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# EFFICIENCY AND DUALITY FOR MULTIOBJECTIVE FRACTIONAL VARIATIONAL PROBLEMS WITH $(\rho,b)$ - QUASIINVEXITY

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**Abstract.** The necessary conditions for (normal) efficient solutions to a class of multi-objective fractional variational problems (MFP) with nonlinear equality and inequality constraints are established using a parametric approach to relate efficient solutions of a fractional problem and a non-fractional problem. Based on these normal efficiency criteria a Mond-Weir type dual is formulated and appropriate duality theorems are proved assuming $(\rho,b)$ - quasi-invexity of the functions involved.

**Keywords:** Duality, fractional variational problem,quasi-invexity.

## 1. INTRODUCTION

For the first results on the necessity of the optimal solutions of the variational problems we cite the Valentine's paper[17]. The papers of Mond and Hanson [9, 10], Bector [1], Mond, Chandra and Husain [12], Mond and Husain [11], Smart and Mond [16] and Preda [14] developed the duality of the scalar variational problems involving convex and generalized convex functions. Mukherjee and Purnachandra [13], Preda and Gramatovici [15] and Mititelu [7] established weak efficiency conditions and developed different types of dualities for multiobjective variational problems generated by various types of generalized convex functions. Kim and Kim [4] used the efficiency property of the multi-objective variational problems in the duality theory. In this paper we will introduce the notion of normal efficient solution and establish the necessary conditions for the normal efficiency of Valentine's type for multiobjective variational problems. Also, we have developed a duality of Mond-Weir type for the multiobjective fractional variational problems that uses the notion of normal efficiency. There are used $(\rho, b)$-quasi-invex functions.

## 2. NOTATION AND STATEMENT OF THE PROBLEM

Let $R^n$ be the $n$-dimensional Euclidean space. Throughout the paper, the following conventions for vectors in $R^n$ will be adopted.

For vectors $v=(v_1,...,v_n)$, $w=(w_1,...,w_n)$ the relations $v=w, v<w, v \leqq w,$ and $v \leq w$ are defined as follows

$$v = w \Leftrightarrow v_i = w_i,\ i=\overline{1,n};\ v < w \Leftrightarrow v_i < w_i,\ i=\overline{1,n};$$

$$v \leqq w \Leftrightarrow v_i \leqq w_i,\ i=\overline{1,n};\ \ v \leq w \Leftrightarrow u \leqq w \text{ and } u \neq v.$$

Let $I=[a,b]$ be a real interval and

$$f=(f_1,...,f_p): I\times R^n \times R^n \to R^p, k=(k_1,...k_p): I\times R^n \times R^n \to R^p,$$
$$g=(g_1,...,g_m): I\times\ R^n \times\ R^n \to\ R^m,\ \ h=(h_1,...,h_q): I\times\ R^n \times\ R^n \to\ R^q$$

be twice continuously differentiable functions.

Consider the vector-valued function $f(t,x,\dot{x})$, where $t \in I$ and $x: I \to R^n$, with derivative $\dot{x}$ with respect to $t$. Denote by $f_x$ and $f_{\dot{x}}$ the $p\times n$ matrices of the first-order partial derivatives of the components with respect to $x$ and $\dot{x}$, i.e. $f_x=(f_{1x},f_{2x},\dots,f_{px})'$ and $f_{\dot{x}}=(f_{1\dot{x}},f_{2\dot{x}},\dots,f_{p\dot{x}})'$, with

$$f_{ix}=\left(\frac{\partial f_i}{\partial x_1},...,\frac{\partial f_i}{\partial x_n}\right) \text{ and } f_{i\dot{x}}=\left(\frac{\partial f_i}{\partial \dot{x}_1},...,\frac{\partial f_i}{\partial \dot{x}_n}\right), i=1,2,...,p.$$

Similarly, $k_x, g_x, h_x$ and $k_{\dot{x}}, g_{\dot{x}}, h_{\dot{x}}$ denote the $p\times n, m\times n, q\times n$ matrices of the first partial derivatives of $k, g$ and $h$ respectively, with respect to $x$ and $\dot{x}$. Let $C(I,\ R^n)$ denote the space of piecewise smooth (continuously differentiable) functions $x$ with the norm $\|x\| := \|x\|_\infty + \|Dx\|_\infty$, where the differential operator $D$ is given by

$$u = Dx \ \Leftrightarrow\ x(t)=x(a)+\int_a^t u(s)\,\mathrm{d}s,$$

where $x(a)$ is a given boundary value. Therefore, $D=\mathrm{d}/\mathrm{d}t$, except at discontinuities.

Consider the multiobjective fractional variational problem

$$\text{(MFP)} \begin{cases} Minimize \left( \dfrac{\int_a^b f_1(t,x,\dot{x})\,\mathrm{d}t}{\int_a^b k_1(t,x,\dot{x})\,\mathrm{d}t}, \ldots, \dfrac{\int_a^b f_p(t,x,\dot{x})\,\mathrm{d}t}{\int_a^b k_p(t,x,\dot{x})\,\mathrm{d}t} \right) \\ subject\ to \\ \quad x(a) = a_0,\ x(b) = b_0\ , \\ \quad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x}) = 0,\ \forall t \in I \end{cases} .$$

Assume that $\int_a^b k_i(t,x,\dot{x})\,\mathrm{d}t > 0$ for all $i = 1, 2, \cdots, .p.$ Let

$$\mathrm{D} = \left\{ x \in C(I,\ R^n) \mid x(a) = a_0,\, x(b) = b_0,\, f(t,x,\dot{x}) \leqq 0,\, h(t,x,\dot{x}) = 0\ , \forall t \in I \right\}$$

be the set of all the feasible solutions to (MFP).

## 3. PRELIMINARIES. THE MULTIOBJECTIVE VARIATIONAL PROBLEM

In this section we will recall some basic definitions and auxiliary results that will be needed later in our discussion of efficiency conditions and Mond-Weir duality to (MFP).

