Yugoslav Journal of Operations Research
Volume 19 (2009) Number 2, 263-279
DOI:10.2298/YUJOR0902263N

AN ENUMERATIVE ALGORITHM FOR NON-LINEAR MULTI-LEVEL INTEGER PROGRAMMING PROBLEM [1]

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Received: August 2007 / Accepted: September 2009

Abstract: In this paper a multilevel programming problem, that is, three level programming problem is considered. It involves three optimization problems where the constraint region of the first level problem is implicitly determined by two other optimization problems. The objective function of the first level is indefinite quadratic, the second one is linear and the third one is linear fractional. The feasible region is a convex polyhedron. Considering the relationship between feasible solutions to the problem and bases of the coefficient sub-matrix associated to the variables of the third level, an enumerative algorithm is proposed, which finds an optimum solution to the given problem. It is illustrated with the help of an example.

Keywords: Multilevel programming, indefinite quadratic programming, fractional programming, quasi-concave function, integer programming.

1. INTRODUCTION

There are many planning and/or decision making situations that can be properly represented by a multi level programming model. The most important characteristic of multilevel programming problems is that a planner at a certain level of hierarchy may

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[1] Classification No.:
Primary: 90C20, 90C10;
Secondary: 90C32, 90C08

have his objective function and decision space determined partially by other levels. Mathematically, a multi level programming problem can be formulated as

$$\underset{X_1}{Max}\, f_1(X_1,....,X_k)$$

$$\underset{X_2}{Max}\, f_2(X_1,....,X_k)$$

$$\vdots$$

$$\underset{X_k}{Max}\, f_k(X_1,....,X_k)$$

$$(X_1, X_2, ..., X_k) \in S$$

where S is a convex feasible set and $f_j(X_1, ..., X_k)$, j = 1, 2, ...,k be linear / non-linear with respect to $X_1, X_2, \ldots, X_k$ where $X_1 = (x_{11}, x_{12}, \ldots, x_{1n_1})$, …., $X_k = (x_{k1}, x_{k2}, ..., x_{kn_k})$.

This system has interacting decision making units within a hierarchical structure where each level performs its policies after knowing completely the decisions of superior levels.

A multilevel programming problem can be found in many decision making situations. Candler and Norton [6] presented a version of this problem in an economic policy context.

A number of methods have been developed to solve a multilevel programming problem. The most notable among them are cutting plane method [2, 8, 9, 19], branch and bound method [7, 15, 17] and by ranking the extreme points [3, 12]. These algorithms have been applied to a number of special problems such as the optimization of a concave quadratic function and bilinear programming problems [8]. H. Konno and T. Kuno [9] have proposed an algorithm for solving a linear multiplicative programming problem by the combination of the parametric simplex method and the standard convex minimization method.

H.I. Calvete and C. Gale [4] have shown the existence of an extreme point which solves a bi-level programming problem where the objective functions of both levels are quasi-concave.

(BLPP) has been used by researchers in several fields ranging from economics to transportation engineering. (BLPP) is used to model problems involving multiple decision makers. These problems include traffic signal optimization [16], structural design [14] and genetic algorithms [5]. A parametric method for solving (BLPP) has been discussed by Faisca, Dua, Rustem, Saraiva and Pistikopoulas [13].

A bibliography of references on bi-level and multilevel programming problems, which is updated biannually, can be found in [18].

2. APPLICATIONS

Consider a programming problem in which the government is at first level. During the planning period, the government proposes certain goals. In order to optimize the achievement of such goals, it formulates certain policy measures such as taxes and subsidies. The industries at the second level design their course of action keeping such policy measures in mind so that their objectives are fulfilled. The industries supply their products to the consumers in a certain area. The customers at the third level are at liberty to make their purchases from any industry. In doing so, the customers will consider economic criteria such as cost optimization.

This is a three level programming problem in which the government's objectives are at least in partial conflict with the two sectors industry and consumers, the policy makers face an optimization problem subject to the optimization problems for industries as well as for the consumers.

In this paper, an enumerative algorithm for the three level integer programming problem (TIPP) is developed. The problem is mathematically stated as:

(TIPP): $\underset{X_1}{Max}\, Z_1(X_1, X_2, X_3) = Z_{11}(X_1, X_2, X_3).Z_{12}(X_1, X_2, X_3)$

where X_2 solves

$\underset{X_2}{Max}\, Z_2(X_1, X_2, X_3) = c_2X_1 + d_2X_2 + e_2X_3 + \alpha_2$; for a given X_1

where X_3 solves

$\underset{X_3}{Max}\, Z_3(X_1, X_2, X_3) = \dfrac{c_{31}X_1 + d_{31}X_2 + e_{31}X_3 + \alpha_3}{c_{32}X_1 + d_{32}X_2 + e_{32}X_3 + \beta_3}$; for a given X_1 and X_2

subject to

$A_1X_1 + A_2X_2 + A_3X_3 = b$

$X_1, X_2, X_3 \geq 0$ and integers,

where $Z_{11}(X_1, X_2, X_3) = c_{11}X_1 + d_{11}X_2 + e_{11}X_3 + \alpha_1$

$Z_{12}(X_1, X_2, X_3) = c_{12}X_1 + d_{12}X_2 + e_{12}X_3 + \beta_1$

$X_1 \in \mathbb{R}^{n_1}$, $X_2 \in \mathbb{R}^{n_2}$ and $X_3 \in \mathbb{R}^{n_3}$ are the variables controlled by the leader and the first and second follower respectively.

