, $r = \overline{1,\beta}$, $v_q \in \mathbb{R}^n$, $q = \overline{1,\delta}$, and $\bar{\mu} \in \mathbb{R}_+^p$ which satisfy the relations (3.1) - (3.5).*

**Proof:** *Using (2.1) we get (3.2), and relation (3.4) follows directly from the assumptions.* Since $Z_{\bar{y}}(x_0) = \varnothing$, for any $z \in \mathbb{R}^n$ with: $-z^T \nabla g_j(x_0) \ge 0$, $j \in J(x_0)$, we have

$$z^T\alpha + \sum_{i=1}^{s} \bar{t}_i \left( \sum_{r\in\overline{\mathcal{B}}(x_0)} \sqrt{z^T B_r z} + \sum_{q\in\overline{\mathcal{D}}(x_0)} \sqrt{z^T((k_0)^2 D_q)z} \right) \ge 0.$$

Let us denote: $\lambda = \sum_{i=1}^{s} \overline{t}_i, \quad \gamma = \sum_{i=1}^{s} \overline{t}_i k_0$. Applying Lemma 2.1 considering:

- the rows of matrix $A$ are the vectors $\left[-\nabla g_j(x_0)\right]$, $j \in J(x_0)$;
- $c = \alpha$;
- $M_r^B = \begin{cases} \lambda^2 B_r & \text{if } r \in \overline{\mathcal{B}}(x_0) \\ 0 & \text{if } r \in \mathcal{B}(x_0) \end{cases}$ and $M_q^D = \begin{cases} \gamma^2 D_q & \text{if } q \in \overline{\mathcal{D}}(x_0) \\ 0 & \text{if } q \in \mathcal{D}(x_0) \end{cases}$,

it follows that there exist the scalars $\overline{\mu}_j \ge 0$, $j \in J(x_0)$, and the vectors $\overline{w}_r \in \mathbb{R}^n$, $r \in \overline{\mathcal{B}}(x_0)$, $\overline{v}_q \in \mathbb{R}^n$, $q \in \overline{\mathcal{D}}(x_0)$, such that

$$-\sum_{j \in J(x_0)} \overline{\mu}_j \nabla g_j(x_0) = c + \sum_{r \in \overline{\mathcal{B}}(x_0)} M_r^B \overline{w}_r + \sum_{q \in \overline{\mathcal{D}}(x_0)} M_q^D \overline{v}_q \tag{3.9}$$

and

$$\overline{w}_r^T M_r^B \overline{w}_r \le 1, \quad r \in \overline{\mathcal{B}}(x_0); \qquad \overline{v}_q^T M_q^D \overline{v}_q \le 1, \quad q \in \overline{\mathcal{D}}(x_0). \tag{3.10}$$

Since $g_j(x_0) = 0$ for $j \in J(x_0)$, we have: $\overline{\mu}_j g_j(x_0) = 0$ for $j \in J(x_0)$. If $j \notin J(x_0)$, we put $\overline{\mu}_j = 0$. It follows: $\sum_{j=1}^{p} \overline{\mu}_j g_j(x_0) = 0$, which shows that relation (3.3) holds.

Now we define

$$w_r = \begin{cases} \dfrac{x_0}{\sqrt{x_0^T B_r x_0}}, & \text{if } r \in \mathcal{B}(x_0) \\ \lambda \overline{w}_r, & \text{if } r \in \overline{\mathcal{B}}(x_0) \end{cases} \quad \text{and} \quad v_q = \begin{cases} \dfrac{x_0}{\sqrt{x_0^T D_q x_0}}, & \text{if } q \in \mathcal{D}(x_0) \\ \gamma \overline{v}_q, & \text{if } q \in \overline{\mathcal{D}}(x_0) \end{cases}$$

With this notations, equality (3.9) yields relation (3.1).