Consider the multiobjective variational problem

$$\text{(MP)} \begin{cases} Minimize \int_a^b f(t,x,\dot{x})\,\mathrm{d}t = \left( \int_a^b f_1(t,x,\dot{x})\,\mathrm{d}t, \ldots, \int_a^b f_p(t,x,\dot{x})\,\mathrm{d}t \right) \\ subject\ to \quad x(a) = a_0,\, x(b) = b_0 \\ \qquad\qquad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x}) = 0,\ t \in I. \end{cases}$$

The domain of (MP) is also D.

**Definition 3.1.** *A feasible solution* $x^0 \in D$ *is said to be an efficient solution of (MP) if for all feasible solutions* $x \in D$

$$\int_a^b f(t,x,\dot{x})\,\mathrm{d}t \leqq \int_a^b f(t,x^0,\dot{x}^0)\,\mathrm{d}t \Longrightarrow \int_a^b f(t,x,\dot{x})\,\mathrm{d}t = \int_a^b f(t,x^0,\dot{x}^0)\,\mathrm{d}t$$

Let $s : I \times R^n \times R^m \to R$ e a scalar continuously differentiable function and consider now the scalar variational problem

$$\text{(SP)} \begin{cases} Minimize \int_a^b s(t,x,\dot{x})\,\mathrm{d}t \\ subject\ to\ \ x(a) = a_0,\, x(b) = b_0 \\ \qquad\qquad g(t,x,\dot{x}) \leqq 0,\, h(t,x,\dot{x}) = 0,\ t \in I. \end{cases}$$

**Definition 3.2.** *The optimal solution* $x^0 \in D$ *to (SP) is called normal if* $\lambda \neq 0$.

According to this definition without the loss of generality, in what follows we can take $\lambda = 1$.

The following result gives the necessary Valentine's conditions [17] for the optimality of $x^0$ in (SP):

**Theorem 3.1** *(The necessary Valentine's conditions). Let $x^0$ be a (normal) optimal solution to (SP) and let s, g and h be continuously differentiable functions. Then there exist a scalar $\lambda$ and piecewice smooth functions $\mu^0(t)$ and $\upsilon^0(t)$ satisfying the conditions*

$$\textbf{(VC)}\begin{cases} \lambda s_x(t,x^0,\dot{x}^0)+\mu^0(t)' g_x(t,x^0,\dot{x}^0)+\upsilon^0(t)' h_x(t,x^0,\dot{x}^0)= \\ =\dfrac{d}{dt}[\lambda s_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon^0(t) h_{\dot{x}}(t,x^0,\dot{x}^0)] \\ \mu^0(t)' g(t,x^0,\dot{x}^0)=0,\ \ \mu^0(t)\geqq ,0, \quad \forall t\in I, \quad (\lambda=1). \end{cases}$$

We have

**Lemma 3.2** *(Chankong, Haimes [2]). $x^0\in D$ is an efficient solution to the problem (MP) if and only if $x^0$ is an optimal solution to the scalar problem*

$$\mathrm{P}_i(x^0) \qquad \begin{cases} \textit{Minimize } \displaystyle\int_a^b f_i(t,x,\dot{x})\,\mathrm{d}t \\ \textit{subject to } \ x(a)=a_0,\ x(b)=b_0 \\ \qquad\qquad g(t,x,\dot{x})\leqq 0,\ h(t,x,\dot{x})=0,\ \ t\in I \\ \qquad \displaystyle\int_a^b f_j(t,x,\dot{x})\,\mathrm{d}t \leqq \int_a^b f_j(t,x^0,\dot{x}^0)\,\mathrm{d}t\ ,\ j=\overline{1,p},\ j\neq i. \end{cases}$$

for each $i=1,\dots,p$.

**Lemma 3.3.** *If $x^0$ is a (normal) optimal solution to the scalar problem $P_i(x^0)$, then there exist real scalars $\lambda_{ji}$, $j=\overline{1,p}$ and functions $\mu_i$ and $\upsilon_i$ such that*

$$\begin{cases} \lambda_{ii} f_{ix}(t,x^0,\dot{x}^0)+\sum\limits_{\substack{j=1\\ j\neq i}}^{p}\lambda_{ji} f_{jx}+\mu_i(t)' g_x(t,x^0,\dot{x}^0)+\upsilon_i(t) h_x(t,x^0,\dot{x}^0)= \\ =\dfrac{d}{dt}[\lambda_{ii} f_{i\dot{x}}(t,x^0,\dot{x}^0)+\sum\limits_{\substack{j=1\\ j\neq i}}^{p}\lambda_{ji} f_{j\dot{x}}+\mu_i(t)' g_{i\dot{x}}(t,x^0,\dot{x}^0)+\upsilon_i(t)' h_{\dot{x}}(t,x^0,\dot{x}^0)] \\ \mu_i(t)' g(t,x^0,\dot{x}^0)=0,\ \ \mu_i(t)\geqq 0,\ \ \forall t\in I \\ \lambda_{ji}\geqq 0,\ (\lambda_{ii}=1),\ j=\overline{1,p}\,. \end{cases} \tag{3.1}$$

**Proof:** For $j=\overline{1,p}$, $j\neq i$ we define the functions $\omega_j : I\times R^n\times R^n \to R$ by

$$\int_a^b [f_j(t,x,\dot{x})-f_j(t,x^0,\dot{x}^0)+\omega_j(t,x,\dot{x})]\,\mathrm{dt}=0\ ,$$

where $\omega_j(t,x,\dot{x})\geq 0,\ j=\overline{1,p},\ j\neq i$

Then $x^0$ is a (normal) optimal solution to $P_i(x^0)$ if and only if $x^0$ is a (normal) optimal solution to the problem

$$\begin{cases} Maximize \quad \int_a^b f_i(t,x,\dot{x})\,\mathrm{dt} \\ subject\ \ to \quad x(a)=a_0, x(b)=b_0 \\ \qquad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x})=0,\ \omega_j(t,x,\dot{x}) \geq 0,\ \ j=\overline{1,p}, j\neq i \\ \int_a^b [f_j(t,x,\dot{x}) - f_j(t,x^0,\dot{x}^0) + \omega_j(t,x,\dot{x})]\,\mathrm{dt} = 0,\ j=\overline{1,p}, j\neq i. \end{cases}$$