Here, $c_{11}, c_{12}, c_2, c_{31}, c_{32} \in \mathbb{R}^{n_1}$;

$d_{11}, d_{12}, d_2, d_{31}, d_{32} \in \mathbb{R}^{n_2}$;

$e_{11}, e_{12}, e_2, e_{31}, e_{32}, \in \mathbb{R}^{n_3}$; $\alpha_{11}, \beta_1, \alpha_2, \alpha_3, \beta_3 \in \mathbb{R}$; $A_1 \in \mathbb{R}^{m\times n_1}$;

$A_2 \in \mathbb{R}^{m\times n_2}$; $A_3 \in \mathbb{R}^{m\times n_3}$ and $b \in \mathbb{R}^m$.

Assume that

$(c_{32}X_1 + d_{32}X_2 + e_{32}X_3 + \beta_3) > 0, \forall (X_1, X_2, X_3) \in S$,

where $S = \{(X_1, X_2, X_3);\ A_1X_1 + A_2X_2 + A_3X_3 = b;\ X_1, X_2, X_3 \geq 0\}$ is non-empty and compact. A_3 has full row rank and $m < n_3$.

The projection of S onto $\mathbb{R}^{n_1}$ is denoted by

$$S_1 = \{X_1 \in \mathbb{R}^{n_1} : (X_1, X_2, X_3) \in S\}.$$

The projection of S into $\mathbb{R}^{n_1+n_2}$ is denoted by

$$S_2 = \{(X_1, X_2) \in \mathbb{R}^{n_1+n_2} : (X_1, X_2, X_3) \in S\}.$$

For each $\overline{X}_1 \in S_1$, the feasible region of the first follower's problem is denoted by

$$S(\overline{X}_1) = \{(X_2, X_3) \in \mathbb{R}^{n_2+n_3} : A_2X_2 + A_3X_3 = b - A_1\overline{X}_1;\ X_2, X_3 \geq 0\}.$$

For each $(\overline{X}_1, \overline{X}_2) \in S_2$, the feasible region of the second follower's problem is denoted by

$$S(\overline{X}_1, \overline{X}_2) = \{X_3 \in \mathbb{R}^{n_3} : A_3X_3 = b - A_1\overline{X}_1 - A_2\overline{X}_2;\ X_3 \geq 0\}.$$

It is also assumed that the optimal solution of the first follower and the second follower's problem is singleton.

The inducible region or the feasible region of the relaxed leader's problem is given by

$$IR = \{(\overline{X}_1, \overline{X}_2, \overline{X}_3) : \overline{X}_1 \geq 0;\ X_2, X_3 \text{ solves } \underset{X_2, X_3}{\text{Max}} (c_2\overline{X}_1 + d_2X_2 + e_2X_3 + \alpha_2)$$
$$\text{s.to. } A_1\overline{X}_1 + A_2X_2 + A_3X_3 = b,\ X_2 \geq 0, X_3 \geq 0\}.$$

The inducible region or the feasible region of the first follower's problem is given by

$$IR_1 = \{(\overline{X}_1, \overline{X}_2, \overline{X}_3) : \overline{X}_1 \geq 0;\ \overline{X}_2 \geq 0,\ \ X_3 \text{ solves } \underset{X_3}{\text{Max}} \frac{c_{31}\overline{X}_1 + d_{31}\overline{X}_2 + e_{31}X_3 + \alpha_3}{c_{32}\overline{X}_1 + d_{32}\overline{X}_2 + e_{32}X_3 + \beta_3}$$
$$\text{s.to } A_1\overline{X}_1 + A_2\overline{X}_2 + A_3X_3 = b,\ X_3 \geq 0\}.$$

In the above (TIPP) problem, if we remove the restriction that X_1, X_2, X_3 are integers, then the problem reduces to three level programming problem.

Result: The optimal solution to the three level programming problem (TPP) occurs at an extreme point of S, provided S is regular.

Proof. *Since $Z_{11}(X_1, X_2, X_3)$ and $Z_{12}(X_1, X_2, X_3)$ are positive for all $(X_1, X_2, X_3) \in S$, therefore, $Z_1(X_1, X_2, X_3)$ is both quasi-concave and quasi-convex on S.*

$Z_2(X_1, X_2, X_3)$ is linear; hence it is both convex and concave. Since it is also differentiable, therefore, in particular, it is quasi-concave. $Z_3(X_1, X_2, X_3)$ is a ratio of two affine functions, hence it is quasi-concave.

