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A PARTIAL BACKLOGGING INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH STOCK-DEPENDENT CONSUMPTION RATE UNDER INFLATION

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Abstract: In this paper, we derive a partial backlogging inventory model for non-instantaneous deteriorating items with stock-dependent demand rate under inflation over a finite planning horizon. We propose a mathematical model and theorem to find minimum total relevant cost and optimal order quantity. Numerical examples are used to illustrate the developed model and the solution process. Finally, a sensitivity analysis of the optimal solution with respect to system parameters is carried out.

Keywords: Partial backlogging, non-instantaneous deterioration, stock-dependent demand, inflation.

1. INTRODUCTION

Deterioration is defined as decay, change, damage, spoilage or obsolescence that results in decreasing usefulness from its original purpose. Some kinds of inventory products (*e.g.*, vegetables, fruit, milk, and others) are subject to deterioration. Ghare and Schrader (1963) first established an economic order quantity model having a constant rate of deterioration and constant rate of demand over a finite planning horizon. Covert and Philip (1973) extended Ghare and Schrader's constant deterioration rate to a two-parameter Weibull distribution. Dave and Patel (1981) discussed an inventory model for deteriorating items with time-proportional demand when shortages were not allowed. The related analysis on inventory systems with deterioration have been performed by Sachan (1984), Balkhi and Benkherouf (1996), Wee (1997), Mukhopadhyay *et al*. (2004, 2005), *etc*.

In reality, not all kinds of inventory items deteriorated as soon as they received by the retailer. In the fresh product time, the product has no deterioration and keeps their original quality. Ouyang *et al*. (2006) named this phenomenon as "non-instantaneous deterioration", and they established an inventory model for non-instantaneous deteriorating items with permissible delay in payments.

In some fashionable products, some customers would like to wait for backlogging during the shortage period. But the willingness is diminishing with the length of the waiting time for the next replenishment. The longer the waiting time is, the smaller the backlogging rate would be. The opportunity cost due to lost sales should be considered. Chang and Dye (1999) developed an inventory model in which the demand rate is a time-continuous function and items deteriorate at a constant rate with partial backlogging rate which is the reciprocal of a linear function of the waiting time. Papachristos and Skouri (2000) developed an EOQ inventory model with time-dependent partial backlogging. They supposed the rate of backlogged demand increases exponentially with the waiting time for the next replenishment decreases. Teng *et al*. (2002, 2003) then extended the backlogged demand to any decreasing function of the waiting time up to the next replenishment. The related analysis on inventory systems with partial backlogging have been performed by Teng and Yang (2004), Yang (2005), Dye *et al*. (2006), San José *et al*. (2006), Teng *et al*. (2007), *etc*.

Many articles assume that the demand is constant during the sale period. It needs to be discussed. In real life, the requirements may be stimulated if there is a large pile of goods displayed on shelf. Levin *et al*. (1972) termed that the more goods displayed on shelf, the more customer's demand will be generated. Gupta and Vrat (1986) presented an inventory model for stock-dependent consumption rate on initial stock level rather than instantaneous inventory level. Baker and Urban (1988) established a deterministic inventory system in which the demand rate depended on the inventory level is described by a polynomial function. Wu *et al*. (2006) presented an inventory model for non-instantaneous deteriorating items with stock-dependent. The related analysis on inventory systems with stock-dependent consumption rate have been performed by Datta and Paul (2001), Balkhi and Benkherouf (2004), Chang *et al*. (2007), *etc*.

In all of the above mentioned models, the influences of the inflation and time value of money were not discussed. Buzacott (1975) first established an EOQ model with inflation subject to different types of pricing policies. Chung and Lin (2001) followed the discounted cash flow approach to investigate inventory model with constant demand rate for deteriorating items taking account of time value of money. Hou (2006) established an inventory model with stock-dependent consumption rate simultaneously considered the inflation and time value of money when shortages are allowed over a finite planning horizon.

In this article, we developed a partial backlogging inventory model for non-instantaneous deteriorating items with stock-dependent demand rate, along with the effects of inflation and time value of money that are considered. We extended the model in Hou (2006) to consider non-instantaneous and partial backlogging inventory model. The rest of this paper is organized as follows. In Section 2, we described the assumptions and notations used throughout this paper. In Section 3, we establish the mathematical model and theorem to find the minimum total relevant cost and the optimal order quantity. In Section 4, we use numerical examples to illustrate the theorem and results we proposed. In Section 5, we make a sensitivity analysis to study the effects of changes in the system parameters on the inventory model. Finally, we make a conclusion and provide suggestions for future research in Section 6.

2. ASSUMPTIONS AND NOTATION

We give the following assumptions and notation which will be used throughout the paper.

Assumptions:

(1) Only a single-product item is considered during the planning horizon H .

(2) Replenishment rate is infinite and lead time is zero.

(3) A constant fraction of the on-hand inventory deteriorates per unit of time and there is no repair or replacement of the deteriorated inventory.