If $x^0$ is (normal) optimal to this problem then, according to an Euler's theorem, there exist real scalars $\lambda_{ji}$, $j=\overline{1,p}$, $j\neq i$, such that $x^0$ is (normal) optimal to problem

$$\overline{P}_i(x^0) \begin{cases} Maximize \quad \int_a^b \left\{ f_i(t,x,\dot{x}) + \sum_{\substack{j=1 \\ j\neq i}}^{p} \lambda_{ji} \left[ f_j(t,x^0,\dot{x}^0) + \omega_j(t,x,\dot{x}) \right] \right\} \mathrm{dt} \\ subject\ \ to \quad x(a)=a_0, x(b)=b_0 \\ \qquad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x})=0,\ -\omega_j(t,x,\dot{x}) \leq 0,\ \ j=\overline{1,p}, j\neq i \end{cases}$$

If $x^0$ is (normal) optimal to $\overline{P}_i(x^0)$ then there are real scalars $\gamma_i$ and $\lambda_{ji}$, $j=\overline{1,p}$, $j\neq i$ and the functions $\mu_i$, $\upsilon_i \in C(I,R^n)$ and $\alpha_j : I \to R, j=\overline{1,p}$, $j\neq i$, such that the next Valentine's conditions are true:

$$\begin{cases} \gamma_i\{f_{ix}(t,x^0,\dot{x}^0) + \sum_{\substack{j=1 \\ j\neq i}}^{p} \lambda_{ji}[f_j(t,x,x) - f_j(t,x^0,\dot{x}^0) + \varpi_j(t,x,\dot{x})]'_{x^0}\} + \\ \qquad + \mu_i(t)' g_x(t,x^0,\dot{x}^0) + \upsilon_i(t)' h_x(t,x^0,\dot{x}^0) - \sum_{\substack{j=1 \\ j\neq i}}^{p} \alpha_j(t) \varpi_{jx}(t,x^0,\dot{x}^0) = \\ = \frac{d}{dt} \langle \gamma_i \{f_{i\dot{x}}(t,x^0,\dot{x}^0) + \sum_{\substack{j=1 \\ j\neq i}}^{p} \lambda_{ji}[f_j(t,x,x) - f_j(t,x^0,\dot{x}^0) + \varpi_j(t,x,\dot{x})]'_{\dot{x}^0}\} + \\ \qquad + \mu_i(t)' g_{\dot{x}}(t,x^0,\dot{x}^0) + \upsilon_i(t)' h_{\dot{x}}(t,x^0,\dot{x}^0) - \sum_{\substack{j=1 \\ j\neq i}}^{p} \alpha_j(t) \varpi_{j\dot{x}}(t,x^0,\dot{x}^0) = \\ \mu_i(t)' g(t,x^0,\dot{x}^0) = 0,\ \mu_i(t) \geqq 0, \quad \alpha_j(t) \varpi_j(t,x^0,\dot{x}^0) = 0, \quad \alpha_j(t) \geqq 0 \\ \gamma_i \geqq 0\ (=1),\ \ \lambda_{ji} \geqq 0,\ j=\overline{1,p},\ j\neq i,\ \forall t\in I. \end{cases} \tag{3.2}$$

The first relation of (3.2) becomes (E)

$$\gamma_i f_{ix}(t,x^0,\dot{x}^0)+\sum_{\substack{j=1\\ j\neq i}}^{p}\gamma_i\lambda_{ji}f_{jx}(t,x^0,\dot{x}^0)+$$
$$\sum_{\substack{j=1\\ j\neq i}}^{p}(\gamma_i\lambda_{ji}-\alpha_j)\omega_{jx}(t,x^0,\dot{x}^0)+\mu_i(t)'g_x(t,x^0,\dot{x}^0)+\upsilon_i(t)'h_x(t,x^0,\dot{x}^0)=$$
$$=\frac{d}{dt}[\gamma_i f_{i\dot{x}}(t,x^0,\dot{x}^0)+\sum_{\substack{j=1\\ j\neq i}}^{p}\gamma_i\lambda_{ji}f_{ji}(t,x^0,\dot{x}^0)+\sum_{\substack{j=1\\ j\neq i}}^{p}(\gamma_i\lambda_{ji}-\alpha_j)f_{j\dot{x}}(t,x^0,\dot{x}^0)$$
$$+\mu_i(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon_i(t)'h_{\dot{x}}(t,x^0,\dot{x}^0)$$

(E) is a system with $n$ partial differential equations having $p-1$ coefficients $\lambda_{ji}$, $p-1$ coefficients $\alpha_j\equiv\alpha_j(t)$, $\mu_i(t)$ a vector function with $m$ components and $\upsilon_i(t)$ other vector function with $q$ components. Therefore (E) admits $2(p-1)+m+q$ infinities of coefficients, a number greater than $n$. We choose a set of values for these coefficients putting the conditions $\gamma_i\lambda_{ji}-\alpha_j(t)=0,\quad j=\overline{1,p}\,, j\neq i.$ Then we define $\lambda_{ii}=\gamma_i$ and $\lambda_{ji}\equiv\gamma_i\lambda_{ji}\geqq 0,\ j=\overline{1,p}\,, j\neq i$ and the system (E) becomes

$$\lambda_{ii}f_{ix}(t,x^0,\dot{x}^0)+\sum_{\substack{j=1\\ j\neq i}}^{p}\lambda_{ji}f_{jx}(t,x^0,\dot{x}^0)+\mu_i(t)'g_x(t,x^0,\dot{x}^0)+\upsilon_i(t)'h_x(t,x^0,\dot{x}^0)=$$

$$=\frac{d}{dt}[\lambda_{ii}f_{i\dot{x}}(t,x^0,\dot{x}^0)+\sum_{\substack{j=1\\ j\neq i}}^{p}\lambda_{ji}f_{ji}(t,x^0,\dot{x}^0)+\mu_i(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon_i(t)'h_{\dot{x}}(t,x^0,\dot{x}^0)\,,$$

that is the first relation of (3.1). So, (3.2) take the form (3.1).