Thus, we get that the objective function at each level of (TPP) is quasi concave in nature and maximum of quasi-concave function occurs at an extreme point. If S is regular, there is an extreme point of the feasible region S which is an optimal solution to the quasi-concave (TPP) problem.

3. ALGORITHMIC DEVELOPMENT

In the course of the algorithm, the Gomory cut is applied to obtain optimal, integer solution of (TIPP). For each $\bar{X}_1 \in S_1$, a point of the inducible region (IR) is obtained by solving the linear programming problem:

$$LP(\bar{X}_1) : \underset{X_2}{Max}\, d_2 X_2 + e_2 X_3 + \bar{\alpha}_2$$

subject to $(X_2, X_3) \in S(\bar{X}_1)$

where $\bar{\alpha}_2 = c_2 \bar{X}_1 + \alpha_2$.

For each $(\bar{X}_1, \bar{X}_2) \in S_2$, a point of IR_1 is obtained by solving the linear fractional programming problem:

$$FP(\bar{X}_1, \bar{X}_2) : \qquad \underset{X_3}{Max} \frac{e_{31} X_3 + \bar{\alpha}_3}{e_{32} X_3 + \bar{\beta}_3}$$

subject to $X_3 \in S(\bar{X}_1, \bar{X}_2)$,

$$\bar{\alpha}_3 = c_{31} X_1 + d_{31} X_2 + \alpha_3$$

$$\bar{\beta}_3 = c_{32} X_1 + d_{32} X_2 + \beta_3 .$$

Hence, an extreme point X_3 of $S(\bar{X}_1, \bar{X}_2)$ can be found which solves $FP(\bar{X}_1, \bar{X}_2)$ and the point so obtained $(\bar{X}_1, \bar{X}_2, \bar{X}_3) \in IR_1$. Since a basis B of A_3 is associated to $\bar{X}_3$, we can associate a basis B of A_3 to each point of IR and IR_1. Therefore, we need only to consider it.

To solve $FP(\bar{X}_1, \bar{X}_2)$, we consider the parametric approach. Consider the linear parametric problem:

$$LP(\bar{X}_1, \bar{X}_2) : F(\lambda) = Max[(e_{31} X_3 + \bar{\alpha}_3) - \lambda(e_{32} X_3 + \bar{\beta}_3)]$$

Subject to $X_3 \in S(\bar{X}_1, \bar{X}_2)$.

Consider the basis B of A_3.

To obtain points of IR and IR_1, there must exist $\bar{X}_1 \in S_1$ and $(\bar{X}_1, \bar{X}_2) \in S_2$, such that B is a feasible basis of LP $(\bar{X}_1, \bar{X}_2)$. For some λ, B should also verify the optimality conditions of problem LP $(\bar{X}_1, \bar{X}_2)$ and F(λ) = 0 for at least one of the value of λ.

While verifying the optimality conditions, we will get a lower bound λ^ℓ and an upper bound λ^u for λ. Hence, for at least one λ, F(λ) = 0 implies that

$$\lambda^\ell \le \frac{e_{31} X_{3B} + \bar{\alpha}_3}{e_{32} X_{3B} + \bar{\beta}_3} = Z_3(\bar{X}_1, \bar{X}_2, X_{3B}) \le \lambda^u \qquad (1)$$

where X_{3B} stands for the variables of $\bar{X}_3$ associated with the basis B, i.e.,

$$X_{3B} = B^{-1}(b - A_1\bar{X}_1 - A_2\bar{X}_2);$$
$$\bar{X}_1 \geq 0,\ \bar{X}_2 \geq 0,\ B^{-1}(b - A_1\bar{X}_1 - A_2\bar{X}_2) \geq 0.$$

Since a basis B should verify the optimality conditions of the problem $LP(\bar{X}_1, \bar{X}_2)$, before the start of a new iteration, we have to check the optimality condition:

$$\text{(OC)}\ e_{31}^j - \lambda e_{32}^j - (e_{31}^B - \lambda e_{32}^B)B^{-1}A_3^j \leq 0, \quad \forall j \in V_3,$$

where e_{31}^j and e_{32}^j are the jth components of vectors e_{31} and e_{32}, respectively; e_{31}^B and e_{32}^B are the m-row vectors of e_{31} and e_{32}, associated to the basic variables of B, A_3^j is the j^{th} component of A_3 and V_3 is the set of indices controlled by the third level.

While checking this condition, we obtain the interval $[\lambda^\ell, \lambda^u]$ for the parameter λ. If $\lambda^\ell = -\infty$ or $\lambda^u = \infty$, the interval $[\lambda^\ell, \lambda^u]$ will be open at the extremes. If no such λ exists, so that basis B verifies condition (1), then this base is of no interest because it is impossible to obtain a point of the inducible region corresponding to it.