(4) Shortage are allowed and backlogged partially. The backlogging rate is a decreasing function of the waiting time. Let the backlogging rate be $B(T-t)=e^{-\delta(T-t)}$, where $\delta \geq 0$, and $T-t$ is the waiting time up to the next replenishment.

(5) A Discounted Cash Flow (DCF) approach is used to consider the various costs at various times.

Notation:

H	the planning horizon
T	the replenishment cycle
m	the replenishment number in the planning horizon H
k	the ratio of no-shortage period to scheduling period T in each cycle
t_d	the length of time in which the product has no deterioration
$I_1(t)$	the inventory level at time t during the time interval $[0,t_d]$
$I_2(t)$	the inventory level at time t during the time interval $[t_d,kT]$
$I_3(t)$	the shortage level at time t during the time interval $[kT,T]$
$L(t)$	the amount of lost sale at time t during the time interval $[kT,T]$
I_m	the maximum inventory level for each cycle
S_b	the maximum shortage quantity for each cycle
$D(t)$	the demand rate at time t . $D(t)=\alpha+\beta I(t)$ when $I(t)>0$ and $D(t)=\alpha$ when $I(t)\leq 0$, where $\alpha>0$, β is the stock-dependent consumption rate parameter, $0\leq\beta\leq 1$.
θ	the constant deterioration rate

R the net discount rate of inflation
c_o the ordering cost per order
c_p the purchasing cost per unit
c_h the holding cost per unit per unit time
c_s the backlogging cost per unit per unit time
c_L the unit cost of lost sales. Note that if the objective is to minimize the cost, then $c_L > c_p$.
TC_o the present value of the ordering cost in the planning horizon H
TC_p the present value of the purchasing cost in the planning horizon H
TC_h the present value of the holding cost in the planning horizon H
TC_s the present value of the shortage cost in the planning horizon H
TC_L the present value of the lost sale cost in the planning horizon H
$TC(m,k)$ the present value of the total relevant inventory cost in the planning horizon H
Q^* the optimal order quantity in each cycle

3. MATHEMATICAL MODEL AND SOLUTION

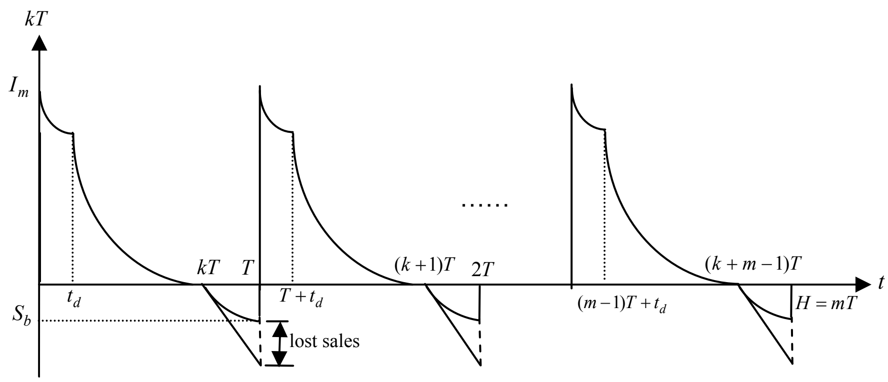

Figure. 1. The graphic representation of inventory model

The inventory model is shown in Fig. 1. The planning horizon H is divided into m equal parts of length $T = H/m$. The *j*th replenishment is made at time jT ($j = 0,1,2,\cdots,m$). The maximum inventory level for each cycle is I_m . During the time interval $[jT, jT + t_d]$ ($j = 0,1,2,\cdots,m-1$) the product has no deterioration, the inventory level is decreasing due to demand only. During the time interval $[jT + t_d, jT + kT]$ ($j = 0,1,2,\cdots,m-1$), the inventory level gradually reduces to zero owing to deterioration and demand. And shortage happens during the time interval $[jT + kT,(j+1)T]$ ($j = 0,1,2,\cdots,m-1$). The quantity received at jT ($j = 1,2,3,\cdots,m-1$)

is used partly to meet the accumulated backorders in the previous cycle from time $(k+j-1)T$ to jT, where k ($t_d / T \le k \le 1$) is the ratio of no-shortage period to scheduling period T in each cycle. The last extra replenishment at time H is needed to replenish shortages generated in the last cycle. The objective of the inventory problem here is to determine the replenishment number m and the ratio k in order to minimize the total relevant cost.