**Definition 3.3.** *$x^0\in D$ is said to be the normal efficient solution to (MP) if it is a normal optimal solution to at least one of the scalar problems $P_i(x^0)$, $i=\overline{1,p}$.*

**Theorem 3.4.** *Let $x^0\in D$ be a normal efficient solution to (MP). Then there exist a vector $\lambda^0\in R^p$ and piecewise smooth functions $\mu^0:I\to R^m$ and $\upsilon^0:I\to R^q$ that satisfy the Valentine's conditions*

$$\text{(MV)}\ \begin{cases}\lambda^{0}{}'f_x(t,x^0,\dot{x}^0)+\mu^0(t)'g_x(t,x^0,\dot{x}^0)+\upsilon^0(t)h_x(t,x^0,\dot{x}^0)=\\ =\dfrac{d}{dt}\left[\lambda^{0}{}'f_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon^0(t)h_x(t,x^0,\dot{x}^0)\right]\\ \mu^0(t)'g(t,x^0,\dot{x}^0)=0,\ \ \mu(t)\geqq 0,\quad \forall t\in I\\ \lambda^0\geq 0,\ \ e'\lambda^0=1,\ \ e=(1,\dots,1)'\in \mathrm{R}^n\,.\end{cases}$$

**Proof:** If $x^0$ is a (normal) efficient solution to (MP), according to Lemma 3.2, $x^0$ is a (normal) efficient solution to the scalar problems $P_i(x^0)$, $i=\overline{1,p}$. According to Lemma

3.3, for each $i$, there exist scalars $\lambda_{ji} \geqq 0, (\lambda_{ii} = 1), j = \overline{1, p}$ and the functions $\mu_i$ and $\upsilon_i$ that satisfy relations (3.1). Summing the relations (3.1) over $i = \overline{1, p}$ and denoting

$$\lambda_j = \sum_{i=1}^{p} \lambda_{ji} \;, \lambda = (\lambda_1 ,..., \lambda_p)' \;, \mu(t) = \sum_{i=1}^{p} \mu_i(t) ,\; \upsilon(t) = \sum_{i=1}^{p} \upsilon_i(t)$$

we obtain

$$\begin{cases} \lambda' f_x(t, x^0, \dot{x}^0) + \mu(t)' g_x(t, x^0, \dot{x}^0) + \upsilon(t)' h_x(t, x^0, \dot{x}^0) = \\ = \dfrac{d}{dt}\left[ \lambda' f_{\dot{x}}(t, x^0, \dot{x}^0) + \mu(t)' g_{\dot{x}}(t, x^0, \dot{x}^0) + \upsilon(t)' h_{\dot{x}}(t, x^0, \dot{x}^0) \right] \\ \mu(t)' g(t, x^0, \dot{x}^0) = 0, \;\; \mu(t) \geqq 0, \quad \forall t \in I \\ \lambda_j \geqq \; 0, j = \overline{1, p} \;\; ,(\lambda_i = 1) . \end{cases} \tag{3.3}$$

Upon dividing the relation (3.3) by $S = \sum_{j=1}^{p} \lambda_j$ and setting $\lambda^0 = \lambda / S$, $\mu^0(t) = \mu(t)/S$ and $\upsilon^0(t) = \upsilon(t)/S$, then the relations (MV) are obtained.

Let $\rho \in R$ and a function $b : X \times X \to [0, \infty)$. Put

$$H(x) = \int_a^b h(t, x, \dot{x}) \, \mathrm{d}t$$

**Definition 3.4.** *The function H is said to be (strictly)* $(\rho, b)$ *-quasi-invex at* $x^0$ *if there exist vector functions* $\eta : I \times X \times X \to R^n$ *with* $\eta(t, x(t), \dot{x}(t)) = 0$ *for* $x(t) = x^0(t)$ *and* $\theta : X \times X \to R^n$ *such that for any* $x (x \neq x^0)$, $H(x) \leqq H(x^0) \Rightarrow$

$$\Rightarrow b(x, x^0) \int_a^b [\eta' h_x(t, x^0, \dot{x}^0) + (D\eta)' h_{\dot{x}}(t, x^0, \dot{x}^0)] \, \mathrm{d}t \; (<) \leqq \; -\rho b(x, x^0) \left\| \theta(x, x^0) \right\|^2 .$$

## 4. NECESSARY EFFICIENCY CONDITIONS FOR (MFP)

Let us now consider the problem

$$(\mathrm{FP})_i(x^0) \begin{cases} \min\limits_x \dfrac{\int_a^b f_i(t, x, \dot{x}) \, \mathrm{d}t}{\int_a^b k_i(t, x, \dot{x}) \, \mathrm{d}t} \\ \textit{subject to} \quad x(a) = a_0, \, x(b) = b_0 \\ \qquad g(t, x, \dot{x}) \leqq 0, \, h(t, x, \dot{x}) = 0, \;\; t \in I \\ \dfrac{\int_a^b f_j(t, x, \dot{x}) \, \mathrm{d}t}{\int_a^b k_j(t, x, \dot{x}) \, \mathrm{d}t} \leqq \dfrac{\int_a^b f_j(t, x^0, \dot{x}^0) \, \mathrm{d}t}{\int_a^b k_j(t, x^0, \dot{x}^0) \, \mathrm{d}t}, \; j = \overline{1, p} \;, j \neq i . \end{cases}$$