Now, points of IR which corresponds to the basis B is obtained by solving the indefinite quadratic programming problem:

IQP(B) : Max $(c_{11}X_1 + c_{12}X_2 + e_{11}^B X_{3B} + \alpha_1)(c_{12}X_1 + d_{12}X_2 + e_{12}^B X_{3B} + \beta_1)$

subject to $A_1X_1 + A_2X_2 + A_3X_{3B} = b$

$X_1, X_2, X_{3B} \geq 0$ and integers.

Note that while B is analyzed, variables of the third level not associated to B are equal to zero. Suppose that IQP (B) is feasible and (OC) is verified. Let the solution so obtained be $X_1 = X_1^*$. Now, points of IR_1 which corresponds to the basis B is obtained by solving the linear programming problem:

LP(B) : Max $d_2X_2 + e_2X_{3B} + \hat{\alpha}_2$

subject to $A_2X_2 + A_3X_{3B} = b - A_1X_1^*$

$X_2, X_{3B} \geq 0$ and integers,

where $\hat{\alpha}_2 = \alpha_2 + c_2X_1^*$.

Again, while B is analyzed, variables of the third level not associated to B are equal to zero.

Suppose LP(B) is feasible and (OC) is verified, then the optimal solution so obtained $X^* = (X_1^*, X_2^*, X_3^*)$ is the best point of IR_1.

Now, we look for a new basis which can improve the values of Z_1 and Z_2 obtained so far.

Let T be the set of indices associated to basis B. Let V_1, V_2 and V_3 be the set of indices controlled by the first level, second level and third level problems, respectively.

Lemma : Any basis from A_3 capable of providing a point of IR_1 better than X^* must include at least one vector whose index belongs to the set

$C = \{j \in V_3 - T : j \in C_1 \text{ and } j \in C_2\}$, where

$C_1 = \{j \in V_3 - T : L_j < 0\}$

and $C_2 = \{j \in V_3 - T : z_j - c_j < 0\}$,

$L_j = Z_{11}(z_j^{12} - d_j) + Z_{12}(z_j^{11} - c_j) - \theta(z_j^{12} - d_j)(z_j^{11} - c_j)$ is the j^{th} reduced cost coefficient in the optimal integer solution of IQP(B) and $(z_j - c_j)$ is the j^{th} reduced cost coefficient in the optimal integer solution of LP(B).

Proof: (i) Let $Z_1(X^*)$ denote the value of the first level objective function at X^*.

According to X^*, the matrix $[A_1\ A_3]$ is decomposed into $[B\ N]$ where B is an (m×m) basis matrix associated to the basic variables of X^*.
Let X_B^* be a basic feasible solution and $\hat{X}_B$ be the new basic feasible solution obtained by entering a_j into the basis and departing b_r. Then,

$$\hat{X}_{B_i} = X_{B_i}^* - X_{B_r}\frac{Y_{ij}}{Y_{rj}} \quad \text{and} \quad \hat{X}_{B_r} = \frac{X_{B_r}^*}{Y_{rj}} > 0$$

i.e., $\hat{X}_{B_i} = X_{B_i}^* - \theta Y_{ij}$ and $\hat{X}_{B_r} = \theta$.

Given, $Z_{11}^* = C_B^T X_B^* + \alpha_1$ and $Z_{12}^* = D_B^T X_B^* + \beta_1$, the new value of the objective function is

$$Z_1(\hat{X}) = Z_{11}(\hat{X}).Z_{12}(\hat{X})$$

$$= \left(\sum_{i=1}^{m} \hat{c}_{B_i}\hat{X}_{B_i} + \alpha_1\right)\left(\sum_{i=1}^{m} \hat{d}_{B_i}\hat{X}_{B_i} + \beta_1\right)$$

$$= \left(\sum_{\substack{i=1\\ i\neq r}}^{m} c_{B_i}(X_{B_i}^* - \theta Y_{ij}) + \hat{c}_{B_r}\theta + \alpha_1\right)\left(\sum_{\substack{i=1\\ i\neq r}}^{m} d_{B_i}(X_{B_i}^* - \theta Y_{ij}) + \hat{d}_{B_i}\theta + \beta_1\right)$$

$$= \left(\sum_{\substack{i=1\\ i\neq r}}^{m} c_{B_i}(X_{B_i}^* - \theta Y_{ij}) + c_j\theta + \alpha_1\right)\left(\sum_{\substack{i=1\\ i\neq r}}^{m} d_{B_i}(X_{B_i}^* - \theta Y_{ij}) + d_j\theta + \beta_1\right)$$

