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PENALTY METHOD FOR FUZZY LINEAR PROGRAMMING WITH TRAPEZOIDAL NUMBERS

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Abstract: In this paper we shall propose an algorithm for solving fuzzy linear programming problems with trapezoidal numbers using a penalty method. We will transform the problem of maximizing a function having trapezoidal fuzzy number values under some constraints into a deterministic multi-objective programming problem by penalizing the objective function for possible constraint violation. Furthermore, the obtained deterministic problem will have only unavoidable inequalities between trapezoidal fuzzy numbers parameters as constraints.

Keywords: Linear programming, trapezoidal fuzzy number, penalty method.

1. INTRODUCTION

Concepts of fuzzy sets theory have "crowded" into a lot of research fields since 1980, because of the great success of fuzzy logic application in the control systems theory. The real advantages of the fuzzy approach to solving optimization problems can be highlighted when its comparison to the stochastic methods is made in order to deal with imprecision [4, 6]. Starting with Zimmermann [8], a lot of papers have been written covering the field of fuzzy linear programming.

Buckley and Feuring dealt with the fully fuzzified linear programming problem (FFLP) by establishing all the coefficients and variables of a linear program as fuzzy quantities [1]. They transformed the fully fuzzified programming problem into a multi-objective deterministic problem (MODP) which is, in the general case treated, non-linear. In such a case, the problem is transformed into a multi-objective fuzzy problem with whose help the authors explore the entire set of Pareto-optimal solutions of the MODP. The multi-objective fuzzy problem is solved by using a genetic algorithm that leads to feasible solutions for the initial problem.

In this paper we have transformed the problem of maximizing the function having trapezoidal fuzzy number values under some constraints into a deterministic multi-objective programming by penalizing the objective for the possible constraint violation. The obtained deterministic problem has constraints such as the unavoidable inequalities between parameters of trapezoidal fuzzy numbers.

Two distinct optimization methods have been combined here in order to develop an algorithm for solving a new programming model. The first one is the penalty method which has replaced a constrained optimization problem with an unconstrained one by adding a penalty function to the objective function that depends on the value of the constraints. The second one is the weighted sum method of solving multiple objective programming problems.

We have applied the extension principle of Zadeh to add fuzzy numbers and an approximate version of the same principle to multiply fuzzy numbers. The application of the results obtained by Jamison and Lodwick [3] have been considered and discussed here.

The paper is organized into 6 sections. In Section 2 the model of the fuzzy linear problem (FLP) has been introduced. The aggregation of trapezoidal fuzzy numbers has been presented in Section 3. In Section 4 we have proposed a method of solving FLP. According to Jamison and Lodwick [3], to solve FLP we need to define a new objective function by penalizing the initial objective for the possible constraint violation. Section 5 has described a numerical example in order to clarify the developed theory. Short concluding remarks have been made in Section 6.

2. THE MODEL OF THE FUZZY LINEAR PROBLEM

Let us consider the following fuzzy linear programming problem

$$\max\left(\overline{Z} = \sum_{j=1}^{n} \overline{C_j X_j}\right) \tag{1}$$

subject to

$$\begin{cases} \sum_{j=1}^{n} \overline{A_{ij} X_j} - \overline{B_i} \le \overline{0}, & i = \overline{1,m} \\ \overline{X_j} \ge \overline{0}, & j = \overline{1,n} \end{cases} \tag{2}$$

1. $\overline{C_j},\ j = \overline{1,n}$ represent the coefficients of the linear objective function,
2. $\overline{A_{ij}}, \overline{B_i},\ i = \overline{1,m},\ j = \overline{1,n}$ represent the coefficients and the right hand side of the linear constraints respectively and
3. Quantities $\overline{X_j},\ j = \overline{1,n}$ represent the decision variables.

4. The notation $\overline{\bullet}$ means that $\bullet$ represents a fuzzy quantity described by a trapezoidal fuzzy number.

As we shall see in the next section, if $\overline{C_j}$, $\overline{X_j}$, $\overline{A_{ij}}$ and $\overline{B_i}$ are trapezoidal fuzzy numbers for each $i = 1,...,m$ and $j = 1,...,n$ then $\overline{Z}$ is a fuzzy number which can be approximated by a trapezoidal fuzzy number.