In the first replenishment cycle, owing to stock-dependent consumption rate only, the inventory level at time t during the time interval $[0, t_d]$ is governed by the following differential equation:

$$\frac{dI_1(t)}{dt} = -[\alpha + \beta I_1(t)] \qquad 0 \le t \le t_d \tag{1}$$

with the boundary condition $I_1(0) = I_m$. The solution of Eq. (1) can be represented by

$$I_1(t) = e^{-\beta t} I_m - \frac{\alpha}{\beta}(1 - e^{-\beta t}) \qquad 0 \le t \le t_d \tag{2}$$

Owing to stock-dependent consumption rate and deterioration, the inventory level at time t during the time interval $[t_d, kT]$ is governed by the following differential equation:

$$\frac{dI_2(t)}{dt} = -\theta I_2 - [\alpha + \beta I_2(t)] \qquad t_d \le t \le kT \tag{3}$$

with the boundary condition $I_2(kT) = 0$. The solution of Eq. (3) can be represented by

$$I_2(t) = \frac{\alpha}{\theta + \beta}[e^{(\theta+\beta)(kT-t)} - 1] \qquad t_d \le t \le kT \tag{4}$$

Because $I_1(t_d) = I_2(t_d)$, the maximum inventory level I_m is

$$I_m = \frac{\alpha}{\theta + \beta}[e^{(\theta+\beta)(kT-t_d)} - 1]e^{\beta t_d} + \frac{\alpha}{\beta}(e^{\beta t_d} - 1) \tag{5}$$

Hence, $I_1(t)$ in Eq. (2) can be represented as

$$I_1(t) = \frac{\alpha}{\theta + \beta}[e^{(\theta+\beta)(kT-t_d)} - 1]e^{-\beta(t-t_d)} + \frac{\alpha}{\beta}[e^{-\beta(t-t_d)} - 1] \tag{6}$$

Since the backlogging rate is a decreasing function of the waiting time, we let the backlogging rate be $B(T-t) = e^{-\delta(T-t)}$, the shortage level at time t during the time interval $[kT, T]$ is governed by the following differential equation:

$$\frac{dI_3(t)}{dt} = \alpha e^{-\delta(T-t)} \qquad kT \le t \le T \tag{7}$$

with the boundary condition $I_3(kT)=0$. The solution of Eq. (7) can be represented by

$$I_3(t)=\frac{\alpha}{\delta}[e^{-\delta(T-t)}-e^{-\delta(1-k)T}] \qquad kT\le t\le T \tag{8}$$

And the amount of lost sale at time t during the time interval $[kT,T]$ is

$$L(t)=\alpha\int_{kT}^{t}[1-e^{-\delta(T-\tau)}]d\tau=\alpha\left\{t-kT-\frac{1}{\delta}[e^{-\delta(T-t)}-e^{-\delta(1-k)T}]\right\} \quad kT\le t\le T \tag{9}$$

Let S_b be the maximum shortage quantity per cycle.

$$S_b=I_3(T)=\frac{\alpha}{\delta}[1-e^{-\delta(T-kT)}] \tag{10}$$

Replenishment is made at time jT ($j=0,1,2,\cdots,m$), the maximum inventory level for each cycle is I_m. The last replenishment at time mT is just to satisfy the backorders generated in the last cycle. There are $m+1$ replenishments in the entire time horizon H. The total relevant inventory cost involves following five factors.

(a) Ordering cost: The present value of the ordering cost in the entire time horizon H is

$$TC_o=c_o\sum_{j=0}^{m}e^{-RjT}=c_o\frac{e^{RH/m}-e^{-RH}}{e^{RH/m}-1} \tag{11}$$

(b) Purchasing cost: The present value of the purchasing cost in the entire time horizon H is

$$\begin{aligned}TC_p&=\sum_{j=0}^{m-1}c_pI_me^{-RjT}+\sum_{j=1}^{m}c_pS_be^{-RjT}\\&=c_p\alpha\left\{\frac{1}{\theta+\beta}[e^{(\theta+\beta)(kT-t_d)}-1]e^{\beta t_d}+\frac{1}{\beta}(e^{\beta t_d}-1)\right\}\\&\times\frac{1-e^{-RH}}{1-e^{-RH/m}}+\frac{c_p\alpha}{\delta}[1-e^{-\delta(1-k)H/m}]\frac{1-e^{-RH}}{e^{RH/m}-1}\end{aligned} \tag{12}$$

(c) Holding cost: The present value of the holding cost in the entire time horizon H is

$$\begin{aligned}TC_h&=\sum_{j=0}^{m-1}c_h\left[\int_0^{t_d}e^{-Rt}I_1(t)dt+\int_{t_d}^{kT}e^{-Rt}I_2(t)dt\right]e^{-RjT}\\&=c_h\alpha\left\{\frac{1}{\theta+\beta}[e^{(\theta+\beta)(kT-t_d)}-1]\frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta}+\frac{1}{\beta}(\frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta}+\frac{e^{-Rt_d}-1}{R})\right.\\&\left.+\frac{1}{\theta+\beta}[\frac{e^{-(\theta+\beta+R)t_d+(\theta+\beta)kT}-e^{-RkT}}{(\theta+\beta+R)}+\frac{e^{-RkT}-e^{-Rt_d}}{R}]\right\}\frac{1-e^{-RH}}{1-e^{-RH/m}}\end{aligned} \tag{13}$$