$[\because \hat{c}_{B_r} = c_j \text{ and } \hat{d}_{B_r} = d_j]$

$$= \left(\sum_{i=1}^{m} c_{B_i}X_{B_i}^* - \theta\sum_{i=1}^{m} c_{B_i}Y_{ij} + c_j\theta + \alpha_1\right)\left(\sum_{i=1}^{m} d_{B_i}X_{B_i}^* - \theta\sum_{i=1}^{m} d_{B_i}Y_{ij} + d_j\theta + \beta_1\right)$$

$$= (Z_{11}^* - \theta z_j^{11} + c_j\theta)(Z_{12}^* - \theta z_j^{12} + d_j\theta)$$

$$= Z_{11}^* Z_{12}^* - \theta(Z_{11}^*(z_j^{12} - d_j) + Z_{12}^*(z_j^{11} - c_j) - \theta(z_j^{11} - c_j)(z_j^{12} - d_j)$$

$$= Z_{11}^* Z_{12}^* - \theta L_j$$

$> Z_{11}^* Z_{12}^*$ ($\because \theta > 0$ and $L_j < 0$)

$= Z_1(X^*)$

Thus, $Z_1(\hat{X}) > Z_1(X^*)$.

Hence, in order to improve the first level, we must consider those a_j's for which $L_j < 0$.

If X^* solves IQP (B), then $L_j \geq 0 \;\; \forall\, j \in V_1$ and $\forall j \in T$.

Let $Z_2(X^*)$ denote the value of the second level objective function at X^*. Again, $[A_2\ A_3]$ can be decomposed into [B N] where B is an $(m\times m)$ basis matrix associated to the basic variables of X^*.

Let X_B^* be the b.f.s. obtained by phase I of the simplex method. Let $\hat{X}_B$ be the new b.f.s. obtained by entering a_j into the basis and departing b_r. Then, the new value of the objective function is

$$Z_2(\hat{X}) = \sum_{\substack{i=1 \\ i\neq r}}^{m} \hat{c}_{B_i} .\hat{X}_{B_i} + \hat{c}_{B_r} \hat{X}_{B_r}$$

$$= \sum_{\substack{i=1 \\ i\neq r}}^{m} c_{B_i} \left(X_{B_i}^* - \frac{X_{B_r}^*}{Y_{rj}} Y_{ij} \right) + c_j \frac{X_{B_r}^*}{Y_{rj}}$$

$$= \sum_{i=1}^{m} c_{B_i} X_{B_i}^* + \frac{X_{B_r}^*}{Y_{rj}} \left(c_j - \sum_{i=1}^{m} c_{B_i} Y_{ij} \right)$$

$= Z_2(X^*) - \theta(z_j - c_j)$

$> Z_2(X^*)$ ($\because \theta > 0$ and $z_j - c_j < 0$)

Thus, $Z_2(\hat{X}) > Z_2(X^*)$.

Hence, in order to improve the second level objective function, we must consider those a_j's for which $(z_j - c_j) < 0$. If X^* solves LP(B), then $(z_j - c_j) \geq 0\ \forall\ j \in V_2$ and $\forall\ j \in T$.

If $C = \phi$, we cannot improve Z_1 and Z_2, hence the current best integer point is optimum to (TIPP). If we have previously built sets $C^1, C^2, ..., C^i$, the new basis B should include at least one index from each sets $C^1, C^2, ..., C^i$.

Let $E_1 = U\ \{C^i\}$. Suppose IQP(B) is not feasible or IQP(B) is feasible but its optimal solution does not verify (1), then this basis is of no longer interest.

If D denotes the set of indices associated to B, then the new basis should not include all vectors with indices in the set D. If we have previously built sets D^1, D^2, ..., D^i, the new basis should not include all vectors with indices in each of these sets. Let $E_2 = U\{D^i\}$.

To select indices which form the new basis, it is suggested to solve for w_j using the following system:

$$(P_I): \sum_j w_j\delta_j \geq 1, \qquad \delta_j = \begin{cases} 1, & j \in C,\ C \in E_1 \\ 0, & otherwise \end{cases}$$

$$\sum_j w_j\delta_j \leq \sum_j \delta_j - 1, \quad \delta_j = \begin{cases} 1, & j \in D,\ D \in E_2 \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_j w_j = m$$

$w_j \in \{0, 1\}$, $j \in V_3$.

The required indices correspond to j where $w_j = 1$.

Remark: 1. If the basis so formed has rank $k < m$, then $B_N = [\hat{B}, N]$ where $\hat{B}$ is a matrix of independent vectors of B and N is a matrix of (m – k) vectors of A_3 is a basis from A_3.

The optimum of the problem

$$\underset{X_1}{\text{Max}}\ Z_1(X_1, X_2, X_3) = (c_{11}X_1 + d_{11}X_2 + e_{11}X_3 + \alpha_1)(c_{12}X_1 + d_{12}X_2 + e_{12}X_3 + \beta_1)$$

where X_2 solves

$$\underset{X_2}{\text{Max}}\ Z_2(X_1, X_2, X_3) = c_2X_1 + d_2X_2 + e_2X_3 + \alpha_2, \text{ for a given } X_1$$

subject to $(X_1, X_2, X_3) \in S$,

is a lower bound on the optimum of (TIPP). Hence, at some stage of the algorithm if an integer point of IR_1 provides this optimum, then that point is the optimum solution to (TIPP).