3. THE AGGREGATION OF TRAPEZOIDAL FUZZY NUMBERS

The purpose of this section is to recall some concepts which will be needed in the sequence. Also, we have defined the way of aggregation of trapezoidal fuzzy numbers.

Definition 3.1 *([9]) Trapezoidal fuzzy number* $\overline{Y}$ *is a quadruple* $(y^1, y^2, y^3, y^4) \in R^4$. *The membership function of* $\overline{Y}$ *is defined in connection with its real parameters as follows:*

$$\overline{Y}(x) = \begin{cases} 0, & x \in \left[y^4, +\infty\right) \\ \dfrac{x - y^3}{y^2 - y^3}, & x \in \left[y^3, y^4\right] \\ 1, & x \in \left[y^2, y^3\right] \\ \dfrac{x - y^1}{y^2 - y^1}, & x \in \left[y^1, y^2\right] \\ 0, & x \in \left(-\infty, y^1\right] \end{cases} \qquad \square$$

The value of the membership function $\overline{Y}(x)$ above evaluated in x represents a number in [0, 1].

The extension principle was formulated by Zadeh in order to extend the known models implying fuzzy elements to the case of fuzzy entities. By applying this principle, the following definitions of the addition and subtraction of trapezoidal fuzzy numbers have resulted.

Definition 3.2 *Being given two trapezoidal fuzzy numbers* $\overline{A}$, $\overline{B}$ *with the real number parameters* $\overline{A} = (a^1, a^2, a^3, a^4)$ *and* $\overline{B} = (b^1, b^2, b^3, b^4)$, *the following operators are defined.*

$$\begin{aligned} \overline{A} + \overline{B} &= (a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4), \\ \overline{A} - \overline{B} &= (a^1 - b^4, a^2 - b^3, a^3 - b^2, a^4 - b^1) \end{aligned} \qquad \square$$

By applying the principle of extension to multiply trapezoidal fuzzy numbers a trapezoidal fuzzy number is not obtained. According to [9] we could use α-cuts method to describe the membership function of the result.

The α-cuts of fuzzy numbers $\overline{A} = (a^1, a^2, a^3, a^4)$ and $\overline{B} = (b^1, b^2, b^3, b^4)$ are the intervals

$$\left[(a^2 - a^1)\alpha + a^1, (a^3 + a^4)\alpha + a^4\right],$$
$$\left[(b^2 - b^1)\alpha + b^1, (b^3 - b^4)\alpha + b^4\right]$$

Considering these intervals with non-negative endpoints and making a multiplication of them, an interval [*l, r*] is obtained, where

$$l = (a^2 - a^1)(b^2 - b^1)\alpha^2 + \left[(a^2 - a^1)b^1 + (b^2 - b^1)a^1\right]\alpha + a^1 b^1,$$

$$r = (a^3 - a^4)(b^3 - b^4)\alpha^2 + \left[(a^3 - a^4)b^4 + (b^3 - b^4)a^4\right]\alpha + a^4 b^4.$$

Consequently, the membership function for fuzzy number $\overline{C} = \overline{A} \cdot \overline{B}$ is

$$\overline{C}(x) = \begin{cases} 0, & x \in \left[a^4 b^4, +\infty\right) \\ p^2 - \sqrt{m^2 + n^2 x}, & x \in \left[a^3 b^3, a^4 b^4\right] \\ 1, & x \in \left[a^2 b^2, a^3 b^3\right] \\ -q^2 + \sqrt{t^2 + s^2 x}, & x \in \left[a^1 b^1, a^2 b^2\right] \\ 0, & x \in \left(-\infty, a^1 b^1\right] \end{cases}$$

Parameters p, m, n, q, t, s are real numbers and could be computed starting from the parameters of trapezoidal fuzzy numbers $\overline{A}$ and $\overline{B}$. Moreover, membership function $\overline{C}(x)$ is convex and it increases on the interval $\left[a^1 b^1, a^2 b^2\right]$. Function $\overline{C}(x)$ is concave and it decreases on the interval $\left[a^3 b^3, a^4 b^4\right]$. We will take into consideration an approximate version for Zadeh's principle in order to obtain a trapezoidal fuzzy number as the result for the multiplication of two trapezoidal fuzzy numbers. We shall work further with Definition 3.3 to solve a linear problem with trapezoidal fuzzy numbers.