(d) Shortage cost: The present value of the shortage cost in the entire time horizon H is

$$TC_s = \sum_{j=0}^{m-1} c_s[\int_{kT}^{T} e^{-Rt} I_3(t)dt]e^{-RjT}$$
$$= \frac{c_s\alpha}{R}[\frac{e^{(R-\delta)(1-k)H/m}-1}{R-\delta}+\frac{e^{-\delta(1-k)H/m}-1}{\delta}]\frac{1-e^{-RH}}{e^{RH/m}-1} \tag{14}$$

(e) Lost sale cost: The present value of the lost sale cost in the entire time horizon H is

$$TC_L = \sum_{j=0}^{m-1}\left[c_L\int_{kT}^{T} e^{-Rt}\alpha[1-e^{-\delta(T-t)}]dt\right]e^{-RjT}$$
$$= c_L\alpha[\frac{e^{R(1-k)H/m}-1}{R}+\frac{1-e^{(R-\delta)(1-k)H/m}}{R-\delta}]\frac{1-e^{-RH}}{e^{RH/m}-1} \tag{15}$$

Hence, the present value of the total relevant inventory cost in the entire time horizon H is

$$TC(m,k) = TC_o + TC_p + TC_h + TC_s + TC_L \tag{16}$$

let

$$U = \frac{e^{RT}-e^{-RH}}{e^{RT}-1} \quad V = \frac{1-e^{-RH}}{1-e^{-RT}} \quad W = \frac{1-e^{-RH}}{e^{RT}-1} \quad T = \frac{H}{m}$$

We substitute Eqs. (11)-(15) into Eq. (16) and obtain

$$TC(m,k) = c_oU + c_p\alpha\left\{\frac{1}{\theta+\beta}[e^{(\theta+\beta)(kT-t_d)}-1]e^{\beta t_d}+\frac{1}{\beta}(e^{\beta t_d}-1)\right\}V$$
$$+c_h\alpha\left\{\frac{[e^{(\theta+\beta)(kT-t_d)}-1](e^{\beta t_d}-e^{-Rt_d})}{(\theta+\beta)(R+\beta)}+\frac{1}{\beta}(\frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta}+\frac{e^{-Rt_d}-1}{R})\right.$$
$$\left.+\frac{1}{\theta+\beta}[\frac{e^{-(\theta+\beta+R)t_d+(\theta+\beta)kT}-e^{-RkT}}{(\theta+\beta+R)}+\frac{e^{-RkT}-e^{-Rt_d}}{R}]\right\}V \tag{17}$$
$$+\alpha\left[(\frac{c_s}{R}-c_L)\frac{e^{(R-\delta)(1-k)T}-1}{R-\delta}+(c_p-\frac{c_s}{R})\frac{1-e^{-\delta(1-k)T}}{\delta}+c_L\frac{e^{R(1-k)T}-1}{R}\right]W$$

There are two variables in the present value of the total inventory cost $TC(m,k)$. One is the replenishment number m which is a discrete variable, the other is the ratio k, where $kT \le t \le T$, which is a continuous variable. For a fixed value of m, the condition for $TC(m,k)$ to be minimized is $dTC(m,k)/dk = 0$. Consequently, we obtain

$$c_p e^{(\theta+\beta)kT-\theta t_d} + c_h\left[\frac{e^{(\theta+\beta)(kT-t_d)}(e^{\beta t_d}-e^{-Rt_d})}{R+\beta} + \frac{e^{(\theta+\beta)(kT-t_d)-Rt_d}-e^{-RkT}}{\theta+\beta+R}\right]$$
$$-\left[(c_p-\frac{c_s}{R})e^{-\delta(1-k)T} + (\frac{c_s}{R}-c_L)e^{(R-\delta)(1-k)T} + c_L e^{R(1-k)T}\right]e^{-RT} = 0 \quad (18)$$

Theorem 1

(a) If

$$c_p e^{\beta t_d} + c_h \frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta} <$$
$$\left[(c_p-\frac{c_s}{R})e^{-\delta(T-t_d)} + (\frac{c_s}{R}-c_L)e^{(R-\delta)(T-t_d)} + c_L e^{R(T-t_d)}\right]e^{-RT} \quad (19)$$

there exists a unique solution k^*, where $t_d < k^*T < T$, such that $TC(m,k^*)$ is the minimum value of k when m is given.

(b) If

$$c_p e^{\beta t_d} + c_h \frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta} >$$
$$\left[(c_p-\frac{c_s}{R})e^{-\delta(T-t_d)} + (\frac{c_s}{R}-c_L)e^{(R-\delta)(T-t_d)} + c_L e^{R(T-t_d)}\right]e^{-RT} \quad (20)$$

$TC(m, mt_d/H)$ is the minimum value when m is given.

Proof: See Appendix 1.

