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GORDON AND NEWELL QUEUEING NETWORKS AND COPULAS

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Abstract: In this paper we have found an analytical formula for a copula that connects the numbers N_i of customers in the nodes of a Gordon and Newell queueing network. We have considered two cases: the first one is the case of the network with 2 nodes, and the second one is the case of the network with at least 3 nodes. The analytical formula for the second case has been found for the most general case (none of the constants from a list is equal to a given value), and the other particular cases have been obtained by limit**.**

Keywords: Gordon and newell queueing networks, copula.

1. INTRODUCTION

A Jackson queueing network (see [7,4]) is an open queueing network with k nodes where the arrivals from outside network at the node i is $\exp(\lambda_i)$, the service at the node i is $\exp(\mu_i)$, and after it finishes its service at the node i, a customer goes to the node j with the probability P_{ij}, or leaves the network with the probability P_{i0}. We know (see [7,4]) that the arrivals from inside or outside network at the node i are independent with the distribution $\exp(\Lambda_i)$, where Λ_i is the solution of the system

$$\sum_{j=1}^{k} P_{ji} \cdot \Lambda_j + \lambda_i = \Lambda_i, i = \overline{1,k} \tag{1}$$

A Gordon and Newell queueing network (see [5]) is a closed queueing network with k nodes and N customers. The service time in the node i has the distribution

$\exp(\mu_i)$, and after the service in this node the customer goes to the node j with the probability P_{ij}. We have noticed that the matrix P as above is the transition matrix of an ergodic Markov chain (see [6]). If we denote by $(p_i)_{i=\overline{1,k}}$ the ergodic probability we know that

$$P\left(N_1 = n_1, ..., N_k = n_k\right) = \alpha_{N,k}(x_1, ..., x_k) \cdot \prod_{j=1}^{k} x_j^{n_j}, \tag{2}$$

where x_j is proportional to $\frac{p_j}{\mu_j}$, and $\alpha_{N,k}(x_1, ..., x_k)$ is computed such that

$$\sum_{\sum_{j=1}^{k} n_j = N} P(N_1 = n_1, ..., N_k = n_k) = 1. \tag{2'}$$

Obviously, because $\sum_{i=1}^{k} N_i = N$ the above random variables N_i are not independent. They depend through a copula, this term having the following definition (see [10,9]).

Definition 1. *A copula is a function $C : [0,1]^n \to [0,1]$ such that:*

a) If there exists i such that $x_i = 0$ then $C(x_1, ..., x_n) = 0$,

b) If $x_j = 1$ for any $j \neq i$ then $C(x_1, ..., x_n) = x_i$ and

c) C is increasing in each argument.

We have the following theorem (see [10,9]).

Theorem 1. *(Sklar). Let $X_1, X_2, ..., X_n$ be random variables with the cumulative distribution functions $F_1, F_2, ..., F_n$ and the common cumulative distribution function H. In this case there exists a copula C such that $H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n))$. The copula C is well defined on the carthesian product of the marginals $F_1, F_2, ..., F_n$.*

Definition 2. ***([10,11,12]).*** *If n=2 the copula C is Archimedean if $C(u,u) < u$ for any $u \in (0,1)$ and $C(C(u,v),w) = C(u, C(v,w))$ for any $u, v, w \in [0,1]$. If $n > 2$ the copula C is Archimedean if there exists a $n-1$ Archimedean copula C_1 and a 2 Archimedean copula C_2 such that $C(u_1, ..., u_n) = C_2(C_1(u_1, ..., u_{n-1}), u_n)$.*

In [11,12], methods to simulate Archimedean copulas have been presented, and in [3], algorithms to simulate queueing systems with a channel with arrivals and services depending copulas have been shown.

For any $n-$copula C we have (see [1])

$$W(x_1, ..., x_n) \le C(x_1, ..., x_n) \le \min(x_1, ..., x_n), \text{ where} \tag{3}$$

$$W(x_1, ..., x_n) = \sum_{i=1}^{n} x_i - n + 1 \tag{3'}$$

is the lower Fréchet bound, and *min* is the upper Fréchet bound.

2. THE COPULA THAT CONNECTS N_i

We now find a copula C such that

$P(N_1 \le n_1,...,N_j \le n_j) = C(F_1(n_1),...,F_j(n_j))$,

where $2 \le j \le k$ and for $i = \overline{1,j}$ F_i is the cdf of the discrete random variable N_i. We have

$$1-C(F_1(n_1),...,F_j(v)) = j-\sum_{i=1}^{j}F_i(n_i)-\sum_{t=2}^{j-1}(-1)^t\cdot\sum_{i_1<...<i_t}P(N_{i_1}>n_{i_1},...,N_{i_t}>n_{i_t}) +(-1)^{j+1}\cdot P(N_1>n_1,...,N_j>n_j),$$

and from here we obtain

$$C(F_1(n_1),...,F_j(n_j)) = \sum_{i=1}^{j}F_i(n_i)-j+1+\sum_{t=2}^{j-1}(-1)^t\cdot\sum_{i_1<...<i_t}P(N_{i_1}>n_{i_1}, ...,N_{i_t}>n_{i_t})+(-1)^j\cdot P(N_1>n_1,...,N_j>n_j). \quad (4)$$