From (3.10) we get: $w_r^T B_r w_r \le 1$ for any $r = \overline{1, \beta}$. Further, if $r \in \overline{\mathcal{B}}(x_0)$, we have $x_0^T B_r x_0 = 0$, which implies $B_r x_0 = 0$, and then $\sqrt{x_0^T B_r x_0} = 0 = x_0^T B_r w_r$. If $r \in \mathcal{B}(x_0)$, we obviously have $x_0^T B_r w_r = \sqrt{x_0^T B_r x_0}$. The same arguments apply to matrices $D_q$, so relation (3.5) holds. Therefore the theorem is proved.

For convenience, if a point $x_0 \in \mathcal{P}$ has the property that the vectors $\nabla g_j(x_0)$, $j \in J(x_0)$, are linear independent and the set $Z_{\overline{y}}(x_0) = \varnothing$, then we say that $x_0 \in \mathcal{P}$ satisfy a *constraint qualification*.

The results of Theorems 3.1 and 3.2 are the necessary conditions for the optimal solution of problem (P). Actually, with some supplementary assumptions, the conditions (3.1) - (3.5) are also the sufficient optimality conditions for (P), which we state the following result for by involving generalized invex functions, being weaker assumptions used by Lai et al. in [3].

**Theorem 3.3** *(Sufficient Conditions) Let* $x_0 \in \mathcal{P}$ *be a feasible solution of (P) for which there exist a positive integer s,* $1 \le s \le n+1$, $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, $k_0 \in \mathbb{R}_+$, *defined by (2.1),* $\bar{t} \in \mathbb{R}_+^s$, $w_r \in \mathbb{R}^n$, $r = \overline{1,\beta}$, $v_q \in \mathbb{R}^n$, $q = \overline{1,\delta}$, *and* $\bar{\mu} \in \mathbb{R}_+^p$ *such that the relations (3.1) - (3.5) are satisfied. If any one of the following four conditions holds:*

*a)* $f(\cdot,\bar{y}_i) + \sum_{r=1}^{\beta} (\cdot)^T B_r w_r$ *is* $(\eta,\rho_i,\theta)$*-invex,* $h(\cdot,\bar{y}_i) - \sum_{q=1}^{\delta} (\cdot)^T D_q v_q$ *is* $(\eta,\rho_i',\theta)$*-incave for* $i = \overline{1,s}$, $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is* $(\eta,\rho_0,\theta)$*-invex, and* $\rho_0 + \sum_{i=1}^{s} \bar{t}_i(\rho_i + \rho_i' k_0) \ge 0$,

*b)* $\bar{\Phi}(\cdot) \overset{def}{=} \sum_{i=1}^{s} \bar{t}_i \left[ f(\cdot,\bar{y}_i) + \sum_{r=1}^{\beta} (\cdot)^T B_r w_r - k_0 \left( h(\cdot,\bar{y}_i) - \sum_{q=1}^{\delta} (\cdot)^T D_q v_q \right) \right]$ *is* $(\eta,\rho,\theta)$*-invex and* $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is* $(\eta,\rho_0,\theta)$*-invex, and* $\rho + \rho_0 \ge 0$,

*c)* $\bar{\Phi}(\cdot)$ *is* $(\eta,\rho,\theta)$*-pseudo-invex,* $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is* $(\eta,\rho_0,\theta)$*-quasi-invex, and* $\rho + \rho_0 \ge 0$,

*d)* $\bar{\Phi}(\cdot)$ *is* $(\eta,\rho,\theta)$*-quasi-invex,* $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is strictly* $(\eta,\rho_0,\theta)$*-pseudo-invex,* $\rho + \rho_0 \ge 0$,

