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DUALITY FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS INVOLVING d-TYPE-I -SET n- FUNCTIONS

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Abstract: We establish duality results under generalized convexity assumptions for a multiobjective nonlinear fractional programming problem involving d-type-I n-set functions. Our results generalize the results obtained by Preda and Stancu-Minasian [24], [25].

Keywords: d-type-I set functions, multiobjective programming, duality results.

1. INTRODUCTION

Consider the multiobjective nonlinear fractional programming problem involving n-set functions

$$\text{minimize}\, F(S) = \left(\frac{F_1(S)}{G_1(S)}, \ldots, \frac{F_p(S)}{G_p(S)} \right) \tag{P}$$

subject to

$$H_j(S) \leqq 0,\ j \in M, S = (S_1, \ldots, S_n) \in \Gamma^n$$

where Γ^n is the n-fold product of a σ- algebra Γ of subsets of a given set X, $M = \{1, 2, \ldots, m\}$, F_i, G_i, $i \in P = \{1, 2, \ldots, p\}$, and $H_j, j \in M$ are differentiable real-valued functions defined on Γ^n with

$$F_i(S) \geqq 0 \text{ and } G_i(S) > 0 \text{, for all } i \in P. \tag{1}$$

Let $S_0 = \{S \mid S \in \Gamma^n, H(S) \leqq 0\}$ be the set of all feasible solutions to (P), where $H = (H_1, \ldots, H_m)$.

The term "minimize" being used in Problem (P) is for finding efficient, weakly and properly efficient solutions.

A feasible solution S^0 to (P) is said to be an *efficient solution* to (P) if there exists no other feasible solution S to (P) so that $F_i(S) \leqq F_i(S^0)$, for all $i \in P$, with strict inequality for at least one $i \in P$.

A feasible solution S^0 to (P) is said to be a *weakly efficient solution* to (P) if there exists no other feasible solution S to (P) so that $F_i(S) < F_i(S^0)$, for all $i \in P$.

The analysis of optimization problems with set or n-set functions i.e. selection of measurable subsets from a given space, has been the subject of several papers. For a historical survey of optimality conditions and duality for programming problems involving set and n-set functions the reader is referred to Stancu-Minasian and Preda's review paper [28]. These problems arise in various applications including fluid flow [3], electrical insulator design [8], regional design (districting, facility location, warehouse layout, urban planning etc.) [10], statistics [11], [21] and optimal plasma confinement [30]. The general theory for optimizing set functions was first developed by Morris [20]. Many results of Morris [20] are only confined to functions of a single set. Corley [9] started to give the concepts of partial derivatives and derivatives of real-valued n-set functions.

Starting from the methods used by Jeyakumar and Mond [12] and Ye [31], Suneja and Srivastava [29] defined some new classes of scalar or vector functions called d-type-I, d-pseudo-type-I, d-quasi-type-I etc. for a multiobjective nondifferentiable programming problem and obtained necessary and sufficient optimality criteria. Also, they established duality between this problem and its Wolfe-type and Mond-Weir-type duals and obtained some duality results considering the concept of a weak minimum.

In particular, multiobjective fractional subset programming problems have been the focus of intense interest in the past few years, and resulted in many papers [1], [2], [4]-[7], [13]-[17], [22], [23], [28], [33]-[35].

In this paper we establish duality results under generalized convexity assumptions for a multiobjective nonlinear fractional programming problem involving generalized d-type-I n-set functions. Our results generalize the results obtained by Preda and Stancu-Minasian [24], [25].

2. DEFINITIONS AND PRELIMINARIES

In this section we introduce the notation and definitions which will be used throughout the paper.

Let $\mathbf{R}^n$ be the n - dimensional Euclidian space and $\mathbf{R}^n_+$ its positive orthant, i.e.

$$\mathbf{R}^n_+ = \{x = (x_j) \in \mathbf{R}^n,\ x_j \geqq 0,\ j = 1,...,n\}.$$

For $x = (x_1,...,x_m), y = (y_1,...,y_m) \in \mathbf{R}^m$ we put $x \leqq y$ iff $x_i \leqq y_i$ for each $i \in M$; $x \leq y$ iff $x_i \leqq y_i$ each $i \in M$, with $x \neq y$; $x < y$ iff $x_i < y_i$ for each $i \in M$ while $x \not< y$ is the negation of $x < y$. We write $x \in \mathbf{R}^n_+$ iff $x \geqq 0$.