Here, we are considering the leader's problem as:

$$(IQP): \underset{X_1}{\text{Max}}\ Z_1(X_1, X_2, X_3)$$

$$= (c_{11}X_1 + d_{11}X_2 + e_{11}X_3 + \alpha_1)\ (c_{12}X_1 + d_{12}X_2 + e_{12}X_3 + \beta_1)$$

subject to $(X_1, X_2, X_3) \in S$.

The first follower's problem is:

(LP): $\underset{X_2}{\text{Max}}\ Z_2(X_1, X_2, X_3) = c_2X_1 + d_2X_2 + e_2X_3 + \alpha_2$, for a given X_1

subject to $(X_1, X_2, X_3) \in S$.

4. STEPWISE DESCRIPTION OF THE ALGORITHM

Step 0 : Solve the problem $\underset{X_1}{Max}\ Z_1(X_1, X_2, X_3)$

subject to $(X_1, X_2, X_3) \in S$

0.1. If it is not feasible, stop. (TIPP) is not feasible.

0.2. If the optimal solution is not an integer solution, then apply Gomory's cutting plane method to find an integer solution. Let $X^* = (X_1^*, X_2^*, X_3^*)$ be an optimal integer solution.

Step 1: Put $X_1 = X_1^*$ in the first follower's problem. Find the integer solution to $LP(X_1^*)$. Let (X_2^1, X_3^1) be its optimal solution.

1.1. If $X_2^* = X_2^1$, go to step 3, otherwise go to step 2.

Step 2: If $X_2^* \neq X_2^1$, find the second best solution of leader's problem (IQP) and go to step 1.

Step 3 : Put $X_1 = X_1^*$ and $X_2 = X_2^*$ in the second follower's problem. Solve it by the parametric approach. Let $\hat{X}_3$ be the optimal integer solution.

If $X_3^* = \hat{X}_3$, then, (X_1^*, X_2^*, X_3^*) is the optimal integer solution for (TIPP).

If $X_3^* \neq \hat{X}_3$, stop. $(X_1^*, X_2^*, \hat{X}_3)$ is the current best integer point of IR_1. Set $E_1 = \phi$, $E_2 = \phi$.

Step 4 : Solve IQP (B).

4.1. If IQP (B) is not feasible or if optimal solution does not verify (1), then compute D. Set $E_2 = E_2 \cup \{D\}$. Go to step 7.

4.2. If IQP (B) is feasible and (1) is verified, then compare this optimal solution with the current best integer point of IR_1 and update if necessary. Let the optimal integer point be (X_1^o, X_2^o, X_3^o). Construct $C_1 = \{j \in V_3 - T : L_j < 0\}$.

Step 5: Solve LP(B) for a given $X_1 = X_1^o$.

5.1. Let its optimal integer solution be (X_2^{**}, X_3^{**}) and (1) is verified.

5.2. If $X_2^{**} \neq X_2^o$, then find the next best solution of IQP (B) and go to step 4.

5.3. If $X_2^{**} = X_2^o$, construct $C_2 = \{j \in V_3 - T: z_j - c_j < 0\}$.

Step 6 : Computer C = {j ∈ V_3 – T: j ∈ C_1 and j ∈ C_2}.

6.1. If C = ϕ, stop. The current best integer point of IR_1 is the optimal solution to (TIPP). Otherwise set $E_1 = E_1 \cup \{C\}$.

Step 7 : Solve P_I.

7.1. If P_I is not feasible, stop. The current best integer point of IR_1 is optimal to (TIPP).

Step 8: If P_I is feasible, construct B and compute $[\lambda^{\ell}, \lambda^{u}]$ by solving (OC). If no solution exist for (OC), compute D, set $E_2 = E_2 \cup \{D\}$ and go to step 8. Otherwise go to step 4.

EXAMPLE : Consider the three level integer programming problem:

(TIPP) : $\underset{x_1}{Max}\, Z_1 = (x_1 + x_2 + 2x_3 + 4)(-x_1 - x_2 + x_3 + 2x_4 + 1)$

where x_2, solves

$$\underset{x_2}{\text{Max}}\ Z_2 = 2x_2 + x_3 + 3x_4$$

where x_3, x_4, x_5, x_6, x_7, x_8 solves, for a given x_1 and x_2

$$Max\, Z_3 = \frac{2x_1 + 3x_2 + 2x_3 - 3x_4}{5x_1 + 11x_2 + x_5 + 29}$$

subject to

$$-3x_1 + 7x_2 + x_3 + x_5 = 10$$

$$14x_1 + 4x_2 + x_6 = 6$$

$$x_1 + x_2 + x_3 - x_4 + x_7 = 5$$

$$2x_1 + x_2 + 2x_4 + x_8 = 8$$

$x_1, x_2, ..., x_8 \geq 0$ and integers.