In the same manner we compute $1-P(N_1>n_1,...,N_j>n_j)$ and we obtain the recurrence formula

$$C(F_1(n_1),...,F_j(n_j)) = (-1)^j\cdot\sum_{i=1}^{j}F_i(n_i)+\sum_{t=2}^{j-1}(-1)^{t+j}\cdot\sum_{i_1<...<i_t}C(F_{i_1}(n_{i_1}), ...,F_{i_t}(n_{i_t}))+(-1)^{j+1}+(-1)^j\cdot P(N_1>n_1,...,N_j>n_j). \quad (4')$$

Proposition 1. If $k=2$ the copula that connects the two nodes is the lower Fréchet bound $C=W$.

Proof: *Consider* F_1 *and* F_2 *the marginals of* N_1 *and* N_2.

First we notice that if $n_1+n_2<N-1$ we have $F_1(n_1)+F_2(n_2)<1$, if $n_1+n_2=N-1$ we have $F_1(n_1)+F_2(n_2)=1$, and if $n_1+n_2>N-1$ we have $F_1(n_1)+F_2(n_2)>1$.

These relations come from the fact that if $n_1+n_2\le N-1$ we have $1-F_1(n_1)-F_2(n_2)=P(n_1<N_1<N-n_2)$ and if $n_1+n_2>N-1$ we have $F_1(n_1)+F_2(n_2)=1-P(n_1\le N_1\le N-n_2)$.

If $n_1+n_2<N$ we have obviously $C(F_1(n_1),F_2(n_2))=P(N_1\le n_1,N_2\le n_2)=0$ because $N_1+N_2=N$ with the probability 1.

If $n_1+n_2\ge N$ we apply the formula (4), and, from the same reason we have $P(N_1>n_1,N_2>n_2)=0$. We obtain $C(F_1(n_1),F_2(n_2))=F_1(n_1)+F_2(n_2)-1$ in this case. It results that $C=W$, and the proposition is proved.

Consider now $k>2$. We compute first $F_i(n_i)$ and $F_i^{-1}(u_i)=h_i(u_i)-1$. We build the Gordon and Newell queueing network with two nodes and the same marginal

F_i for this. First we leave the state corresponding to the node i with the transition probabilities P_{ii} and $1-P_{ii}$, and we group the other states (see [6]). If $(p_j)_{j=\overline{1,k}}$ is the ergodic probability of the $k-states$ Markov chain, we obtain the ergodic Markov chain with the ergodic probability $(p_i, 1-p_i)$. Next, if we denote by $\mu = \sum_{j=1}^{k} \mu_j$, we set the services in these two nodes to $\exp(\mu_i)$ and $\exp(\mu - \mu_i)$. We obtain $P(N_i = n_i) = \alpha_{N,2}(x_i, \tilde{x}_i) \cdot x_i^{n_i} \cdot \tilde{x}_i^{n_i}$, where $x_i = \frac{p_i}{\mu_i}$ and $\tilde{x}_i = \frac{1-p_i}{\mu-\mu_i}$. Obviously, $x_i = \tilde{x}_i$ is equivalent to $x_i = \frac{1}{\mu}$. We denote by $y_i = \frac{x_i}{\tilde{x}_i}$. It results that

$$F_i(n_i) = \begin{cases} \frac{y_i^{n_i+1}-1}{y_i^{N+1}-1} = \frac{y_i^{n_i+1}-1}{\beta_i}, \ if \ y_i \neq 1 \\ \frac{n_i+1}{N+1}, \ if \ y_i = 1 \end{cases}, \text{ and} \tag{5}$$

$$h_i(u_i) = \begin{cases} \frac{\ln(\beta_i \cdot u_i + 1)}{\ln y_i}, \ if \ y_i \neq 1 \\ (N+1) u_i, \ if \ y_i = 1 \end{cases} \tag{5'}$$

We want to find an analytical form for the copula $C(u_1, ..., u_j)$ with $2 \leq j \leq k$. First we can see that

$$F_i(n_i) = x_i^{n_i+1} \cdot \frac{\alpha_{N,k}(x_1, ..., x_k)}{\alpha_{N-n_i-1,k}(x_1, ..., x_k)}. \tag{5''}$$

It results that

$$P(N_1 > n_1, ..., N_j > n_j) = P\left(N_l \geq \sum_{i=1}^{j} (n_i + 1)\right), \tag{6}$$

where l is a node of the network (not necessarily between the j nodes).