Let (X, Γ, μ) be a finite non-atomic measure space with $L_1(X, \Gamma, \mu)$ separable, and let d be the pseudometric on Γ^n defined by:

$$d(S,T) = \left[\sum_{k=1}^{n} \mu^2 (S_k \Delta T_k) \right]^{1/2}$$

for $S = (S_1,...,S_n)$, $T = (T_1,...,T_n) \in \Gamma^n$, where Δ denotes the symmetric difference. Thus (Γ^n, d) is a pseudometric space, which will serve as the domain for most of the functions that will be used in this paper.

For $h \in L_1(X, \Gamma, \mu)$, the integral $\int_S h \,\mathrm{d}\mu$ will be denoted by $\langle h, I_S \rangle$, where I_S is the indicator (characteristic) function of $S \in \Gamma$.

We next introduce the notion of differentiability for n-set functions. This was originally introduced by Morris [20] for set functions and subsequently extended by Corley [9] to n-set functions.

A function $\varphi : \Gamma \to \mathbf{R}$ is said to be differentiable at $S^0 \in \Gamma$ if there exist $D\varphi(S^0) \in L_1(X, \Gamma, \mu)$, called the derivative of φ at S^0, and $\psi : \Gamma \times \Gamma \to \mathbf{R}$ such that for each $S \in \Gamma$,

$$\varphi(S) = \varphi(S^0) + \langle D\varphi(S^0), I_S - I_{S^0} \rangle + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $o(d(S, S^0))$, that is, $\lim_{d(S,S^0)\to 0} \dfrac{\psi(S,S^0)}{d(S,S^0)} = 0$.

A function $F : \Gamma^n \to \mathbf{R}$ is said to have a partial derivative at $S^0 = (S^0_1,...,S^0_n)$ with respect to its k-th argument if the function

$$\varphi(S_k) = F(S^0_1,...,S^0_{k-1}, S_k, S^0_{k+1},...,S^0_n)$$

has derivative $D\varphi(S^0_k)$ and we define $D_k F(S^0) = D\varphi(S^0_k)$. If $D_k F(S^0)$, $1 \leqq k \leqq n$, all exist, then we put $DF(S^0) = (D_1 F(S^0),...,D_n F(S^0))$.

A function $F:\Gamma^n \to \mathbf{R}$ is said to be differentiable at S^0 if there exist $DF(S^0)$ and $\psi:\Gamma^n \times \Gamma^n \to \mathbf{R}$ such that

$$F(S) = F(S^0) + \sum_{k=1}^{n} \left\langle D_k F(S^0), I_{S_k} - I_{S_k^0} \right\rangle + \psi(S, S^0),$$

where $\psi(S,S^0)$ is $o(d(S,S^0))$, for all $S \in \Gamma^n$.

Along the lines of Jeyakumar and Mond [12] and Suneja and Srivastava [29], Preda and Stancu-Minasian [24] defined new classes of n-set functions, called *d*-type-I, *d*-quasi type-I, *d*-pseudo type-I, *d*-quasi-pseudo type-I, *d*-pseudo-quasi type-I.

In [18] Mishra extended the generalized d-type-I vector-valued functions of Preda and Stancu-Minasian [24] to new generalized d-type-I n-set functions and establish optimality and Mond-Weir type duality results.

Definition 1. *[24] We say that (F,G) is of d-type-I at $S^0 \in \Gamma^n$ if there exist functions $\alpha_i, \beta_j : \Gamma^n \times \Gamma^n \to \mathbf{R}_+ \setminus \{0\}$, $i \in P, j \in M$, such that for all $S \in S_0$, we have*

$$F_i(S) - F_i(S^0) \geqq \alpha_i(S,S^0) \sum_{k=1}^{n} \left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \right\rangle, \ i \in P \tag{2}$$

and

$$-H_j(S^0) \geqq \beta_j(S,S^0) \sum_{k=1}^{n} \left\langle D_k H_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle, \ j \in M. \tag{3}$$

We say that (F,H) is of *d*-semistrictly type-I at S^0 if in the above definition we have $S \neq S^0$ and (2) is a strict inequality.