Solution : Solve (IQP) given by

(IQP) :Max $Z_1 = (x_1 + x_2 + 2x_3 + 4)\,(-x_1 - x_2 + x_3 + 2x_4 + 1)$

subject to

$$-3x_1 + 7x_2 + x_3 + x_5 = 10$$

$$14x_1 + 4x_2 + x_6 = 6$$

$$x_1 + x_2 + x_3 - x_4 + x_7 = 5$$

$$2x_1 + x_2 + 2x_4 + x_8 = 8$$

$x_1, ..., x_8 \geq 0$ and integers.

The optimal table of (IQP) is given by

$c_j \rightarrow$				1	1	2	0	0	0	0	0
$d_j \rightarrow$				–1	–1	1	2	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	0	x_5	1	–5	11/2	0	0	1	0	–1	–1/2
0	0	x_6	6	14	4	0	0	0	1	0	0
2	1	x_3	9	2	3/2	1	0	0	0	1	½
0	2	x_4	4	1	½	0	1	0	0	0	½
$Z_{11} = 22$		$z_j^{11} - c_j \rightarrow$		3	2	0	0	0	0	2	1
$Z_{12} = 18$		$z_j^{12} - d_j \rightarrow$		5	7/2	0	0	0	0	1	3/2
		$L_j \rightarrow$		164-5θ	113-7θ	0	0	0	0	58-2θ	51-3/2θ

Here, $L_j \geq 0 \quad \forall$ j.
The optimal integer solution is given by

$$X^* = (X_1^*, X_2^*, X_3^*) = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*) = (0,0,9,4,1,6,0,0)$$

Put $x_1^* = 0$ and solve the follower's problem:

$LP(x_1^*)$ = Max $Z_2 = 2x_2 + x_3 + 3x_4$
subject to

$7x_2 + x_3 + x_5 = 10$

$4x_2 + x_6 = 6$

$x_2 + x_3 - x_4 + x_7 = 5$

$x_2 + 2x_4 + x_8 = 8$

$x_2, ..., x_8 \geq 0$ and integers.

The optimal integer solution is

$$(X_2^1, X_3^1) = (x_2^1, x_3^1, x_4^1, x_5^1, x_6^1, x_7^1, x_8^1) = (0,\ 9,\ 4,\ 1,\ 6,\ 0,\ 0).$$

We get $x_2^1 = x_2^* = 0$.
Put $x_1^* = 0$ and $x_2^* = 0$ in the second follower's problem:

FP (x_1^*, x_2^*) : $Max\, Z_3 = \dfrac{2x_3 - 3x_4}{x_5 + 29}$

subject to

$$\begin{aligned} & x_3 + x_5 = 10 \\ & + x_6 = 6 \\ & x_3 - x_4 + x_7 = 5 \\ & 2x_4 + x_8 = 8 \\ & x_3, ..., x_8 \geq 0 \text{ and integers} \end{aligned} \qquad (1)$$

Solve $FP(x_1^*, x_2^*)$ by parametric approach.
The linear parametric problem is given by

$$FP(x_1^*, x_2^*) : F(\lambda) = Max\ Z_3 = (2x_3 - 3x_4) - \lambda(x_5 + 29)$$

subject to the constraints (1).

Optimal integer solution of $LP(x_1^*, x_2^*)$ is given by

$$\hat{X}_3 = (\hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8) = (5,\ 0,\ 5,\ 6,\ 0,\ 8) \text{ with}$$

$[\lambda^{\ell}, \lambda^{u}] = [-2,\ 1]$ and $F(\lambda) = 0$ for $\lambda = 10/34 = 0.29$.

Here, $X_3^* \neq \hat{X}_3$.

The current best integer point of IR_1 is (0, 0, 5, 0, 5, 6, 0, 8).
Set $E_1 = \phi$ and $E_2 = \phi$.
Basis B_1 is given by the vectors with indices 3, 5, 6, and 8.

Solve IQP (B_1) given by
IQP (B_1) : Max $(x_1 + 2x_3 + 4)(-x_1 + x_3 + 1)$
subject to

$-3x_1 + x_3 + x_5 = 10$

$14x_1 + x_6 = 6$

$x_1 + x_3 = 5$

$2x_1 + x_8 = 8$

$x_1, x_3, x_5, x_6, x_8 \geq 0$ and integers.

The optimal integer solution is

$X^o = (x_1^o, x_3^o, x_5^o, x_6^o, x_8^o) = (0, 5, 5, 6, 8)$, with $Z_1 = 84$. Since $Z_3(X^o) = \frac{10}{34} \in [-2, 1]$, condition (1) is verified. Note that variables of the third level not associated to B_1 which are not to be considered while solving problem IQP(B_1) are included in the construction of the set C_1. The reduced cost L_j of x_4 is negative in the optimal solution of problem IQP(B_1), therefore, we have

$C_1 = \{j \in V_3 - T: L_j < 0\} = \{4\}$.