Definition 3.3 *Being given two trapezoidal fuzzy numbers* $\overline{A}$, $\overline{B}$ *with real number parameters* $\overline{A} = (a^1, a^2, a^3, a^4)$, $\overline{B} = (b^1, b^2, b^3, b^4)$ *we define*

$$\overline{A} \cdot \overline{B} = (a^1 b^1, a^2 b^2, a^3 b^3, a^4 b^4). \qquad \square$$

In order to compare one trapezoidal fuzzy number with trapezoidal fuzzy number $\overline{0}$ Definition 3.4 has to be taken into consideration. It is based on the idea of making a comparison between the left and the right surfaces of trapezes in which initial trapeze is split by the axe *Oy* when a graphical representation of the fuzzy number is made.

Definition 3.4 *Being given the trapezoidal fuzzy number* $\overline{A}$ *parametrically described by* (a^1, a^2, a^3, a^4) *we define*

$$\max(\overline{A}, \overline{0}) = \begin{cases} \overline{A}, & a^1 + a^2 + a^3 + a^4 \le 0 \\ \overline{0}, & \text{otherwise} \end{cases}. \qquad \square$$

4. THE SOLVING ALGORITHM

In Problem (1)-(2) variables were denoted by $\overline{X} = (\overline{X_j})_{j=1,\dots,n}$. Without including any of their descriptions, it was mentioned that they are trapezoidal fuzzy numbers. From now on, the parametric description of the trapezoidal number X_j will be denoted by $(x_j^1, x_j^2, x_j^3, x_j^4)$ for each $j = 1,\dots,n$. Also, for $(x_1^i, x_2^i, \dots, x_n^i)$ notation X^i will be used for each $i = 1,2,3,4$. In the same way, for each $i = 1,2,3,4$, A^i represent the matrix of i-th components of trapezoidal fuzzy numbers which are components of matrix *A*.

The solving method is the following one. Firstly, we have transformed the Problem (1)-(2) of maximizing a function having fuzzy number values under some constraints into an unconstrained fuzzy programming problem (3) by penalizing the objective function for possible constraint violation.

$$\max(\overline{CX} - \overline{d}\max\left\{0, \overline{AX} - \overline{B}\right\}) \tag{3}$$

Secondly, we have aggregated the trapezoidal fuzzy numbers according to Definition 3.1, Definition 3.2 and Definition 3.3. Thus, the function to be optimized in (3) is a trapezoidal fuzzy number and unavoidable constraints representing inequalities between parameters have to be added. Problem (4)-(5) has been obtained.

$$\max \overline{Z} \tag{4}$$

subject to

$$0 \le X_1^i \le X_2^i \le X_3^i \le X_4^i, \quad i = 1,\dots,n \tag{5}$$

where

$$\overline{Z} = (z^1, z^2, z^3, z^4) \tag{6}$$

$$z^1 = C^1X^1 - d^4 \max\left\{0, A^4X^4 - B^1\right\}, \tag{7}$$

$$z^2 = C^2X^2 - d^3 \max\left\{0, A^3X^3 - B^2\right\}, \tag{8}$$

$$z^3 = C^3X^3 - d^2 \max\left\{0, A^2X^2 - B^3\right\}, \tag{9}$$

$$z^4 = C^4X^4 - d^1 \max\left\{0, A^1X^1 - B^4\right\}. \tag{10}$$

Thirdly, we have changed the maximization of the fuzzy objective function with the deterministic maximization of the four components of the fuzzy number, as described by Buckley and Feuring in [1], and using again notations (7)-(10), we have obtained multi-objective programming problem (11) subject to (5).

$$\max(z^1, z^2, z^3, z^4) \tag{11}$$

Consequently, Problem (1)-(2) has been reduced to a deterministic multiple criteria programming problem which can be solved using classical methods [6].

Solving Problem (11)-(5) will allow us to obtain solution $(x_j^1, x_j^2, x_j^3, x_j^4)^*_{j=\overline{1,n}}$, namely the trapezoidal fuzzy numbers $(\overline{X_j})^*_{j=\overline{1,n}}$ representing the solution of Problem (1)-(2).

The algorithm's steps are described below.