Denoting

$$R_i^0 = \frac{\int_a^b f_i(t, x^0, \dot{x}^0)\,\mathrm{d}t}{\int_a^b k_i(t, x^0, \dot{x}^0)\,\mathrm{d}t} = \min_x \frac{\int_a^b f_i(t, x, \dot{x})\,\mathrm{d}t}{\int_a^b k_i(t, x, \dot{x})\,\mathrm{d}t}, \quad i = \overline{1, p}.$$

problem $(\mathrm{FP})_i(x^0)$ can be written as

$$(\mathrm{FPR})_i \begin{cases} \min_x \dfrac{\int_a^b f_i(t, x, \dot{x})\,\mathrm{d}t}{\int_a^b k_i(t, x, \dot{x})\,\mathrm{d}t} \qquad \left[= R_i^0\right] \\ \text{subject to} \quad x(a) = a_0, \ x(b) = b_0 \\ \qquad g(t, x, \dot{x}) \leqq 0, \ h(t, x, \dot{x}) = 0, \quad t \in I \\ \qquad \int_a^b [f_j(t, x, \dot{x}) - R_j^0 k_j(t, x, \dot{x})]\,\mathrm{d}t \leqq 0, \ j = \overline{1, p}, \ j \neq i. \end{cases}$$

Consider now the problem

$$(\mathrm{SPR})_i \begin{cases} \min_x \int_a^b [f_i(t, x, \dot{x}) - R_i^0 k_i(t, x, \dot{x})]\,\mathrm{d}t \\ \text{subject to} \quad x(a) = a_0, x(b) = b_0 \\ \qquad g(t, x, \dot{x}) \leqq 0, \ h(t, x, \dot{x}) = 0, \quad t \in I \\ \qquad \int_a^b [f_j(t, x, \dot{x}) - R_j^0 k_j(t, x, \dot{x})]\,\mathrm{d}t \leqq 0, \ j = \overline{1, p}, \ j \neq i. \end{cases}$$

**Lemma 4.1** *(Jagannathan [3]). $x^0 \in D$ is optimal to $(FPR)_i$ if and only if $x^0$ is optimal to $(SPR)_i$.*

**Theorem 4.2.** *$x^0 \in D$ is an efficient solution to (MFP) if and only if it is an optimal solution to each of the problems $(SPR)_i, i = \overline{1, p}$.*

**Proof:** $x^0$ is efficient in (MFP) if and only if it is optimal to the problems $(\mathrm{FPR})_i, i = \overline{1, p}$ (according to Lemma 3.2 ) and also, for each $i$ $(i = \overline{1, p})$, $x^0$ is optimal in $(FPR)_i$ if it is optimal in $(SPR)_i$ (according to Lemma 4.1).

The point $x^0 \in D$ is a normal efficient solution to (MFP) if it is a normal optimal solution to at least one of the scalar problems $(FP)_i(x^0)$, $i = \overline{1, p}$, or equivalently $(\mathrm{SPR})_i, i = \overline{1, p}$.

Now the main result of this section follows.

**Theorem 4.3** *(The necessary efficiency conditions). Let $x^0 \in D$ be a normal efficient solution to problem (MFP). Then there exist $\lambda^0 \in R^p$ and piecewise smooth functions $\mu^0 : I \to R^m$ and $\upsilon^0 : I \to R^q$ that satisfy the conditions*

$$(\text{MFV})\begin{cases}\sum_{i=1}^{p}\lambda_i^0[f_{ix}(t,x^0,\dot{x}^0)-R_i^0k_{ix}(t,x^0,\dot{x}^0)]+\mu^0(t)'g_x(t,x^0,\dot{x}^0)\\ +\ \upsilon^0(t)'h_x(t,x^0,\dot{x}^0)=\dfrac{d}{dt}\{\sum_{i=1}^{p}\lambda_i^0[f_{i\dot{x}}(t,x^0,\dot{x}^0)-R_i^0k_{i\dot{x}}(t,x^0,\dot{x}^0)]\\ +\mu^0(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon^0(t)'h_{\dot{x}}(t,x^0,\dot{x}^0)\}\\ \mu(t)'g(t,x^0,\dot{x}^0)=0,\mu(t)\geqq 0,\ \forall t\in I,\lambda_i^0\geqq\ 0,\ i=\overline{1,p}\ ,(\lambda_i^0\geqq 1).\end{cases}$$

**Proof:** Suppose that $x^0$ is a (normal) optimal solution to $(\text{SPR})_i$. In what follows the proof is similar to those in Theorem 3.4, where for $i=\overline{1,p}$ , $f_i(t,x,\dot{x})$ is replaced by $f_i(t,x,\dot{x})-R_i^0k_i(t,x,\dot{x})$.

We denote

$$F_i(x^0)=\int_a^b f_i(t,x^0,\dot{x}^0)\,\mathrm{d}t\,,K_i(x^0)=\int_a^b k_i(t,x^0,\dot{x}^0)\,\mathrm{d}t.$$

Then we obtain

$$R_i^0=\frac{F_i(x^0)}{K_i(x^0)},\quad i=\overline{1,p}. \tag{4.1}$$

Taking the relations (4.1) into account, Theorem 4.3 becomes:

**Theorem 4.4** *(The necessary efficiency conditions). Let $x^0$ be a normal efficient solution to problem (MFP). Then there exist $\lambda^0\in R^n$ and piecewise smooth functions $\mu^0:I\to R^q$ and $\upsilon^0:I\to R^n$ that satisfy the conditions*

$$(\text{MFV})_0\begin{cases}\sum_{i=1}^{p}\lambda_i^0[K_i(x^0)f_{ix}(t,x^0,\dot{x}^0)-F_i(x^0)k_{ix}(t,x^0,x^0)]+\\ +\mu^0(t)'g_x(t,x^0,\dot{x}^0)+\upsilon^0(t)'h_x(t,x^0,\dot{x}^0)=\\ =\dfrac{d}{dt}\{\sum_{i=1}^{p}\lambda_1^0[K_i(x^0)f_{i\dot{x}}(t,x^0,\dot{x}^0)-F_i(x^0)k_{i\dot{x}}(t,x^0,\dot{x}^0)]\\ +\mu^0(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon^0(t)h_{\dot{x}}(t,x^0,\dot{x}^0)\}\\ \mu^0(t)'g(t,x^0,\dot{x}^0)=0,\quad \mu^0(t)\geqq 0,\quad \forall t\in I\\ \lambda^0\geq 0,\ e'\lambda^0=1\,.\end{cases}$$