Using the formulae (4) and (6) we obtain

$$\begin{aligned} C(u_1, ..., u_j) = C(F_1(n_1), ..., F_j(n_j)) = 1 + \sum_{t=1}^{j} (-1)^t \cdot \\ \sum_{\sum_{r=1}^{t} h_{i_r}(u_{i_r}) < N+1} P\left(N_l \geq \sum_{r=1}^{t} h_{i_r}(u_{i_r})\right) \cdot \prod_{r=1}^{t} \left(\frac{x_{i_r}}{x_l}\right)^{n_{i_r}+1} \end{aligned}. \tag{7}$$

Suppose that we have no $i = \overline{1, j}$ such that $x_i = \frac{1}{\mu}$. We take the value of l between 1 and j, or, in the contrary case ($l > j$), the property $x_l \neq \frac{1}{\mu}$ is fulfilled too. Denote by $\gamma_{i,l}$ and $\delta_{i,l}$ the real numbers such that $y_i^{\gamma_{i,l}} = \frac{x_i}{x_l}$ and $y_i^{\delta_{i,l}} = \frac{x_i \cdot y_l}{x_l}$. Obviously, $\gamma_{i,i} = 0$ and $\delta_{i,i} = 1$. From the formula (5) we obtain

$$\begin{cases} (\frac{x_i}{x_l})^{n_i+1} = (\beta_i \cdot u_i + 1)^{\gamma_{i,l}} \\ (\frac{x_i \cdot y_l}{x_l})^{n_i+1} = (\beta_i \cdot u_i + 1)^{\delta_{i,l}} \end{cases} . \quad (8)$$

We have the following proposition.

Proposition 2. If there exists no $i = \overline{1, j}$ such that $x_i = \frac{1}{\mu}$ we have

$$C(u_1,...,u_j) = 1 + \sum_{t=1}^{j} (-1)^t \cdot \sum_{\sum_{r=1}^{t} h_{i_r}(u_{i_r}) < N+1} \left[\frac{y_l^{N+1}}{\beta_l} \cdot \prod_{r=1}^{t} (\beta_{i_r} \cdot u_{i_r} + 1)^{\gamma_{i_r,l}} - \frac{1}{\beta_l} \cdot \prod_{r=1}^{t} (\beta_{i_r} \cdot u_{i_r} + 1)^{\delta_{i_r,l}} \right].$$

Proof: For the formula from the enunciation we use the formulae (7) and (8). It remains to prove that C is a copula.

The random variables N_i can be considered as continuous random variables on $[0, N+1)$ with the cumulative density function having the same values in the integer arguments. We denote by $m_i = h_i(u_i)$ and we prove that $C(m_1,...,m_j)$ is increasing in each m_1. This property can be proved first if we consider $m_i \in \mathbb{Q} \cap [0, N+1)$ as follows. We consider for each m_1 one rational value, except one i for that we want to prove the monotony. For this i we consider two distinct rational values. We reduce all the involved fractions to the same denominator p and we build a Gordon and Newell queueing network with $N \cdot p$ customers, k nodes and the same x_i. It results that for this network we have $C(m_1,...,m_j) = P(N_1 \le m_1 \cdot p - 1, ..., N_j \le m_j \cdot p - 1)$.

Therefore we have proved the monotony on $\mathbb{Q} \cap [0, N+1)$, and the monotony on $\mathbb{R} \cap [0, N+1)$ results by limit.

If we have an $i = \overline{1, j}$ with $u_i = 0$ we have $h_i(0) = 0$, and each term from the enunciation that contains this u_i appears two times with opposites signs: once with $(\beta_i \cdot u_i + 1)$ and twice without this factor.

This is true including the term $-\left[\frac{y_l^{N+1}}{\beta_l} \cdot (\beta_i \cdot u_i + 1)^{\gamma_{i,l}} - \frac{1}{\beta_l} \cdot (\beta_i \cdot u_i + 1)^{\delta_{i,l}} \right] = -1$, which can be reduced with l from the beginning of the formula.

It results that $C(u_1,...,u_j) = 0$.

If $u_r = 1$ for all $r = \overline{1, j}$ with $r \ne i$ we obtain

$$C(u_1,...,u_j) = 1 - \left[\frac{y_l^{N+1}}{\beta_l} \cdot (\beta_i \cdot u_i + 1)^{\gamma_{i,l}} - \frac{1}{\beta_l} \cdot (\beta_i \cdot u_i + 1)^{\delta_{i,l}} \right].$$

We take now $l = i$ and we obtain

$$C(u_1,...,u_j) = 1 - \frac{y_i^{N+1} - y_i^{n_i+1}}{y_i^{N+1} - 1} = \frac{y_i^{n_i+1} - 1}{y_i^{N+1} - 1} = u_i .$$

It results that C is indeed a copula, and the proposition is proved.

Remark 1. The copula from the above proposition is indeed a continuous function because we can prove in the same way as for the monotony that if $\sum_{r=1}^{t} h_{i_r}(u_{i_r}) = N+1$ the involving term becomes zero. From this we obtain the left continuity, and the right continuity is obvious because the involving term does not appear.