Now, we introduce

Definition 2. *[32] A feasible solution S^0 to (P) is said to be a regular feasible solution if there exists $\hat{S} \in \Gamma^n$ such that*

$$H_j(S^0) + \sum_{k=1}^{n} \left\langle D_k H_j(S^0), I_{\hat{S}_k} - I_{S_k^0} \right\rangle < 0, \ j \in M.$$

Now, for each $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbf{R}_+^p$ we consider the parametric problem

$$\text{minimize}(\mathrm{F}_1(S) - \lambda_1 G_1(S), \dots, \mathrm{F}_p(S) - \lambda_p G_p(S)) \tag{P_λ}$$

subject to

$$H_j(S) \leqq 0, \ j \in M, S = (S_1, \dots, S_n) \in \Gamma^n.$$

The problem (P_λ) is equivalent to the problem (P) in the sense that for particular choices of λ_i, $i \in P$, the two problems have the same set of efficient solutions. This equivalence is stated in the following lemma which is well known in fractional programming [27].

Lemma 3. *An S^0 is an efficient solution to (P) if and only if is an efficient solution to* (P_{λ^0}) *with* $\lambda_i^0 = \dfrac{F_i(S^0)}{G_i(S^0)}$, $i = 1,\dots,p$.

In this paper the proofs of the duality results for Problem (P) will invoke the following necessary efficiency result for (P_λ) (see Zalmai [32], Theorem 3.2).

Theorem 4. *[32] Let S^0 be a regular efficient (or weakly efficient) solution to (P) and assume that F_i, G_i, $i \in P$ and H_j, $j \in M$, are differentiable at S^0. Then there exist $u^0 \in \mathbf{R}_+^p$, $\sum_{i=1}^{p} u_i^0 = 1$, $v^0 \in \mathbf{R}_+^m$, and $\lambda^0 \in \mathbf{R}_+^p$ such that*

$$\sum_{k=1}^{n}\left\langle \sum_{i=1}^{p} u_i^0 \left(D_k F_i(S^0) - \lambda_i^0 D_k G_i(S^0) + \sum_{i=1}^{m} v_j^0 D_k H_j(S^0), I_{S_k} - I_{S_k^0} \right)\right\rangle \geqq 0, \textit{for all},$$
$$S \in \Gamma^n \tag{4}$$

$$u_i^0 (F_i(S^0) - \lambda_i^0 G_i(S^0)) \geqq 0 , \ i \in P \tag{5}$$

$$v_j^0 H_j(S^0) = 0 , \ j \in M . \tag{6}$$

3. DUALITY

In this section, in the differentiable case, based on the equivalence of (P) and (P_λ) a dual for (P_λ) is defined and some duality results in *d*-type-I assumptions are stated. With (P_λ) we associate a dual stated as

$$\text{maximize } (\lambda_1,\dots,\lambda_p) \tag{D}$$

subject to

$$\sum_{i=1}^{p}\sum_{k=1}^{n} u_i \left\langle D_k F_i(T) - \lambda_i D_k G_i(T), I_{S_k} - I_{T_k} \right\rangle + \sum_{j=1}^{m}\sum_{k=1}^{n} v_j \left\langle D_k H_j(T), I_{S_k} - I_{T_k} \right\rangle \geqq 0 ,$$
$$S \in \Gamma^n \tag{7}$$

$$u_i \, (F_i(T) - \lambda_i \, G_i(T)) \geqq 0 , \ i \in P , \tag{8}$$

$$v_j H_j(T) \geqq 0 , \ j \in M , \tag{9}$$

$$u \in \mathbf{R}_+^p, \ \sum_{i=1}^{p} u_i = 1 , v \in \mathbf{R}_+^m , \lambda \in \mathbf{R}_+^p . \tag{10}$$

Let D_0 be the set of feasible solutions to (D). Let us prove the duality theorems.