## 5. MOND-WEIR DUALITY TYPE

Let $\{J_1,...,J_r\}$ be a partition of $\{1,...,m\}$ and $\{K_1,...,K_r\}$ a partition of $\{1,...,q\}$. We consider the functions $y,v\in C(I,R^n)$ and associate the multiobjective variational problem with (MFP)

$$
\text{(MFD)}\begin{cases}
Maximize \left( \dfrac{\int_a^b f_1(t,y,\dot{y})\,\mathrm{d}t}{\int_a^b k_1 t,y,\dot{y})\,\mathrm{d}t}, ..., \dfrac{\int_a^b f_p t,y,\dot{y})\,\mathrm{d}t}{\int_a^b k_p t,y,\dot{y})\,\mathrm{d}t} \right) \\
subject \quad to \quad y(a)=a_0,\ y(b)=b_0 \\
\quad \sum_{i=1}^{p} \lambda_i [K_i(y) f_{iy}(t,y,\dot{y}) - F_i(y) k_{iy}(t,y,\dot{y})] + \\
\quad + \mu(t)' g_y(t,y,\dot{y}) + \upsilon(t)' h_y(t,y,\dot{y}) = \\
= \dfrac{d}{dt} \ \{ \sum_{i=1}^{p} \lambda_i [K_i(y) f_{i\dot{y}}(t,y,\dot{y}) - F_i(y) k_{i\dot{y}}(t,y,y)] + \\
\mu(t)' g_{\dot{y}}(t,y,\dot{y}) + \upsilon(t)' h_{\dot{y}}(t,y,\dot{y}) \} \\
\mu_{J_\alpha}(t)' g_{J_\alpha}(t,y,\dot{y}) + \upsilon_{K_\alpha}(t) h_{K_\alpha}(t,y,\dot{y}) \geqq 0, \quad \alpha = \overline{1,r}, \quad \forall t \in I \\
\lambda \geq 0,\ e'\lambda = 1.
\end{cases}
$$

Denote by $\pi(x)$ the value of problem (MFP) at $x \in D$ and let $\delta(y,\lambda,\eta,\upsilon)$ be the value of the dual (MFD) at $(y,\lambda,\eta,\upsilon) \in \Delta$, where $\Delta$ is the domain of (MFD).

THEOREM 5.1 (Weak duality). Let $x$ and $(y,\lambda,\mu,v)$ be the feasible points of problems (MFP) *and* (MFD). *Assume that the following conditions are satisfied*:

1. for each $i = \overline{1,p}$ we have $F_i(x) > 0,\ K_i(x) > 0, \forall x \in X$.
2. for each $i = \overline{1,p}$, $F_i(x)$ is $(\rho_i', b)$-quasi-invex at $y$ and $-K_i(x)$ is $(\rho_i'', b)$-quasi-invex at $y$, all with respect to $\eta$ and $\theta$.
3. $\int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t,x,\dot{x}) + \upsilon_{K_\alpha}(t)' h_{K_\alpha}(t,x,\dot{x})]\,\mathrm{d}t$ is $(\rho_\alpha''', b)$-quasi-invex at $y$ with respect to $\eta$ and $\theta$.
4. one of the functions of b)-c) is strictly $(\rho, b)$-quasi-invex
5. $\sum_{i=1}^{p} \lambda_i [\rho_i' K_i(y) + \rho_i'' F_i(y)] + \sum_{\alpha=1}^{r} \rho_\alpha''' \geqq 0$.
6. Then π (x) $\pi(x) \leq \delta\ (y,\lambda,\mu,v)$ is false.

**Proof:** From b) it results

$$
\begin{aligned}
&F_i(x) \leqq F_i(y) \Rightarrow b(x,y) \int_a^b \{ \eta' f_{iy}(t,y,\dot{y}) + (D\eta)' f_{i\dot{y}}(t,y,\dot{y}) \}\,\mathrm{d}t \\
&\leqq -\rho_i' b(x,y) \|\theta(x,y)\|^2,
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
&-K_i(x) \leqq -K_i(y) \Rightarrow b(x,y) \int_a^b \{ -\eta' k_{iy}(t,y,\dot{y}) - (D\eta)' k_{i\dot{y}}(t,y,\dot{y}) \}\,\mathrm{d}t \\
&\leqq -\rho_i'' b(x,y) \|\theta(x,y)\|^2
\end{aligned} \tag{5.2}
$$

By multiplying the relations (5.1) by $K_i(y) > 0$ and (4.2) by $F_i(y) > 0$, by summing the obtained implications side by side we obtain:

$$F_i(x)K_i(y)-K_i(x)F_i(y)\leqq 0 \Rightarrow b(x,y)\int_a^b\{\eta'[K_i(y)f_{iy}(t,y,\dot{y})$$
$$-F_i(y)k_{iy}(t,y,\dot{y})]+(D\eta)'[K_i(y)f_{i\dot{y}}(t,y,\dot{y})-F_i(y)k_{i\dot{y}}(t,y,\dot{y})]\}\,\mathrm{d}t \tag{5.3}$$
$$\leqq -[\rho_i'K_i(y)+\rho_i''F_i(y)]b(x,y)\|\theta\|^2$$

By multiplying (5.3) by $\lambda_i, i=\overline{1,p}, (\lambda \geq 0)$ and by summing over $i=\overline{1,p}$, we get the result

$$\sum_{i=1}^{p}\lambda_i[F(x)K_i(y)-K_i(x)F(y)]\leqq 0 \Rightarrow$$
$$b(x,y)\int_a^b\{\eta'\sum_{i=1}^{p}\lambda_i[K_i(y)f_{iy}(t,y,\dot{y})-F_i(y)k_{iy}(t,y,\dot{y})]+$$
$$+(D\eta)'\sum_{i=1}^{p}\lambda_i[K_i(y)f_{i\dot{y}}(t,y,\dot{y})-F_i(y)k_{i\dot{y}}(t,y,\dot{y})]\}\,\mathrm{d}t \tag{5.4}$$
$$\leqq -b(x,y)\|\theta\|^2\sum_{i=1}^{p}\lambda_i[\rho_i'K_i(y)+\rho_i''F(y)]$$