1 **Step 1.** The input parameters of fuzzy coefficients for Problem (1)-(2).

2 **Step 2.** Constructing parameters z^1, z^2, z^3, z^4 for Problem (11) using formulas (7)-(10).

3 **Step 3.** Solving multiple criteria Problem (11)-(5) by the weighted sum method.

4 **Step 4.** Constructing the fuzzy solution of Problem (1)-(2) from the deterministic solutions of Problem (11)-(5).

5. NUMERICAL EXAMPLE

In order to illustrate the algorithm for solving linear programming with trapezoidal fuzzy numbers using a penalty method, let us consider the following deterministic linear program with one single objective function

$$\max(z = 7x_1 + 5x_2) \tag{12}$$

subject to

$$\begin{cases} 2x_1 + x_2 \le 4, \\ x_1 + x_2 \le 3, \\ x_1, x_2 \ge 0. \end{cases} \tag{13}$$

The optimal solution of the crisp problem is $x_1 = 1$, $x_2 = 2$ and the optimal value is $z = 17$. Now we have attached to this problem a problem with trapezoidal fuzzy numbers, considering its real number coefficient c as being symmetric trapezoidal fuzzy number $\overline{c}$ of spread 1.2, having the following form

$$\overline{c} = (c^1, c^2, c^3, c^4) = (c - 0.6, c - 0.2, c + 0.2, c + 0.6).$$

Thus the linear programming problem with trapezoidal fuzzy numbers which we want to solve is

$$\max(\overline{c_1 x_1} + \overline{c_2 x_2})$$

subject to

$$\begin{cases} \overline{a_{11}x_1} + \overline{a_{12}x_2} - \overline{b_1} \le \overline{0}, \\ \overline{a_{21}x_1} + \overline{a_{22}x_2} - \overline{b_2} \le \overline{0}, \\ \overline{a_{31}x_1} + \overline{a_{32}x_2} - \overline{b_3} \le \overline{0}, \\ \overline{x_1}, \overline{x_2} \ge \overline{0}. \end{cases}$$

The coefficients' values are

$$\overline{a} = \begin{bmatrix} (1.4,1.8,2.2,2.6)(0.4,0.8,1.2,1.6) \\ (0.4,0.8,1.2,1.6)(0.4,0.8,1.2,1.6) \end{bmatrix},$$
$$\overline{b} = \begin{bmatrix} (3.4,3.8,4.2,4.6) \\ (2.4,2.8,3.2,3.6) \end{bmatrix}, \qquad .$$
$$\overline{c} = \left[(6.4,6.8,7.2,7.6)(4.4,4.8,5.2,5.6)\right]$$

In order to obtain a synthesis function of the four objective functions and applying to it the results presented in [7], we have used the importance coefficients $\pi^1 = 0.25$, $\pi^2 = 0.25$ $\pi^3 = 0.25$ and $\pi^4 = 0.25$ respectively.

The optimum of the synthesis function $\pi^1 z^1 + \pi^2 z^2 + \pi^3 z^3 + \pi^4 z^4$ is reached in

$$X = \overline{x_1} = (1,1,1,1), \overline{x_2} = (2,2,2,2).$$

The value of the objective function in this point is $Z = (15.2,16.4,17.6,18.8)$.

Thus $\overline{x_1}$ and $\overline{x_2}$ as trapezoidal fuzzy numbers describe the real numbers 1 and 2 respectively, which represents the solution X to Problem (12)-(13). Also, the trapezoidal fuzzy number Z describes very closely the real number 17 which represents the optimal value of Problem (12)-(13).

6. CONCLUDING REMARKS

In this paper we have proposed a method of solving a linear programming problem with trapezoidal fuzzy numbers as parameters and variables.

We have transformed the problem of maximizing a function having trapezoidal fuzzy number values into a deterministic multiple criteria programming problem. We have applied the extension principle of Zadeh and an approximate version of it to aggregate trapezoidal fuzzy numbers. The results obtained by Buckley and Feuring in 2000 and Jamison and Lodwiwick in 2001 have been taken into consideration here. An illustrative numerical example has been given and solved to clarify the developed theory.

In our opinion, there are many other points of research which should be studied later on. One of these points is to use different order penalty functions (for example as in [3]) in the optimization model and to analyse the convergence of the algorithms.

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