Theorem 5. *(Weak duality) Let S and (T, u, v, λ) be feasible solutions to problem (P) and (D), respectively such that (i_1) for each $i \in P$ and $j \in M$, $(F_i(\cdot) - \lambda_i G_i(\cdot), H_j(\cdot))$ is* *of d-type-I at* T *; (i₂)* $u_i > 0$ *for any* $i \in P$*, and for some* $i \in P$ *and* $j \in M$*,* $(F_i(\cdot) - \lambda_i G_i(\cdot), H_j(\cdot))$ *is of d-semistrictly type-I at* T.

Then for any $S \in S_0$ one cannot have

$$\frac{F_i(S)}{G_i(S)} \leqq \lambda_i \text{ for any } i \in P, \tag{11}$$

$$\frac{F_j(S)}{G_j(S)} < \lambda_j \text{ for some } j \in P. \tag{12}$$

Proof: Let us suppose the contrary that (11) and (12) hold. Then there exists S, a feasible solution for (P_λ), such that (11) and (12) hold.

If hypothesis (i_2) holds, then $u_i > 0$ for any $i = 1,...,p$. From (1), (11) and (12) we get

$$\sum_{i=1}^{p} u_i \,(F_i(S) - \lambda_i \, G_i(S)) < 0. \tag{13}$$

Using the feasibility of S, and the relations (9) and (10), we have

$$v_j H_j(S) \leqq 0 \leqq v_j H_j(T) \;\; \forall\, j = 1,...,m. \tag{14}$$

Since $\alpha_i(S,T) > 0, i \in P$, and $\beta_j(S,T) > 0, j \in M$, combining (8), (13) and (14) we obtain

$$\sum_{i=1}^{p} \frac{u_i}{\alpha_i(S,T)}(F_i(S) - \lambda_i \, G_i(S)) < \sum_{i=1}^{p} \frac{u_i}{\alpha_i(S,T)}(F_i(T) - \lambda_i \, G_i(T)) + \sum_{i=1}^{m} \frac{v_j H_j(T)}{\beta_j(S,T)}. \tag{15}$$

We claim that $S \neq T$ for if it is not true, then, from $u_i > 0$, $i \in P$, the feasibility of S and (8) we obtain a contradiction with (11) and (12).

One the other hand, from $S \neq T$, (i_1) and (i_2), it follows that

$$(F_i(S) - \lambda_i G_i(S)) - (F_i(T) - \lambda_i G_i(T)) \geqq \alpha_i(S,T) \sum_{k=1}^{n} \left\langle D_k F_i(T) - \lambda_i D_k G_i(T), I_{S_k} - I_{T_k} \right\rangle \tag{16}$$

for any $i \in P$, with strict inequality for some i, and

$$-H_j(T) \geqq \beta_j(S,T) \sum_{k=1}^{n} \left\langle D_k H_j(T), I_{S_k} - I_{T_k} \right\rangle, \; j \in M. \tag{17}$$

By dividing by $\alpha_i(S,T) > 0$ and $\beta_j(S,T) > 0$, respectively, the above inequalities reduce to the following

$$\frac{F_i(S)-\lambda_i G_i(S)}{\alpha_i(S,T)}-\frac{F_i(T)-\lambda_i G_i(T)}{\alpha_i(S,T)} \geqq \sum_{k=1}^{n}\left\langle D_k F_i(T)-\lambda_i D_k G_i(T), I_{S_k}-I_{T_k}\right\rangle \qquad (18)$$

for any $i \in P$, with strict inequality for some i, and

$$-\frac{H_j(T)}{\beta_j(S,T)} \geqq \sum_{k=1}^{n}\left\langle D_k H_j(T), I_{S_k}-I_{T_k}\right\rangle,\ j \in M \qquad (19)$$