From theorem 1, we can use Newton-Raphson method to find the optimal value k^* when the replenishment number m is given. However, since the high-power expression of the exponential function in $TC(m,k)$, it is difficult to show analytic solution of m such that it makes $TC(m,k)$ minimized. Following the optimal solution procedure proposed by Montgomery (1982), we let (m^*,k^*) denote the optimal solution to $TC(m,k)$ and let $(m,k^*(m))$ denote the optimal solution to $TC(m,k)$ when m is given. If m^* is the smallest integer such that $TC(m^*,k^*(m^*))$ less than each value of $TC(m,k(m))$ in the interval $m^*+1 \le m \le m^*+10$. Then we take $(m^*,k^*(m^*))$ as the optimal solution to $TC(m,k(m))$. And we can obtain the maximum inventory level I_m as

$$I_m = \frac{\alpha}{\theta+\beta}[e^{(\theta+\beta)(\frac{k^*H}{m^*}-t_d)}-1]e^{\beta t_d} + \frac{\alpha}{\beta}(e^{\beta t_d}-1) \quad (21)$$

Also the optimal order quantity Q^* is

$$Q^* = I_m + S_b$$
$$= \frac{\alpha}{\theta+\beta}[e^{(\theta+\beta)(\frac{k^* H}{m^*}-t_d)} - 1]e^{\beta t_d} + \frac{\alpha}{\beta}(e^{\beta t_d} - 1) + \frac{\alpha}{\delta}[1 - e^{-\delta(1-k^*)\frac{H}{m^*}}] \tag{22}$$

4. NUMERICAL EXAMPLES

To illustrate the proposed model, let us consider the following parametric data as examples.

Example 1: Let $c_0 = \$250.00/\text{order}$, $c_p = \$5/\text{unit}$, $c_h = \$1.75/\text{unit}/\text{year}$, $c_s = \$3/\text{unit}/\text{year}$, $c_L = \$20/\text{unit}$, $\alpha = 600\ \text{units}/\text{year}$, $\beta = 0.05$, $\theta = 0.20$, $\delta = 0.02$, $R = 0.20$, $H = 10$ year, $t_d = 0.05$ year. The above data satisfy *Theorem 1(a)*. Following the optimal solution procedure proposed by Montgomery (1982), *Table 1* shows the optimal replenishment number $m^* = 13$, the ratio $k^* = 0.351$, the optimal order quantity $Q^* = 464.11$ and the minimum present value of total relevant cost $TC(m^*, k^*) = \$15929.2$. The relation between $TC(m, k^*)$ and m under different policies in *Table 1* are shown in *Figure 2*.

Table 1 Different policies with respect to total cost for example 1

m	k^*	k^*T	T	Q^*	$TC(m,k^*)$
2	0.234	1.168	5.00	3019.30	22206.5
3	0.271	0.903	3.33	2025.21	19533.4
4	0.293	0.732	2.50	1519.54	18161.7
5	0.307	0.614	2.00	1214.58	17358.8
6	0.318	0.529	1.67	1011.03	16851.7
7	0.325	0.465	1.43	865.661	16517.0
8	0.332	0.414	1.25	756.717	16291.6
9	0.337	0.374	1.11	672.06	16139.5
10	0.341	0.341	1.00	604.10	16039.0
11	0.345	0.313	0.91	549.09	15976.2
12	0.348	0.290	0.83	503.04	15941.8
13*	0.351*	0.270*	0.77*	464.11*	15929.2*
14	0.353	0.252	0.71	430.77	15933.9
15	0.356	0.237	0.67	401.89	15952.3
16	0.358	0.224	0.63	376.63	15981.9
17	0.360	0.212	0.59	354.36	16020.8
18	0.362	0.201	0.56	334.58	16067.3
19	0.363	0.191	0.53	316.88	16120.4
20	0.365	0.182	0.50	300.97	16178.9
21	0.367	0.175	0.48	286.57	16242.2
22	0.368	0.167	0.45	273.49	16309.6
23	0.370	0.161	0.43	261.55	16380.6
24	0.371	0.155	0.42	250.61	16454.6

**Optimal solution*

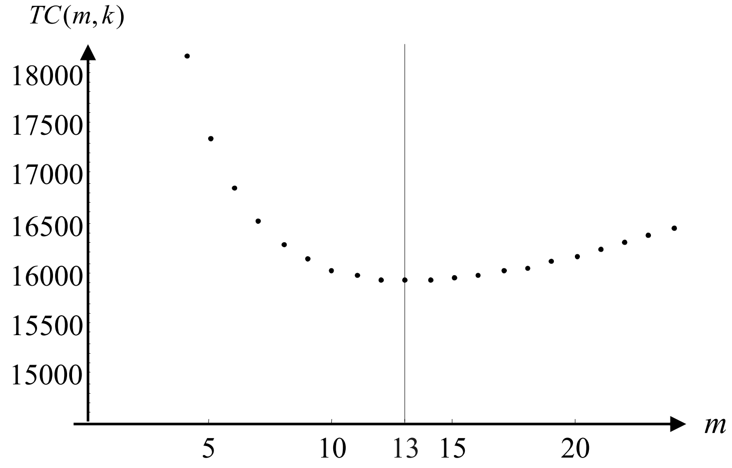

Figure 2 Relation between $TC(m,k)$ and m. The condition satisfies Theorem 1(a).