According to c), we have

$$\int_a^b[\mu_{J_\alpha}(t)'g_{J\alpha}(t,x,\dot{x})+\upsilon_{K_\alpha}h_{K_\alpha}(t,x.,\dot{x})]\,\mathrm{d}t\leqq$$
$$\int_a^b[\mu_{J_\alpha}(t)'g_{J_\alpha}(t,y,v)+\upsilon_{K_\alpha}(t)'h_{K_\alpha}(t,y,\dot{y})]\,\mathrm{d}t\Rightarrow$$
$$b(x,y)\int_a^b\{\eta'[\mu_{J\alpha}(t)'g_{J_\alpha y}(t,y,\dot{y})+\upsilon_{K_\alpha}(t)'h_{K_\alpha y}]+(D\eta)'[\mu_{J_\alpha}(t)'g_{J_\alpha y}(t,y,\dot{y}) \tag{5.5}$$
$$+\upsilon_{K_\alpha}(t)'h_{K_\alpha}(t,y,\dot{y})]\}\,\mathrm{d}t$$
$$\leqq -\rho_\alpha'''b(x,y)\|\theta\|^2.$$

By summing side by side over $\alpha=\overline{1,r}$ the twice implications (5.5) and equivalently, we obtain

$$\int_a^b[\mu(t)'g(t,x,\dot{x})+\upsilon(t)'h(t,x(t,x,\dot{x})]\,\mathrm{d}t-$$
$$\int_a^b[\mu(t)'g(t,y,\dot{y})+\upsilon(t)'h(t,y,\dot{y})]\,\mathrm{d}t\leqq 0\Rightarrow b(x,y)$$
$$\int_a^b\{\eta'[\mu(t)'g_y(t,y,\dot{y})+\upsilon(t)'h_y(t,y,\dot{y})]+(D\eta)'[\mu(t)'g_{\dot{y}}(t,y,\dot{y}) \tag{5.6}$$
$$+\upsilon(t)'h_{\dot{y}}(t,y,\dot{y})]\}\mathrm{d}t\leqq -b(x,y)\|\theta\|^2\sum_{\alpha=1}^{r}\rho_\alpha'''$$

By summing now the double implications (5.4) and (5.6) side by side, the result gained is

$$\sum_{i=1}^{p}\lambda_i[F_i(x,u)K_i(y,v)-K_i(x,u)F_i(y,v)]+$$
$$\int_a^b[\mu(t)'g(t,x,\dot{x})+\nu(t)'h(t,x,\dot{x})]\,\mathrm{d}t-$$
$$\int_a^b[\mu(t)'g(t,y,\dot{y})+\nu(t)'h(t,y,\dot{y})]\,\mathrm{d}t\leqq 0$$
$$\Rightarrow b(x,y)\int_a^b\eta'\{\sum_{i=1}^{p}\lambda_i[K_i(y)f_{iy}(t,y,\dot{y})-F_i(y)k_{iy}(t,y,\dot{y})]+ \tag{5.7}$$
$$\mu(t)'g_y(t,y,\dot{y})+\upsilon(t)'h_y(t,y,\dot{y})\}+(D\eta)'\{\sum_{i=1}^{p}\lambda_i[K_i(y)f_{i\dot{y}}(t,y,\dot{y})$$
$$-F(y)k_{i\dot{y}}(t,y,\dot{y})]+\mu(t)'g_{\dot{y}}(t,y,v)+\upsilon(t)'h_{\dot{y}}(t,y,v)\}\,\mathrm{d}t<$$
$$<-b(x,y)\|\theta\|^2\{\sum_{i=1}^{p}\lambda_i[\rho_i'K_i(y)+\rho_i''F_i(y)]+\sum_{\alpha=1}^{r}\rho_\alpha'''\}.$$

From the second implication of (5.7) it results that $b(x,y)>0.$
Then the second implication of (5.7) shortly, becomes

$$\int_a^b[\eta'V_y+(D\eta)'V_{\dot{y}}]\,\mathrm{d}t<-\|\theta\|^2\{\sum_{i=1}^{p}\lambda_i[\rho_i'K_i(y)+\rho_i''F_i(y)]+\sum_{\alpha=1}^{r}\rho_\alpha'''\} \tag{5.8}$$

where `

$$V_y=\sum_{i=1}^{p}\lambda_i\left[K_i(y)f_{iy}(t,y,v)-F_i(y,v)k_{iy}(t,y,v)\right]+\mu(t)'g_y(t,y,v)+\upsilon(t)'h_y(t,y,v)\ \}$$

We bring in the second term of under-integral and (5.8) becomes

$$\eta'V_{\dot{y}}\Big|_a^b+\int_a^b\eta'[V_y-\frac{d}{dt}V_{\dot{y}}]\,\mathrm{d}t<-\|\theta\|^2\{\sum_{i=1}^{p}\lambda_i[\rho_i'K_i(y)+\rho_i''F_i(y)]+\sum_{\alpha=1}^{r}\rho_\alpha'''\} \tag{5.9}$$

But $\eta(a,y(a),\dot{y}(a))=\eta(b,y(b),\dot{y}(b))=0$ and taking the first constraint of problem (MFD) into account, relation (5.9) becomes

$$0<-\|\theta\|^2\{\sum_{i=1}^{p}\lambda_i[\rho_i'K_i(y,v)+\rho_i''F_i(y,v)]+\sum_{\alpha=1}^{r}\rho_\alpha'''\} \tag{5.10}$$

By taking the hypothesis e) into account, the inequality (5.10) becomes $0<0,$ that is false.