Multiplying the inequality (18) by $u_i > 0$, $\forall i \in P$, and (19) by $v_j \geqq 0$, $\forall j \in M$, and summing after all i and j, respectively, yields

$$\sum_{i=1}^{p}\frac{u_i}{\alpha_i(S,T)}(F_i(S)-\lambda_i G_i(S))-\sum_{i=1}^{p}\frac{u_i}{\alpha_i(S,T)}(F_i(T)-\lambda_i G_i(T))-\sum_{i=1}^{p}\frac{v_j H_j(T)}{\beta_j(S,T)} >$$
$$> \sum_{i=1}^{p}\sum_{k=1}^{n} u_i\left\langle D_k F_i(T)-\lambda_i D_k G_i(T), I_{S_k}-I_{T_k}\right\rangle + \sum_{j=1}^{m}\sum_{k=1}^{n} v_j\left\langle D_k H_j(T), I_{S_k}-I_{T_k}\right\rangle. \qquad (20)$$

Now, by (15) it follows

$$\sum_{i=1}^{p}\sum_{k=1}^{n} u_i\left\langle D_k F_i(T)-\lambda_i D_k G_i(T), I_{S_k}-I_{T_k}\right\rangle + \sum_{j=1}^{m}\sum_{k=1}^{n} v_j\left\langle D_k H_j(T), I_{S_k}-I_{T_k}\right\rangle < 0 .$$

This inequality contradicts (7). Thus the theorem is proved.

Corollary 6. Let S^0 and (S^0,u^0,v^0,λ^0) be feasible solutions to (P_{λ^0}) and (D), respectively. If the hypotheses of Theorem 5 are satisfied, then S^0 is an efficient solution to (P_{λ^0}) and (S^0,u^0,v^0,λ^0) is an efficient solution to (D).

Proof: We proceed by contradiction. If S^0 is not an efficient solution to (P_{λ^0}) then there exists a feasible solution S' to (P_{λ^0}) such that

$F_i(S') \leqq \lambda_i^0 G_i(S')$, $\forall i \in P$,

and (21)

$F_j(S') < \lambda_j^0 G_j(S')$, for some $j \in P$.

Since (S^0,u^0,v^0,λ^0) is a feasible solution to (D) by (21), and Theorem 5 we obtain a contradiction. Hence S^0 is an efficient solution to (P_{λ^0}). In the same way we obtain that (S^0,u^0,v^0,λ^0) is an efficient solution to (D).

Theorem 7. *(Strong duality) Let S^0 be a regular efficient solution to (P). Then there exist $u^0 \in \mathbf{R}_+^p$, $\sum_{i=1}^{p} u_i^0 = 1$, $v^0 \in \mathbf{R}_+^m$, and $\lambda^0 \in \mathbf{R}_+^p$, such that (S^0,u^0,v^0,λ^0) is a feasible solution to (D). Further, if the conditions of Weak Duality Theorem 5 also hold, then*

$(S^0, u^0, v^0, \lambda^0)$ *is an efficient solution to (D) and the values of the objective functions of (P) and (D) are equal at* S^0 *and* $(S^0, u^0, v^0, \lambda^0)$ *respectively.*

Proof: Using Theorem 4 we obtain that there exist $u^0 \in \mathbf{R}_+^p$, $\sum_{i=1}^{p} u_i^0 = 1$, $v^0 \in \mathbf{R}_+^m$, and (4) and (5) hold. Thus, $(S^0, u^0, v^0, \lambda^0)$ satisfies (7) – (10). Hence, $(S^0, u^0, v^0, \lambda^0)$ is a feasible solution to (D). Further, if Theorem 5 holds then, by Corollary 6 we obtain that this solution $(S^0, u^0, v^0, \lambda^0)$ is also an efficient solution to (D), and the values of the objective functions of (P) and (D) are equal at S^0 and $(S^0, u^0, v^0, \lambda^0)$ respectively.

Now we give a strict converse duality theorem of Mangasarian type [19] for (P_λ) and (D).