Suppose now that we can have $i = \overline{1,j}$ such that $x_i = \frac{1}{\mu}$, but there exists $l = \overline{1,k}$ such that $x_l \neq \frac{1}{\mu}$. In this case we replace in the above proposition for each i with $u_i = \frac{1}{\mu}$ the expression $(\beta_i \cdot u_i + 1)^{\gamma_{i,l}}$ by $(\frac{x_i}{x_l})^{(N+1)u_i}$, and the expression $(\beta_i \cdot u_i + 1)^{\delta_{i,l}}$ by $(\frac{x_i \cdot y_l}{x_l})^{(N+1)u_i}$. By limit we can prove in this case that C is also a copula.

Finally, we consider the case with $x_i = \frac{1}{\mu}$ for any $i = \overline{1,k}$. Because in fact x_i is only proportional to $\frac{p_i}{\mu_i}$, we can consider $x_i = 1$. We use now the formulae (5') and (7) we obtain

$$C(u_1,...,u_j) = 1 + \sum_{t=1}^{j}(-1)^t \cdot \sum_{\sum_{r=1}^{t} u_{i_r} < 1} \left(1 - \sum_{r=1}^{t} u_{i_r}\right) \tag{9}$$

By limit we can also prove in this case that C is a copula.

3. SOME PROPERTIES FOR THE COPULA THAT CONNECTS N_i

We will compute now for the copula from the previous section the value ρ of Spearman (see [8]):

$$\rho = 12 \cdot \int_0^1\int_0^1 C(u,v)dudv - 3\,. \tag{10}$$

In the case $k = 2$ we have $C = W$, hence

$$\rho = -1\,. \tag{11}$$

In the case $k > 2$ we use the proposition 2 and the recurrence relation (4') . We obtain

$$C(u_1,u_2) = u_1 + u_2 - 1 + \frac{y_1^{N+1}}{\beta_1}(\beta_2 u_2 + 1)^{\gamma_{2,1}} - \frac{1}{\beta_1}(\beta_1 u_1 + 1)(\beta_2 u_2 + 1)^{\delta_{2,1}} \tag{12}$$

if $h_1(u_1) + h_2(u_2) \leq N+1$, and, in the contrary case

$$C(u_1,u_2) = u_1 + u_2 - 1\,. \tag{12'}$$

Suppose first that $y_1 \neq 1$ and $y_2 \neq 1$.

Because $\int_0^1\int_0^1 (u_1+u_2-1)du_1du_2=0$, it results that $I=\int_0^1\int_0^1 C(u_1,u_2)du_1du_2=\frac{y_1^{N+1}}{\beta_1}\cdot$ $\int_{h_1(u_1)+h_2(u_2)\le N+1}(\beta_2u_2+1)^{\gamma_{2,1}}du_1du_2-\frac{1}{\beta_1}\cdot\int_{h_1(u_1)+h_2(u_2)\le N+1}(\beta_1u_1+1)(\beta_2u_2+1)^{\delta_{2,1}}du_1du_2$.

We use now the substitutions $u_i=\frac{y_i^{t_i}-1}{\beta_i}$, $du_i=\frac{\ln y_i}{\beta_i}y_i^{t_i}dt_i$, $(\beta_2u_2+1)^{\gamma_{2,1}}=(\frac{x_2}{x_1})^{t_2}$ and $(\beta_2u_2+1)^{\delta_{2,1}}=(\frac{x_2\cdot y_1}{x_1})^{t_2}$. We obtain

$$I=\frac{y_1^{N+1}\ln y_1\ln y_2}{\beta_1^2\beta_2}\cdot\int_0^{N+1}(\tfrac{x_2y_2}{x_1})^{t_2}\left(\int_0^{N+1-t_2}y_1^{t_1}dt_1\right)dt_2-\frac{\ln y_1\ln y_2}{\beta_1^2\beta_2}\cdot\int_0^{N+1}(\tfrac{x_2y_1y_2}{x_1})^{t_2}\left(\int_0^{N+1-t_2}y_1^{2t_1}dt_1\right)dt_2=$$
$$\frac{y_1^{N+1}\ln y_2}{\beta_1^2\beta_2}\cdot\int_0^{N+1}(\tfrac{x_2y_2}{x_1})^{t_2}(y_1^{N+1-t_2}-1)dt_2-\frac{\ln y_2}{2\beta_1^2\beta_2}\cdot\int_0^{N+1}(\tfrac{x_2y_1y_2}{x_1})^{t_2}(y_1^{2N+2-2t_2}-1)dt_2=$$
$$\frac{y_1^{2N+2}\ln y_2}{2\beta_1^2\beta_2}\cdot\int_0^{N+1}(\tfrac{x_2y_2}{x_1y_1})^{t_2}dt_2-$$
$$\frac{y_1^{N+1}\ln y_2}{\beta_1^2\beta_2}\cdot\int_0^{N+1}(\tfrac{x_2y_2}{x_1})^{t_2}dt_2+\frac{\ln y_2}{2\beta_1^2\beta_2}\cdot\int_0^{N+1}(\tfrac{x_2y_1y_2}{x_1})^{t_2}dt_2.$$