*then* $x_0$ *is an optimal solution of (P).*

**Proof:** *On contrary, let us suppose that* $x_0$ *is not an optimal solution of (P). Then there exists an* $x_1 \in \mathcal{P}$ *such that*

$$\sup_{y \in Y} \Psi \left( \frac{f(x_1,y) + \sum_{r=1}^{\beta} \sqrt{x_1^T B_r x_1}}{h(x_1,y) - \sum_{q=1}^{\delta} \sqrt{x_1^T D_q x_1}} \right) < \sup_{y \in Y} \Psi \left( \frac{f(x_0,y) + \sum_{r=1}^{\beta} \sqrt{x_0^T B_r x_0}}{h(x_0,y) - \sum_{q=1}^{\delta} \sqrt{x_0^T D_q x_0}} \right)$$

We note that, for $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, we have

$$\sup_{y \in Y} \Psi \left( \frac{f(x_0,y) + \sum_{r=1}^{\beta} \sqrt{x_0^T B_r x_0}}{h(x_0,y) - \sum_{q=1}^{\delta} \sqrt{x_0^T D_q x_0}} \right) = \Psi \left( \frac{f(x_0,\bar{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_0^T B_r x_0}}{h(x_0,\bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_0^T D_q x_0}} \right) = \Psi(k_0),$$

and

$$\Psi\left(\frac{f(x_1,\overline{y}_i)+\sum_{r=1}^{\beta}\sqrt{x_1^T B_r x_1}}{h(x_1,\overline{y}_i)-\sum_{q=1}^{\delta}\sqrt{x_1^T D_q x_1}}\right)\le \sup_{y\in Y}\Psi\left(\frac{f(x_1,y)+\sum_{r=1}^{\beta}\sqrt{x_1^T B_r x_1}}{h(x_1,y)-\sum_{q=1}^{\delta}\sqrt{x_1^T D_q x_1}}\right).$$

Since $\Psi'(x)>0$, $\Psi$ is an increasing function and we get

$$f(x_1,\overline{y}_i)+\sum_{r=1}^{\beta}\sqrt{x_1^T B_r x_1}-k_0\left(h(x_1,\overline{y}_i)-\sum_{q=1}^{\delta}\sqrt{x_1^T D_q x_1}\right)<0,\ \text{for } i=\overline{1,s}. \tag{3.11}$$

From the generalized Schwarz inequality $x^T Mv\le\sqrt{x^T Mx}\sqrt{v^T Mv}$, it follows that $v^T Mv\le 1 \Rightarrow x^T Mv\le\sqrt{x^T Mx}$, where $M$ is an arbitrary symmetric positive semi definite matrix. Using now the relations (3.5), (3.11), (3.2), and (3.4), we obtain

$$\overline{\Phi}(x_1)<\overline{\Phi}(x_0). \tag{3.12}$$

1. If hypothesis a) holds, then for $i=\overline{1,s}$, we have

$$\begin{aligned}&f(x_1,\overline{y}_i)+\sum_{r=1}^{\beta}x_1^T B_r w_r-f(x_0,\overline{y}_i)-\sum_{r=1}^{\beta}x_0^T B_r w_r\ge\\&\ge\eta(x_1,x_0)^T\left(\nabla f(x_0,\overline{y}_i)+\sum_{r=1}^{\beta}B_r w_r\right)+\rho_i\theta(x_1,x_0),\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&-h(x_1,\overline{y}_i)+\sum_{q=1}^{\delta}x_1^T D_q v_q+h(x_0,\overline{y}_i)-\sum_{q=1}^{\delta}x_0^T D_q v_q\ge\\&\ge\eta(x_1,x_0)^T\left(-\nabla h(x_0,\overline{y}_i)+\sum_{q=1}^{\delta}D_q v_q\right)+\rho_i'\theta(x_1,x_0).\end{aligned} \tag{3.14}$$

Now, multiplying (3.13) by $\overline{t}_i$, (3.14) by $\overline{t}_i k_0$, and then sum up these inequalities, we obtain

$$\begin{aligned}&\overline{\Phi}(x_1)-\overline{\Phi}(x_0)\ge\sum_{i=1}^{s}\overline{t}_i(\rho_i+k_0\rho_i')\theta(x_1,x_0)+\\&+\eta(x_1,x_0)^T\sum_{i=1}^{s}\overline{t}_i\left[\nabla f(x_0,\overline{y}_i)+\sum_{r=1}^{\beta}B_r w_r-k_0\left(\nabla h(x_0,\overline{y}_i)-\sum_{q=1}^{\delta}D_q v_q\right)\right]\end{aligned}$$