Some factors that influence the total relevant cost are shown in *Table 2*.

Table 2 Some special cases of the inventory model in *Example 1*.

Conditions	m^*	k^*	k^*T^*	T^*	Q^*	$TC^*(m^*,k^*)$	remark
our example	13	0.351	0.270	0.77	464.11	15929.2	TC^*
$R=0$	14	0.522	0.373	0.71	436.47	37012.0	TC_1^*
$\beta=0$	13	0.367	0.282	0.77	463.41	15892.4	TC_2^*
$\theta=0$	12	0.411	0.342	0.83	500.33	15814.2	TC_3^*
$\delta=0$	13	0.322	0.247	0.77	464.85	15804.8	TC_4^*
$t_d=0$, $\delta=0$	13	0.310	0.238	0.77	465.89	15846.0	TC_5^*
$t_d=0$, $\delta=0$, $\beta=0$	12	0.323	0.269	0.83	504.43	15816.8	TC_6^*

The present value of total relevant cost TC_5^* is the same as Hou (2006), and TC_6^* is the same as Chung and Lin (2001). The other numerical results can make following comparative conclusions:

(1) If the inflation and time value of money are not considered, $R=0$, the present value of total relevant cost TC_1^* is far larger than the total relevant cost TC^*. *i.e.*, $TC_1^* >> TC^*$.

(2) If the stock-dependent consumption rate is not considered, $\beta = 0$, the present value of total relevant cost TC_2^* is smaller than the total relevant cost TC^*. *i.e.*, $TC_2^* < TC^*$.

(3) If deterioration isn't considered, $\theta = 0$, the present value of total relevant cost TC_3^* is smaller than the total relevant cost TC^*. *i.e.*, $TC_3^* < TC^*$.

(4) If the backlogging is complete, $\delta = 0$, the present value of total relevant cost TC_4^* is smaller than the total relevant cost TC^*. *i.e.*, $TC_4^* < TC^*$.

Example 2: All data are the same as example 1 except $c_p = \$20/\text{unit}$ and $c_L = \$5/\text{unit}$. The above data satisfy *Theorem 1(b)*. From *Table 3* we obtain the optimal replenishment number $m^* = 1$, the ratio $k^* = 0.005$, the optimal order quantity $Q^* = 5443.54$ and the minimum present value of total relevant cost $TC(m^*, k^*) = \$39905.1$. The relation between $TC(m, k^*)$ and m under different policies in *Table 3* is shown in *Figure 3*.

Table 3 Different policies with respect to total cost for example 2

m	k^*	k^*T	T	Q^*	$TC(m,k^*)$
1*	0.005*	0.05*	10.0*	5443.5*	39905.1*
2	0.010	0.05	5.00	2857.8	45193.4
3	0.015	0.05	3.33	1936.7	47456.1
4	0.020	0.05	2.50	1464.6	48734.1
5	0.025	0.05	2.00	1177.5	49575.5
6	0.030	0.05	1.67	984.52	50186.1
7	0.035	0.05	1.43	845.88	50660.1
8	0.040	0.05	1.25	741.47	51046.6
9	0.045	0.05	1.11	660.00	51373.8
10	0.050	0.05	1.00	594.66	51658.9
11	0.055	0.05	0.91	541.09	51913.0
12	0.060	0.05	0.83	496.38	52143.6
13	0.065	0.05	0.77	458.49	52356.2
14	0.070	0.05	0.71	425.97	52554.4

* *Optimal solution*

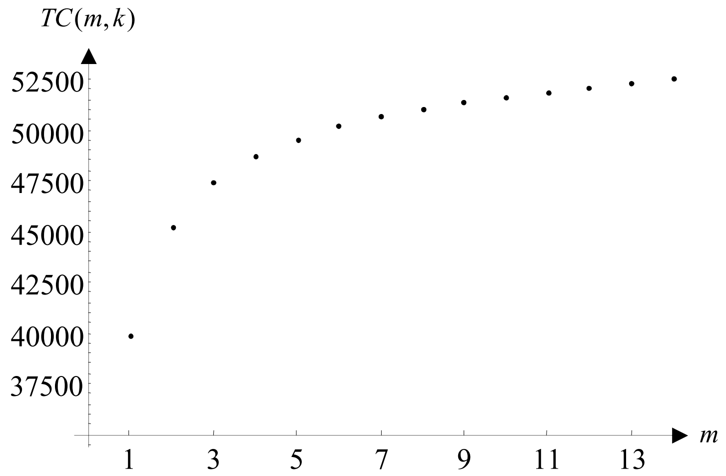

Figure 3 Relation between $TC(m,k)$ and m. The condition satisfies Theorem 1(b).