If we have also $x_1y_1\ne x_2y_2$, $x_1\ne x_2y_2$ and $x_1\ne x_2y_1y_2$ we obtain

$$I=\frac{y_1^{2N+2}\ln y_2}{2\beta_1^2\beta_2(\ln x_2+\ln y_2-\ln x_1-\ln y_1)}\cdot((\tfrac{x_2y_2}{x_1y_1})^{N+1}-1)-\frac{y_1^{N+1}\ln y_2}{\beta_1^2\beta_2(\ln x_2+\ln y_2-\ln x_1)}\cdot((\tfrac{x_2y_2}{x_1})^{N+1}-1)+$$
$$\frac{\ln y_2}{2\beta_1^2\beta_2(\ln x_2+\ln y_1+\ln y_2-\ln x_1)}\cdot((\tfrac{x_2y_1y_2}{x_1})^{N+1}-1)$$, and from here

$$\begin{aligned}\rho=&\frac{6y_1^{2N+2}\ln y_2}{\beta_1^2\beta_2(\ln x_2+\ln y_2-\ln x_1-\ln y_1)}\cdot((\tfrac{x_2y_2}{x_1y_1})^{N+1}-1)-\\&\frac{12y_1^{N+1}\ln y_2}{\beta_1^2\beta_2(\ln x_2+\ln y_2-\ln x_1)}\cdot((\tfrac{x_2y_2}{x_1})^{N+1}-1)+\\&\frac{6\ln y_2}{\beta_1^2\beta_2(\ln x_2+\ln y_1+\ln y_2-\ln x_1)}\cdot((\tfrac{x_2y_1y_2}{x_1})^{N+1}-1)-3\end{aligned}. \qquad (13)$$

In the same way we obtain

$$\rho=\frac{6(N+1)\,y_1^{2N+2}\ln y_2}{\beta_1^2\beta_2}-\frac{(9y_1^{N+1}-3)\ln y_2}{\beta_1\beta_2\ln y_1}-3\ if\ x_1y_1=x_2y_2 \qquad (14)$$

$$\rho=\frac{6(y_1^{N+1}+1)\ln y_2}{\beta_1\beta_2\ln y_1}-\frac{12(N+1)\,y_1^{N+1}\ln y_2}{\beta_1^2\beta_2}-3\ if\ x_1=x_2y_2, and \qquad (15)$$

$$\rho=\frac{3(y_1^{N+1}-3)\ln y_2}{\beta_1\beta_2\ln y_1}+\frac{6(N+1)\ln y_2}{\beta_1^2\beta_2}-3\ if\ x_1=x_2y_1y_2. \qquad (16)$$

Suppose now that $y_1\ne 1$ and $y_2=1$. It results that $x_1\ne\frac{1}{\mu}$, $x_2=\frac{1}{\mu}$ and

$$I=\frac{y_1^{2N+2}}{2(N+1)\beta_1^2}\cdot\int_0^{N+1}(\frac{x_2}{x_1y_1})^{t_2}\,dt_2-\frac{y_1^{N+1}}{(N+1)\beta_1^2}\cdot\int_0^{N+1}(\frac{x_2}{x_1})^{t_2}\,dt_2+\frac{1}{2(N+1)\beta_1^2}\cdot\int_0^{N+1}(\frac{x_2y_1}{x_1})^{t_2}\,dt_2\,.$$

If we have also $x_1y_1\neq x_2$ and $x_1\neq x_2y_1$ we obtain

$$I=\frac{y_1^{2N+2}}{2(N+1)\beta_1^2(\ln x_2-\ln x_1-\ln y_1)}\cdot((\frac{x_2}{x_1y_1})^{N+1}-1)-\frac{y_1^{N+1}}{(N+1)\beta_1^2(\ln x_2-\ln x_1)}\cdot((\frac{x_2}{x_1})^{N+1}-1)+$$
$$+\frac{1}{2(N+1)\beta_1^2(\ln x_2+\ln y_1-\ln x_1)}\cdot((\frac{x_2y_1}{x_1})^{N+1}-1)\,,$$

and from here

$$\rho=\frac{6y_1^{2N+2}}{(N+1)\beta_1^2(\ln x_2-\ln x_1-\ln y_1)}\cdot((\frac{x_2}{x_1y_1})^{N+1}-1)-\frac{12y_1^{N+1}}{(N+1)\beta_1^2(\ln x_2-\ln x_1)}\cdot((\frac{x_2}{x_1})^{N+1}-1)+$$
$$\frac{6}{(N+1)\beta_1^2(\ln x_2+\ln y_1-\ln x_1)}\cdot((\frac{x_2y_1}{x_1})^{N+1}-1)-3. \tag{13'}$$

In the same way we obtain

$$\rho=\frac{6y_1^{2N+2}}{\beta_1^2}-\frac{6}{(N+1)\ln y_1}-3\ if\ x_1y_1=x_2,\ and \tag{14'}$$

$$\rho=\frac{3(y_1^{N+1}-3)}{(N+1)\beta_1\ln y_1}+\frac{6}{\beta_1^2}-3\ if\ x_1=x_2y_1 \tag{16'}$$

If $y_1=1$ and $y_2\neq 1$ we switch the indexes *1* and *2* in (13'), (14') and (16').