Further, by (3.1) and $(\eta,\rho_0,\theta)$-invexity of $\sum_{j=1}^{p}\overline{\mu}_j g_j(\cdot)$, we get

$$\overline{\Phi}(x_1) - \overline{\Phi}(x_0) \geq -\eta(x_1, x_0)^T \sum_{j=1}^{p} \overline{\mu}_j \nabla g_j(x_0) + \sum_{i=1}^{s} \overline{t}_i(\rho_i + k_0\rho_i')\theta(x_1, x_0)$$

$$\geq -\sum_{j=1}^{p} \overline{\mu}_j g_j(x_1) + \sum_{j=1}^{p} \overline{\mu}_j g_j(x_0) + \left(\rho_0 + \sum_{i=1}^{s} \overline{t}_i(\rho_i + k_0\rho_i')\right)\theta(x_1, x_0).$$

Since $x_1 \in \mathscr{P}$, we have $g_i(x_1) \leq 0$, $i = \overline{1, s}$, and using (3.3) it follows

$$\overline{\Phi}(x_1) - \overline{\Phi}(x_0) \geq \left(\rho_0 + \sum_{i=1}^{s} \overline{t}_i(\rho_i + k_0\rho_i')\right)\theta(x_1, x_0) \geq 0,$$

which contradicts the inequality (3.12).

The conclusion follows similarly by using the assumptions b), c) and d).

## 4. DUALITY

Let $H(s,t,y)$ be the set consisting of all $(z, \mu, k, v, w) \in \mathbb{R}^n \times \mathbb{R}_+^p \times \mathbb{R}_+ \times \mathbb{R}^{n\delta} \times \mathbb{R}^{n\beta}$, where $v = (v_1, \cdots, v_\delta)$, $v_q \in \mathbb{R}^n$, $q = \overline{1, \delta}$, and $w = (w_1, \cdots, w_\beta)$, $w_r \in \mathbb{R}^n$, $r = \overline{1, \beta}$, which satisfy the following conditions:

$$\sum_{i=1}^{s} t_i \Psi'(k)\left[\nabla f(z, y_i) + \sum_{r=1}^{\beta} B_r w_r - k\left(\nabla h(z, y_i) - \sum_{q=1}^{\delta} D_q v_q\right)\right] + \sum_{j=1}^{p} \mu_j \nabla g_j(z) = 0, \quad (4.1)$$

$$\sum_{i=1}^{s} t_i \left[f(z, y_i) + \sum_{r=1}^{\beta} z^T B_r w_r - k\left(h(z, y_i) - \sum_{q=1}^{\delta} z^T D_q v_q\right)\right] \geq 0, \quad (4.2)$$

$$\sum_{j=1}^{p} \mu_j g_j(z) \geq 0, \quad (4.3)$$

$$(s, t, y) \in K(z), \quad (4.4)$$

$$w_r^T B_r w_r \leq 1, \quad r = \overline{1, \beta}, \qquad \text{and} \qquad v_q^T D_q v_q \leq 1, \quad q = \overline{1, \delta}. \quad (4.5)$$

The optimality conditions, stated in the preceding section for the minimax problem (P), suggest us to define the following dual problem:

$$\max_{(s,t,y)\in K(z)} \sup\left\{\Psi(k) \mid (z, \mu, k, v, w) \in H(s, t, y)\right\} \quad \text{(DP)}$$

If, for a triplet $(s, t, y) \in K(z)$, the set $H(s, t, y) = \varnothing$, then we define the supremum over $H(s, t, y)$ to be $-\infty$. Further, we denote

$$\Phi(\cdot) = \sum_{i=1}^{s} t_i \left[f(\cdot, y_i) + \sum_{r=1}^{\beta} (\cdot)^T B_r w_r - k\left(h(\cdot, y_i) - \sum_{q=1}^{\delta} (\cdot)^T D_q v_q\right)\right]$$

Now, we can state the following weak duality theorem for (P) and (DP).