For $x_1 = x_1^o = 0$, solve LP(B_1) given by

LP(B_1) : Max $2x_2 + x_3$

subject to $7x_2 + x_3 + x_5 = 10$

$4x_2 + x_6 = 6$

$x_2 + x_3 = 5$

$x_2 + x_8 = 8$

$x_2, x_3, x_5, x_6, x_8 \geq 0$ and integers.

The optimal integer solution obtained is

$X_2^{**} = (x_2^{**}, x_3^{**}, x_5^{**}, x_6^{**}, x_8^{**}) = (0, 5, 5, 6, 3)$, with $Z_2 = 5$.

Since $Z_3(X_2^{**}) = \frac{10}{34} \in [-2,1]$, condition (1) is verified. Here, $x_2^o = x_2^{**} = 0$. Again, the variables of the third level not associated to B_1, are not considered while solving problem LP(B_1) and are included in order to construct the set C_2.

$C_2 = \{j \in V_3 - T : z_j - c_j < 0\} = \{4\}$.

Thus, from above, we get

$C = \{j \in V_3 - T: j \in C_1 \text{ and } j \in C_2] = \{4\}$.

$E_I = E_I \cup \{C\} = \{\{4\}\}$.

Solve the problem (P_I) to obtain a new base which may provide us a better point of IR_1,

(P_I): $w_4 \geq 1$

$w_3 + w_4 + w_5 + w_6 + w_7 + w_8 = 4$

$w_i \in \{0, 1\}$.

Choose $w_3 = w_4 = w_5 = w_6 = 1$.
Here, the new basis B_2 associated to variables x_3, x_4, x_5 and x_6 and having rank 4.

To find $[\lambda^{\ell}, \lambda^{u}]$, that is, to check if the new base B_2 satisfies the optimality condition, solve (OC):-

(OC) : $e_{31}^{j} - \lambda e_{32}^{j} - (e_{31}^{B} - \lambda e_{32}^{B}) B^{-1} A_3^j \leq 0 \quad \forall j \in V_3 = \{3,\ 4,\ 5,\ 6,\ 7,\ 8\}$.

$$\Rightarrow\ 0 - [(2,-3,0,0) - \lambda(0,0,1,0)] \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \leq 0$$

Solving, we get $\lambda \in [0, 1]$.
A new iteration begins for the base B_2.
Solve IQP (B_2) given by –IQP(B_2) : Max $(x_1 + 2x_3 + 4)(-x_1 + x_3 + 2x_4 + 1)$

subject to $-3x_1 + x_3 + x_5 = 10$

$14x_1 + x_6 = 6$

$x_1 + x_3 - x_4 = 5$

$2x_1 + 2x_4 = 8$

$x_1, x_3, x_4, x_5, x_6 \geq 0$ and integers.

The optimal integer solution is given by

$X^o = (x^o, x_3^o, x_4^o, x_5^o, x_6^o) = (0,\ 9,\ 4,\ 1,\ 6)$ with

$Z_1 = 612 > 84$.

$Z_3(X^0) = \frac{1}{5} \in [0,1]$, therefore, (1) is verified. Update the current best integer point of inducible region as (0, 0, 9, 4, 1, 6, 0, 0).
Here $C_1 = \{j \in V_3 - T : L_j < 0\} = \{\phi\}$.

For, $x_1^{\circ} = 0$, solve LP (B_2) :

LP(B_2) : Max $2x_2 + x_3 + 3x_4$

subject to $7x_2 + x_3 + x_5 = 10$

$4x_2 + x_6 = 6$

$x_2 + x_3 - x_4 = 5$

$x_2 + 2x_4 = 8$

$x_2, x_3, x_4, x_5, x_6 \geq 0.$

The optimal integer solution is

$X_2^{**} = (x_2^{**}, x_3^{**}, x_4^{**}, x_5^{**}, x_6^{**}) = (0, 9, 4, 1, 6)$, with $Z_2 = 21 > 5$.

$Z_3(X_2^{**}) = \frac{1}{5} \in [0,1]$, therefore, (1) is verified. Update the current best integer point of IR_1 as (0, 0, 9, 4, 1, 6, 0, 0). Here $x_2^{**} = x_2^{o} = 0$.

$C_2 = \{j \in V_3 - T : z_j - c_j < 0\} = \{\phi\}$.

Thus, $C = \{j \in V_3 - T : j \in C_1 \text{ and } j \in C_2\} = \{\phi\}$.

The current best integer point is (0, 0, 9, 4, 1, 6, 0, 0) with Max $Z_1 = 612$, Max $Z_2 = 21$ and Max $Z_3 = 1/5$. This is the optimal integer solution for (TIPP).

Acknowledgement

I am thankful to the referees for their valuable comments which helped in improving the paper.

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