Then from (5.7) it results

$$\sum_{i=1}^{p}\lambda_i[F_i(x)K_i(y)-K_i(x)F_i(y)]+$$
$$\int_a^b[\mu(t)'g(t,x,\dot{x})+\nu(t)'h(t,x,\dot{x})]\,\mathrm{d}t- \tag{5.11}$$
$$\int_a^b[\mu(t)'g(t,y,\dot{y})+\nu(t)'h(t,y,\dot{y})]\,\mathrm{d}t>0$$

But

$$\int_a^b\left[\mu(t)'g(t,x,\dot{x})+\nu(t)'h(t,x,\dot{x})\right]\mathrm{d}t\leqq 0,$$
$$-\int_a^b[\mu(t)'g(t,y,\dot{y})+\nu(t)'h(t,y,\dot{y})]\,\mathrm{d}t\leqq 0$$

and then relation (5.11) becomes

$$\sum_{i=1}^{p}\lambda_i[F_i(x)K_i(y)-K_i(x)F_i(y)]>0\,.$$

which can be written under the form

$$\sum_{i=1}^{p}\lambda_i K_i(x)K_i(y)\left[\frac{F_i(x)}{K_i(x)}-\frac{F_i(y)}{K_i(y)}\right]>0\,.$$

Since $\lambda_i K_i(x)K_i(y)>0,\ i=\overline{1,s}\ (1\le s\le p)$, from this inequality we infer that

$$\left(\frac{F_1(x)}{K_1(x)}-\frac{F_1(y)}{K_1(y)},...,\frac{F_p(x)}{K_p(x)}-\frac{F_p(y)}{K_p(y)}\right)\le(0,...,0)$$

Or

$$\left(\frac{F_1(x)}{K_1(x)},...,\frac{F_p(x)}{K_p(x)}\right)\le\left(\frac{F_1(y)}{K_1(y)},...,\frac{F_p(y)}{K_p(y)}\right)$$

Therefore $\pi(x)\le\delta(y,\lambda,\mu,\upsilon)$ is false.

**Theorem 5.2** *(Direct duality). Let $x^0$ be a normal efficient solution to the primal (MFP) and assume the hypotheses of Theorem 5.1. Then there exists $\lambda^0\in R^p$ and piecewise smooth functions $\mu^0:I\to R^m$ and $\upsilon^0:I\to R^r$ such that $(x^0,\lambda^0,\mu^o,\upsilon^0)$ is an efficient solution to the dual (MFD) and, moreover, $\pi(x^0)=\delta(x^0,\lambda^0,\mu^0,\upsilon^0)$.*

**Proof:** Because $x^0$ is a regular efficient solution to (MFP), according to Theorem 3.4 there are vector $\lambda^0\in Rn$ and the piecewise nonsmooth functions $\mu^0:I\to R^m$ and $\upsilon^0:I\to R^q$ that satisfy relations $(MFV)_0$. Also, $\upsilon_s^0(t)h_s(t,x^0,\dot{x}^0)=0,\ s=\overline{1,m}$. Then it results that $(x^0,\lambda^0,\mu^0,\upsilon^0)\in\Delta$ and in addition, $\pi(x^0)=\delta(x^0,\lambda^0,\mu^0,\upsilon^0)$.

**Theorem 5.3** *(Converse duality). Let* $(x^0, \lambda^0, \mu^0, \upsilon^0)$ *be an efficient solution to the dual (MFD) and suppose satisfied the following conditions:*

1. $\bar{x}$ is a normal efficient solution to the primal (MFP).
2. $a^0$) For each $i = \overline{1, p}$, $F_i(x^0) > 0$, $K_i(x^0) > 0$.
3. $b^0$) For each $i = \overline{1, p}$, $F_i(x)$ is $(\rho_i', b)$-quasiinvex at $x^0$, and $-K_i(x)$ is $(\rho_i'', b)$-quasiinvex at $x^0$, all with respect to $\eta$ and $\theta$.
4. $c^0$) $\int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x, \dot{x}) + \upsilon_{K_\alpha}(t)' h_{K_\alpha}(t, x, \dot{x})]\,\mathrm{d}t$ is $(\rho_\alpha''', b)$-quasiinvex at $x^0$ with respect to $\eta$ and $\theta$.
5. $d^0$) One of the functions of b $b^0) - c^0)$ is strictly $(\rho, b)$-quasiinvex.
6. $e^0$) $\sum_{i=1}^{p} \lambda_i^0 [\rho_i' K_i(x^0) + \rho_i'' F_i(x^0)] + \sum_{\alpha=1}^{r} \rho_\alpha''' \geqq 0.$

Then $\bar{x} = x^0$ and, moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \upsilon^0)$.

**Proof:** On the contrary, suppose that $\bar{x} \neq x^0$ and we will obtain a contradiction. Because $\bar{x}$ is a normal efficient solution to (MFP) then, according to Theorem 3.4 there are vector $\bar{\lambda} \in R^p$ and vector functions $\bar{\mu} : I \to R^m$, $\bar{\upsilon} : I \to R^q$ that satisfy conditions $(MFV)_0$. It results

$$\bar{\mu}(t)' g(t, \bar{x}, \dot{\bar{x}}) + \bar{\upsilon}(t) h(t, \bar{x}, \dot{\bar{x}}) = 0$$

and so, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\upsilon}) \in \Delta$ .Moreover, $\pi(\bar{x}) = \delta(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\upsilon})$. According to Theorem 5.1 the relation $\pi(\bar{x}) \leq \delta(x^0, \lambda^0, \mu^0, \upsilon^0)$, is false. It results that the relation $\delta(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\upsilon}) \leq \delta(x^0, \lambda^0, \mu^0, \upsilon^0)$, is false. Therefore, the maximal efficiency of $(x^0, \lambda^0, \mu^0, \upsilon^0)$ is contradicted. It results that the supposition $\bar{x} \neq x^0$, above made, is false and we have $\pi(\bar{x}) = \delta(x^0, \lambda^0, \mu^0, \upsilon^0)$.

Note that if $p = 1$ problems (MFP) and (MFD) become the pair of Mond-Weir type dual variational problems [18].

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