If we take $p_1=p_2$ and $\mu_1=\mu_2$ we obtain $x_1=x_2$, $y_1=y_2$ and $\beta_1=\beta_2$. We can consider $y_i\neq 1$. It results that

$$I=\int_{h_1(u_1)+h_2(u_2)\le N+1}(1-u_1-u_2-\beta_1u_1u_2)du_1du_2=\int_0^{N+1}\int_0^{N+1-t_1}(1-\frac{y_1^{t_1}-1}{\beta_1}-\frac{y_1^{t_2}-1}{\beta_1}-\frac{y_1^{t_1+t_2}-y_1^{t_1}-y_1^{t_2}+1}{\beta_1})\cdot=$$
$$\frac{\ln^2 y_1}{\beta_1^2}y_1^{t_1+t_2}dt_2dt_1(\frac{\ln^2 y_1}{\beta_1^2}+\frac{\ln^2 y_1}{\beta_1^3})\int_0^{N+1}\int_0^{N+1-t_1}y_1^{t_1+t_2}dt_2dt_1-\frac{\ln^2 y_1}{\beta_1^3}\cdot\int_0^{N+1}\int_0^{N+1-t_1}y_1^{2t_1+2t_2}dt_2dt_1=$$
$$\left(\frac{\ln y_1}{\beta_1^2}+\frac{\ln y_1}{\beta_1^3}\right)\int_0^{N+1}\left(y_1^{N+1}-y_1^{t_1}\right)dt_1-\frac{\ln y_1}{2\beta_1^3}\int_0^{N+1}\left(y_1^{2N+2}-y_1^{2t_1}\right)dt_1=\frac{2(N+1)y_1^{2N+2}\ln y_1-3y_1^{2N+2}+4y_1^{N+1}-1}{4\left(y_1^{N+1}-1\right)^3}$$

We consider now $y_1\to 1$ and we use l' Hôpital, and we obtain $I=\frac{1}{6}$, and from here

$$\rho=-1\,. \tag{17}$$

We will compute now for the copula from the previous section the value τ of Kendall (see [8]):

$$\tau=P((X_1-X_2)(Y_1-Y_2)>0)-P((X_1-X_2)(Y_1-Y_2)<0)=$$
$$4\int_0^1\int_0^1 C(u,v)\frac{\partial^2 C}{\partial u\partial v}dudv-1=1-4\int_0^1\int_0^1\frac{\partial C}{\partial u}\cdot\frac{\partial C}{\partial v}dudv. \tag{18}$$

As for Spearman's ρ, we consider first the case $y_i\neq 1$.

Using the substitutions $u_i = \frac{y_i^{t_i}-1}{\beta_i}$ and (18) we obtain

$$\tau = 1 - 4\int_0^{N+1}\int_0^{N+1} \frac{\partial C}{\partial t_1}\cdot\frac{\partial C}{\partial t_2} dt_1 dt_2 . \quad (18')$$

But for $h_1(u_1) + h_2(u_2) \le N+1$ we have

$$C(t_1,t_2) = u_1 - u_1(\beta_2 u_2 + 1)^{\delta_{2,1}} = \frac{y_1^{t_1}-1}{\beta_1}\cdot\left(1 - \left(\frac{x_2 y_1}{x_1}\right)^{t_2}\right).$$

Therefore $\frac{\partial C}{\partial t_1}\cdot\frac{\partial C}{\partial t_2} = -\frac{(\ln x_2 + \ln y_1 - \ln x_1)\ln y_1}{\beta_1} y_1^{t_1}\left(\frac{x_2 y_1}{x_1}\right)^{t_2}$ in this case.

If $h_1(u_1) + h_2(u_2) > N+1$ we have also

$C(t_1,t_2) = u_1 - \frac{y_1^{N+1}}{\beta_1}(\beta_2 u_2 + 1)^{\gamma_{2,1}} + \frac{(\beta_2 u_2 + 1)^{\delta_{2,1}}}{\beta_1} = \frac{y_1^{t_1}-1}{\beta_1} - \frac{y_1^{N+1}}{\beta_1}\left(\frac{x_2}{x_1}\right)^{t_2} + \frac{1}{\beta_1}\left(\frac{x_2 y_1}{x_1}\right)^{t_2}$, and from here $\frac{\partial C}{\partial t_1}\cdot\frac{\partial C}{\partial t_2} = \frac{(\ln x_2 + \ln y_1 - \ln x_1)\ln y_1}{\beta_1^2} y_1^{t_1}\left(\frac{x_2 y_1}{x_1}\right)^{t_2} - \frac{y_1^{N+1}(\ln x_2 - \ln x_1)\ln y_1}{\beta_1^2} y_1^{t_1}\left(\frac{x_2}{x_1}\right)^{t_2}$. It results that