**Theorem 4.1** *(Weak Duality) Let* $x \in \mathcal{P}$ *be a feasible solution of (P) and* $(x, \mu, k, v, w, s, t, y)$ *be a feasible solution of (DP). If any of the following four conditions holds:*

*a)* $f(\cdot, y_i) + \sum_{r=1}^{\beta} (\cdot)^T B_r w_r$ *is* $(\eta, \rho_i, \theta)$*-invex,* $h(\cdot, y_i) - \sum_{q=1}^{\delta} (\cdot)^T D_q v_q$ *is* $(\eta, \rho_i', \theta)$*-incave for* $i = \overline{1, s}$, $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is* $(\eta, \rho_0, \theta)$*-invex, and* $\rho_0 + \sum_{i=1}^{s} t_i (\rho_i + \rho_i' k) \geq 0,$

*b)* $\Phi(\cdot)$ *is* $(\eta, \rho, \theta)$*-invex and* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is* $(\eta, \rho_0, \theta)$*-invex, and* $\rho + \rho_0 \geq 0,$

*c)* $\Phi(\cdot)$ *is* $(\eta, \rho, \theta)$*-pseudo-invex,* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is* $(\eta, \rho_0, \theta)$*-quasi-invex, and* $\rho + \rho_0 \geq 0,$

*d)* $\Phi(\cdot)$ *is* $(\eta, \rho, \theta)$*-quasi-invex,* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is strictly* $(\eta, \rho_0, \theta)$*-pseudo-invex,* $\rho + \rho_0 \geq 0,$

*then* $$\sup_{y \in Y} \Psi\left[ \left( f(x, y) + \sum_{r=1}^{\beta} \sqrt{x^T B_r x} \right) \Big/ \left( h(x, y) - \sum_{q=1}^{\delta} \sqrt{x^T D_q x} \right) \right] \geq \Psi(k).$$

The proof of this theorem uses similar arguments as in the proof of Theorem 3.3.

**Theorem 4.2** *(Strong Duality) Let* $x^*$ *be an optimal solution of problem (P). Assume that* $x^*$ *satisfies a constraint qualification for problem (P). Then there exist* $(s^*, t^*, y^*) \in K(x^*)$ *and* $(x^*, \mu^*, k^*, v^*, w^*) \in H(s^*, t^*, y^*)$ *such that* $(x^*, \mu^*, k^*, v^*, w^*, s^*, t^*, y^*)$ *is a feasible solution of (DP). If the hypotheses of Theorem 4.1 are also satisfied, then* $(x^*, \mu^*, k^*, v^*, w^*, s^*, t^*, y^*)$ *is an optimal solution for (DP), and both problems (P) and (DP) have the same optimal values.*

**Proof:** By Theorems 3.1 and 3.2, there exist $(s^*, t^*, y^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, v^*, w^*) \in H(s^*, t^*, y^*)$ such that $(x^*, \mu^*, k^*, v^*, w^*, s^*, t^*, y^*)$ is a feasible solution of (DP), and

$$\Psi(k^*) = \Psi\left[ \left( f(x^*, y_i^*) + \sum_{r=1}^{\beta} \sqrt{(x^*)^T B_r x^*} \right) \Big/ \left( h(x^*, y_i^*) - \sum_{q=1}^{\delta} \sqrt{(x^*)^T D_q x^*} \right) \right].$$

The optimality of this feasible solution for (DP) follows from Theorem 4.1.