5. SENSITIVITY ANALYSIS

In this section, we discuss system parameters that influence the minimum total relevant cost TC^*, the optimal order quantity Q^* and the replenishment number m.

Theorem 2 *TC^* is the minimum total relevant cost of $TC(m,k)$. If $c_p < c_s / R < c_L$, then TC^* is a strictly increasing function of δ for $\delta \geq 0$.*

Proof: See Appendix 2.

Theorem 3 TC^* is the minimum total relevant cost of $TC(m,k)$ and Q^* is the optimal order quantity. If $t_d < k^* T^* < T^*$, i.e., it satisfies Theorem 1(a), then TC^* and Q^* are strictly decreasing functions of t_d for $t_d \geq 0$.

Proof: See Appendix 3.

Theorem 4 *Q^* is the optimal order quantity. Q^* is a strictly decreasing function of δ for $\delta \geq 0$.*

Proof: See Appendix 4.

Using Example 1 data to study the effects of change in the system parameters on the optimal order quantity Q^*, the minimum total relevant cost TC^* and the replenishment number m. The sensitivity analysis is performed by changing (increasing or decreasing) the parameter by 50%, 20% taking at a time, keeping the remaining parameters at their original values. Let the estimated values of the optimal order quantity, the minimum total relevant cost and the replenishment number be Q', TC' and m'

respectively, the three values in *Example 1* are Q^*, TC^* and m^*. The following inferences can be observed from the sensitivity analysis based on Table 4.

Table 4. Sensitivity analysis of Example 1

Parameter		Percentage of under-estimation and over-estimation parameters				
		-50	-20	0	+20	+50
α	Q'/Q^*	0.7240	0.8671	1	1.1138	1.2173
	TC'/TC^*	0.5417	0.8187	1	1.1795	1.4462
	m'	9	12	13	14	16
β	Q'/Q^*	0.9993	0.9997	1	1.0003	1.0006
	TC'/TC^*	0.9989	0.9996	1	1.0004	1.0011
	m'	13	13	13	13	13
θ	Q'/Q^*	0.9981	0.9994	1	1.0005	1.0012
	TC'/TC^*	0.9967	0.9988	1	1.0012	1.0028
	m'	13	13	13	13	13
δ	Q'/Q^*	1.0007	1.0003	1	0.9998	0.9994
	TC'/TC^*	0.9962	0.9985	1	1.0015	1.0036
	m'	13	13	13	13	13
t_d	Q'/Q^*	1.0011	1.0004	1	0.9996	0.9990
	TC'/TC^*	1.0014	1.0005	1	0.9995	0.9988
	m'	13	13	13	13	13
R	Q'/Q^*	0.9336	1.0024	1	0.9979	1.0781
	TC'/TC^*	1.4683	1.1564	1	0.8738	0.7262
	m'	14	13	13	13	12
c_0	Q'/Q^*	0.6828	0.8659	1	1.0839	1.1831
	TC'/TC^*	0.9430	0.9793	1	1.0189	1.0448
	m'	19	15	13	12	11
c_p	Q'/Q^*	1.0098	1.0033	1	0.9973	1.0769
	TC'/TC^*	0.5932	0.8387	1	1.1597	1.3964
	m'	13	13	13	13	12
c_h	Q'/Q^*	1.0043	1.0015	1	0.9987	0.9257
	TC'/TC^*	0.9915	0.9969	1	1.0028	1.0063
	m'	13	13	13	13	14
c_s	Q'/Q^*	1.4285	1.0793	1	0.9314	0.87271
	TC'/TC^*	0.9428	0.9828	1	1.0131	1.02800
	m'	9	12	13	14	15
c_L	Q'/Q^*	0.9986	0.9995	1	1.0005	0.9293
	TC'/TC^*	0.9946	0.9979	1	1.0021	1.0049
	m'	13	13	13	13	14

(1) The optimal order quantity increases as α, β, θ or c_0 increases. But it decreases as δ, t_d, c_h or c_s increases.

(2) The optimal order quantity is more sensitive on the change in α , c_0 or c_s to other parameters.
(3) The minimum present value of total relevant cost increases as α , β , θ , δ , c_0 , c_p , c_h , c_s or c_L increases. But it decreases as t_d or R increases.
(4) The minimum total relevant cost is more sensitive on the change in α , R or c_p to other parameters
(5) The replenishment number increases as α or c_s increases. But it decreases as c_0 increases.
(6) The replenishment number is insensitive on the change in β , θ , δ or t_d to other parameters
(7) TC^* is a increasing function of δ for $\delta \geq 0$, it obeys *Theorem 2*. And Q^* is a decreasing function of δ for $\delta \geq 0$, it obeys *Theorem 4*.
(8) TC^* and Q^* are decreasing functions of t_d for $t_d \geq 0$, it obeys *Theorem 3*.