$$I = \int_0^{N+1}\int_0^{N+1} \frac{\partial C}{\partial t_1}\cdot\frac{\partial C}{\partial t_2} dt_1 dt_2 = \int_0^{N+1}\int_0^{N+1-t_1} \frac{\partial C}{\partial t_1}\cdot\frac{\partial C}{\partial t_2} dt_2 dt_1 + \int_0^{N+1}\int_{N+1-t_1}^{N+1} \frac{\partial C}{\partial t_1}\cdot\frac{\partial C}{\partial t_2} dt_2 dt_1 =$$

$$-\frac{(\ln x_2 + \ln y_1 - \ln x_1)\ln y_1}{\beta_1}\int_0^{N+1} y_1^{t_1}\int_0^{N+1-t_1}\left(\frac{x_2 y_1}{x_1}\right)^{t_2} dt_2 dt_1 + \frac{(\ln x_2 + \ln y_1 - \ln x_1)\ln y_1}{\beta_1^2}\int_0^{N+1} y_1^{t_1}\int_{N+1-t_1}^{N+1}\left(\frac{x_2 y_1}{x_1}\right)^{t_2} dt_2 dt_1$$

$$-\frac{y_1^{N+1}(\ln x_2 - \ln x_1)}{\beta_1^2}\int_0^{N+1} y_1^{t_1}\int_{N+1-t_1}^{N+1}\left(\frac{x_2}{x_1}\right)^{t_2} dt_2 dt_1 .$$

If $x_1 \ne x_2$ and $x_1 \ne x_2 y_1$ we obtain

$$I = \frac{\ln y_1}{\beta_1}\int_0^{N+1} y_1^{t_1} dt_1 - \left(\frac{x_2 y_1}{x_1}\right)^{N+1}\frac{\ln y_1}{\beta_1}\int_0^{N+1}\left(\frac{x_1}{x_2}\right)^{t_1} dt_1 + \left(\frac{x_2 y_1}{x_1}\right)^{N+1}\frac{\ln y_1}{\beta_1^2}\int_0^{N+1} y_1^{t_1} dt_1 -$$
$$\left(\frac{x_2 y_1}{x_1}\right)^{N+1}\frac{\ln y_1}{\beta_1^2}\int_0^{N+1}\left(\frac{x_1}{x_2}\right)^{t_1} dt_1 - \left(\frac{x_2 y_1}{x_1}\right)^{N+1}\frac{\ln y_1}{\beta_1^2}\int_0^{N+1} y_1^{t_1} dt_1 + \left(\frac{x_2 y_1}{x_1}\right)^{N+1}\frac{\ln y_1}{\beta_1^2}\int_0^{N+1}\left(\frac{x_1 y_1}{x_2}\right)^{t_1} dt_1 .$$

We notice that the above condition $x_1 \ne x_2 y_1$ can be avoided by limit. If $x_1 \ne x_2$ and $x_1 y_1 \ne x_2$ we obtain

$I = 1 - \frac{\ln y_1}{\beta_1^2(\ln x_1 - \ln x_2)}\left(y_1^{2N+2} - \left(\frac{x_2 y_1^2}{x_1}\right)^{N+1}\right) + \frac{\ln y_1}{\beta_1^2(\ln x_1 + \ln y_1 - \ln x_2)}\left(y_1^{2N+2} - \left(\frac{x_2 y_1}{x_1}\right)^{N+1}\right)$, and from here

$$\tau = \frac{4\ln y_1}{\beta_1^2(\ln x_1 - \ln x_2)}\left(y_1^{2N+2} - \left(\frac{x_2 y_1^2}{x_1}\right)^{N+1}\right) - \frac{4\ln y_1}{\beta_1^2(\ln x_1 + \ln y_1 - \ln x_2)}\left(y_1^{2N+2} - \left(\frac{x_2 y_1}{x_1}\right)^{N+1}\right) - 3 . \quad (19)$$

In the same way we obtain

$$\tau = \frac{4N y_1^{N+1}\ln y_1}{\beta_1} + \frac{4(N+1)y_1^{N+1}\ln y_1}{\beta_1^2} - 3,\ \textit{if}\ x_1 = x_2\ \textit{and} \quad (19')$$

$$\tau = 4\beta_1 + \frac{4y_1^{N+1}}{\beta_1} - \frac{4(N+1)y_1^{2N+2}\ln y_1}{\beta_1^2} - 1,\ if\ x_1y_1 = x_2 \tag{19''}$$

If $y_1 = 1$ and $y_2 \neq 1$ we have

$$\tau = \frac{4}{(N+1)^2(\ln x_1 - \ln x_2)^2} - \frac{4}{(N+1)^2(\ln x_1 - \ln x_2)^2}\left(\frac{x_2}{x_1}\right)^{N+1} - \frac{4}{(N+1)(\ln x_1 - \ln x_2)}\left(\frac{x_2}{x_1}\right)^{N+1} - 3, \tag{19'''}$$

and in the case $y_1 \neq 1$ and $y_2 = 1$ we change the indexes in the above formula.