***Theorem 4.3*** *(Strict Converse Duality) Let* $x^*$ *and* $(\bar{z},\bar{\mu},\bar{k},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y})$ *be the optimal solutions of (P) and (DP), respectively, and that the hypotheses of Theorem 4.2 are fulfilled. If any one of the following three conditions holds:*

*a)* *one of* $f(\cdot,\bar{y}_i)+\sum_{r=1}^{\beta}(\cdot)^T B_r\bar{w}_r$ *is strictly* $(\eta,\rho_i,\theta)$*-invex,* $h(\cdot,\bar{y}_i)-\sum_{q=1}^{\delta}(\cdot)^T D_q\bar{v}_q$ *is strictly* $(\eta,\rho_i',\theta)$*-incave for* $i=\overline{1,s}$*, or* $\sum_{j=1}^{p}\bar{\mu}_j g_j(\cdot)$ *is strictly* $(\eta,\rho_0,\theta)$*-invex, and* $\rho_0+\sum_{i=1}^{s}\bar{t}_i(\rho_i+\rho_i'\bar{k})\geq 0$;

*b)* *either* $\sum_{i=1}^{s}\bar{t}_i\left[f(\cdot,\bar{y}_i)+\sum_{r=1}^{\beta}(\cdot)^T B_r\bar{w}_r-\bar{k}\left(h(\cdot,\bar{y}_i)-\sum_{q=1}^{\delta}(\cdot)^T D_q\bar{v}_q\right)\right]$ *is strictly* $(\eta,\rho,\theta)$*-invex or* $\sum_{j=1}^{p}\bar{\mu}_j g_j(\cdot)$ *is strictly* $(\eta,\rho_0,\theta)$*-invex, and* $\rho+\rho_0\geq 0$;

*c)* $\sum_{i=1}^{s}\bar{t}_i\left[f(\cdot,\bar{y}_i)+\sum_{r=1}^{\beta}(\cdot)^T B_r\bar{w}_r-\bar{k}\left(h(\cdot,\bar{y}_i)-\sum_{q=1}^{\delta}(\cdot)^T D_q\bar{v}_q\right)\right]$ *is strictly* $(\eta,\rho,\theta)$*-pseudo-invex and* $\sum_{j=1}^{p}\bar{\mu}_j g_j(\cdot)$ *is* $(\eta,\rho_0,\theta)$ *-quasi-invex, and* $\rho+\rho_0\geq 0$;

*then* $x^*=\bar{z}$*, that is,* $\bar{z}$ *is an optimal solution for problem (P) and*

$$\sup_{y\in Y}\Psi\left[\left(f(\bar{z},y)+\sum_{r=1}^{\beta}\sqrt{\bar{z}^T B_r\bar{z}}\right)\Big/\left(h(\bar{z},y)-\sum_{q=1}^{\delta}\sqrt{\bar{z}^T D_q\bar{z}}\right)\right]=\Psi(\bar{k}).$$

**Proof:** *Suppose on the contrary that* $x^*\neq\bar{z}$*. From Theorem 4.2 we know that there exist* $(s^*,t^*,y^*)\in K(x^*)$ *and* $(x^*,\mu^*,k^*,v^*,w^*)\in H(s^*,t^*,y^*)$ *such that* $(x^*,\mu^*,k^*,v^*,w^*,s^*,t^*,y^*)$ *is a feasible solution for (DP) with the optimal value* $\Psi(k^*)$*. Now, if we proceed similarly as in the proof of Theorem 3.3, we arrive at the strict inequality*

$$\sup_{y\in Y}\Psi\left[\left(f(x^*,y)+\sum_{r=1}^{\beta}\sqrt{(x^*)^T B_r x^*}\right)\Big/\left(h(x^*,y)-\sum_{q=1}^{\delta}\sqrt{(x^*)^T D_q x^*}\right)\right]>\Psi(\bar{k}).$$

But this contradicts the fact $\Psi(k^*)=\Psi(\bar{k})$, and we conclude that $x^*=\bar{z}$.

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