Theorem 8. *(Strict converse duality) Let* S^* *and* $(S^0, u^0, v^0, \lambda^0)$ *be efficient solutions to* (P_{λ^0}) *and (D), respectively. Assume that*

(j1) $\sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*, S^0)} (F_i(S^*) - \lambda_i^0 G_i(S^*)) \leqq \sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*, S^0)} (F_i(S^0) - \lambda_i^0 G_i(S^0))$;

(j2) for any $i \in P$ *and* $j \in M$, $(F_i(\cdot) - \lambda_i^0 G_i(\cdot), H_j(\cdot))$ *is of* d*-semistrictly type-I at* S^**.Then,* $S^0 = S^*$.

Proof: We assume that $S^0 \neq S^*$ and exhibit a contradiction. Using (j2) we obtain

$$(F_i(S^*) - \lambda_i^0 G_i(S^*)) - (F_i(S^0) - \lambda_i^0 G_i(S^0)) >$$
$$> \alpha_i(S^*, S^0) \sum_{k=1}^{n} \left\langle D_k F_i(S^0) - \lambda_i^0 D_k G_i(S^0), I_{S_k^*} - I_{S_k^0} \right\rangle$$

for any $i \in P$, and

$$-H_j(S^0) \geqq \beta_j(S^*, S^0) \sum_{k=1}^{n} \left\langle D_k H_j(T), I_{S_k^*} - I_{S_k^0} \right\rangle, \; j \in M .$$

By dividing by $\alpha_i(S^*, S^0) > 0$ and $\beta_j(S^*, S^0) > 0$, respectively, the above inequalities reduce to the following

$$\frac{F_i(S^*) - \lambda_i^0 G_i(S^*)}{\alpha_i(S^*, S^0)} - \frac{F_i(S^0) - \lambda_i^0 G_i(S^0)}{\alpha_i(S^*, S^0)} > \sum_{k=1}^{n} \left\langle D_k F_i(S^0) - \lambda_i^0 D_k G_i(S^0), I_{S_k^0} - I_{S_k^*} \right\rangle \tag{22}$$

for any $i \in P$, and

$$-\frac{H_j(S^0)}{\beta_j(S^*, S^0)} \geqq \sum_{k=1}^{n} \left\langle D_k H_j(S^0), I_{S_k^*} - I_{S_k^0} \right\rangle, \; j \in M \tag{23}$$

Multiplying the inequality (22) by $u^0 \geqq 0$, $\sum_{i=1}^{p} u_i^0 = 1$, $\forall i \in P$, and (23) by $v^0 \geqq 0$, $\forall j \in M$, and summing after all i and j, respectively, yields

$$\sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*,S^0)}(F_i(S^*) - \lambda_i^0 G_i(S^*)) - \sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*,S^0)}(F_i(S^0) - \lambda_i^0 G_i(S^0))$$
$$-\sum_{j=1}^{m} \frac{v_j^0 H(S_0)}{\beta_j(S^*,S^0)} > \sum_{i=1}^{p}\sum_{k=1}^{n} u_i^0 \left\langle D_k F_i(S^0) - \lambda_i^0 D_k G_i(S^0), I_{S_k^*} - I_{S_k^*} \right\rangle \tag{24}$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{n} v_j^0 \left\langle D_k H_j(S^0), I_{S_k^*} - I_{S_k^0} \right\rangle.$$

Now, because $(S^0, u^0, v^0, \lambda^0)$ is a feasible solution to (D) by (7) we get

$$\sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*,S^0)}(F_i(S^*) - \lambda_i^0 G_i(S^*)) - \sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*,S^0)}(F_i(S^0) - \lambda_i^0 G_i(S^0)) -$$
$$-\sum_{i=1}^{m} \frac{v_j^0 H(S^0)}{\beta_i(S^*,S^0)} > 0. \tag{25}$$

Since $v_j^0 H_j(S^0) \geqq 0$ for any $j \in M$, by (25) we obtain

$$\sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*,S^0)}(F_i(S^*) - \lambda_i^0 G_i(S^*)) > \sum_{i=1}^{p} \frac{u_i^0}{\alpha_i(S^*,S^0)}(F_i(S^0) - \lambda_i^0 G_i(S^0))$$

which contradicts the assumption (j_1). Thus the theorem is proved.

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