Because $\rho = -1$, in the case $x_i = x_j$ for any $i \neq j$ we have (see [8])

$$\tau = -1. \tag{20}$$

Now we check when the two „$\leq$" from (3) become „$=$". For W we have from the recurrence formula (4') $C(u_1,u_2) = u_1 + u_2 - 1 + P(N_1 > n_1, N_2 > n_2)$. It results that

$$C(u_1,u_2) = u_1 + u_2 - 1 + \frac{y_l^{N+1}}{\beta_l}(\beta_1 u_1 + 1)^{\gamma_{1,l}}(\beta_2 u_2 + 1)^{\gamma_{2,l}} - \frac{1}{\beta_l}(\beta_1 u_1 + 1)^{\delta_{1,l}}(\beta_2 u_2 + 1)^{\delta_{2,l}}\ if\ h_1(u_1) + h_2(u_2) < N+1,\ and \tag{21}$$

$$C(u_1,u_2) = u_1 + u_2 - 1\ if\ h_1(u_1) + h_2(u_2) \geq N+1. \tag{21'}$$

It results that in the case $j = 2$ we have $C = W$ if $h_1(u_1) + h_2(u_2) \geq N+1$. Because the Boole inequality is proved by induction, if there exists i_1 and i_2 such that $h_{i_1}(u_{i_1}) + h_{i_2}(u_{i_2}) < N+1$ we have $C(u_1,...,u_j) > W(u_1,...,u_j)$.

If $h_{i_1}(u_{i_1}) + h_{i_2}(u_{i_2}) \geq N+1$ for any i_1 and i_2 we obtain by using the proposition *2* and the result for $j = 2$ that $C(u_1,...,u_j) = W(u_1,...,u_j)$.

In the case of the upper Fréchet bound *min* and $j = 2$ there exists for any u_1 and u_2 in $(0,1)$ the number u_2' such that $u_2 < u_2' < 1$ and $h_1(u_1) + h_2(u_2') \geq N+1$. It results that

$C(u_1,u_2) \leq C(u_1,u_2') = u_1 + u_2' - 1 < u_1$, and analogously $C(u_1,u_2) < u_2$.

Therefore $C(u_1,u_2) = \min(u_1,u_2)$ iff at least one of the arguments is O or I. In the case $j > 2$ we denote by $u_{i_1} = \min(u_1,...,u_j)$. If $u_{i_1} \in (0,1)$ and there exists another u_{i_2} such that $u_{i_2} \in (0,1)$ we obtain $C(u_1,...,u_j) \leq C(u_{i_1},u_{i_2}) < \min(u_1,...,u_j)$.

In this way we have proved that $C(u_1,...,u_j) = \min(u_1,...,u_j)$ if there exists $u_i = 0$ or, in the contrary case, $u_i = 1$ for $j-1$ indexes.

4. CONCLUSIONS

As we can notice, when we deduce the analytical form from the copula that connects N_i, we can see that we have to consider two cases of the Gordon and Newell queueing network: the network with two nodes, and the network with at least three nodes. This partition into cases can be explained by the fact that for the first case the copula is the lower Fréchet bound W, which is only a $2-copula$, not an $n-copula$ for $n \geq 3$.

For the queueing network with *2* nodes we have $C = W$ in the general case, not only for the Gordon and Newell queueing network. This can be explain by the fact that $N_1 + N_2 = N$, hence the variables are strongly antithetic, as we know for $C = W$.

For the queueing network with at least *3* nodes we have not $C = W$ even if $j = 2$. This is because N_1 and N_2 can increase together by means of decreasing another N_i. But for $j = 2$, C can tend to W if the services in the other nodes tend to infinity (they are very fast reporting to those of the two nodes), because the N customers tend to stay in the two nodes.

For $2 \leq j < k$ we set $N \to \infty$ and we build a Jackson queueing network with the first $k-1$ nodes (the last node of the Gordon and Newell queueing network can be considered as an outside network). The node $i \neq k$ has the same services and the transition probabilities, and the arrivals from outside the network $\exp(P_{ki} \cdot \mu_k)$. If Λ_i from (1) is less than μ_i the obtained Jackson queueing network is stable, and any copula from this paper involving $j < k$ from the first $k-1$ nodes tends to the copula *Prod* (N_i tends to be independent).

An open problem is to find an analytical form for the copula that connects N_i for a more general queueing network then the Gordon and Newell queueing network if $k \geq 3$. We can start with the Buzen queueing network, where the service in the node i depends on the number N_i of customers in that node: its distribution is $\exp(a_i(n_i) \cdot \mu_i)$, where $\mu_i > 0$ and a_i is a given function (see [2]).

As we can notice, for $k = 2$ and the corresponding limit case for $k \geq 3$, the copula $C = W$ is Archimedean, and the same thing we can say about the corresponding limit case for *C=Prod*. Another open problem is to study if in the other cases the obtained copula C is Archimedean and, if not, to obtain another copula that is Archimedean (in the discrete case, we know that the copula is not unique).

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