6. CONCLUSION

In this article, we establish an inventory model for non-instantaneous deteriorating items with stock-dependent consumption rate to determine the optimal order quantity, the minimum present value of total relevant cost and replenishment number. The effects of inflation and time value of money are also considered. We present the condition of the unique solution of the total relevant cost when replenishment number is given in *Theorem 1*. We also discuss the minimum total relevant cost and the optimal order quantity with respect to backlogging parameter and non-instantaneous deteriorating time from *Theorem 2* to *Theorem 4* respectively.

From the sensitivity analysis, the optimal order quantity is more sensitive on the change in the parameter α , c_0 or c_s . The minimum present value of total relevant cost is more sensitive on the change in the parameter α , c_p or R . It helps retailer to make decisions in different replenishment policies.

Finally, the proposed model can be extended in several ways. For example, we could extend the deterministic model to varying cycle length. Also we could generalize the model to allow for quantity discounts or others.

APPENDEX 1

Proof of Theorem 1 part (a).

$$dTC(m,k)/dk = \alpha T\{c_p e^{(\theta+\beta)kT-\theta t_d}$$

$$+c_h \frac{e^{(\theta+\beta)(kT-t_d)}(e^{\beta t_d}-e^{-Rt_d})}{R+\beta}+c_h\frac{[e^{(\theta+\beta+R)(kT-t_d)}-1]e^{-RkT}}{\theta+\beta+R}$$

$$-[(c_p-\frac{c_s}{R})e^{-\delta(1-k)T}+(\frac{c_s}{R}-c_L)e^{(R-\delta)(1-k)T}+c_L e^{R(1-k)T}]e^{-RT}\}V$$

$$d^2TC(m,k)/dk^2=\alpha T^2\{c_p[(\theta+\beta)e^{(\theta+\beta)kT-\theta t_d}-\delta e^{-\delta(1-k)T-RT}]$$

$$+c_h(\theta+\beta)\left[\frac{e^{(\theta+\beta)(kT-t_d)}(e^{\beta t_d}-e^{-Rt_d})}{R+\beta}+\frac{e^{(\theta+\beta)(kT-t_d)-Rt_d}+\mathrm{Re}^{-RkT}}{\theta+\beta+R}\right]$$

$$+\left[[\frac{c_s}{R}-c_L](R-\delta)e^{(R-\delta)(1-k)T}+\frac{c_s}{R}\delta e^{-\delta(1-k)T}+c_L\,\mathrm{Re}^{R(1-k)T}\right]e^{-RT}\}V>0$$

Clearly, $dTC(m,k)/dk$ is a strictly increasingly function of k. Besides,

$$\frac{dTC(m,1)}{dk}=\alpha T\{c_p[e^{(\theta+\beta)T-\theta t_d}-e^{-RT}]+c_h[\frac{e^{(\theta+\beta)(T-t_d)}(e^{\beta t_d}-e^{-Rt_d})}{R+\beta}$$

$$+\frac{e^{(\theta+\beta+R)(T-t_d)}-1}{\theta+\beta+R}e^{-RT}]\}V>0$$

$$\frac{dTC(m,t_d/T)}{dk}=\alpha T\{c_p e^{\beta t_d}+c_h\frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta}$$

$$-\left[(c_p-\frac{c_s}{R})e^{-\delta(T-t_d)}+(\frac{c_s}{R}-c_L)e^{(R-\delta)(T-t_d)}+c_L e^{R(T-t_d)}\right]e^{-RT}\}V$$

if

$$c_p e^{\beta t_d}+c_h\frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta}<\left[(c_p-\frac{c_s}{R})e^{-\delta(T-t_d)}+(\frac{c_s}{R}-c_L)e^{(R-\delta)(T-t_d)}+c_L e^{R(T-t_d)}\right]e^{-RT}$$

, then $dTC(m,t_d/T)/dk<0$. From the Intermediate Value Theorem, there will exist a unique solution k^* that satisfies $dTC(m,k^*)/dk=0$, where $t_d<k^*T<T$. Because $d^2TC(m,k)/dk^2>0$, $TC(m,k)$ is a convex function of k for a fixed value m. Hence, $TC(m,k^*)$ is the minimum value of k when m is given.

Proof of Theorem 1 part (b).

if

$$c_p e^{\beta t_d}+c_h\frac{e^{\beta t_d}-e^{-Rt_d}}{R+\beta}>\left[(c_p-\frac{c_s}{R})e^{-\delta(T-t_d)}+(\frac{c_s}{R}-c_L)e^{(R-\delta)(T-t_d)}+c_L e^{R(T-t_d)}\right]e^{-RT}$$

, owing to $dTC(m,1)/dk>0$ and $d^2TC(m,k)/dk^2>0$, $TC(m,k)$ is a strictly increasing function of k in the interval $mt_d / H \le k \le 1$ when m is given. Consequently, the minimum value of $TC(m,k)$ will happen at $k = mt_d / H$ when m is given.

APPENDIX 2

Proof of Theorem 2

Now, we consider the relation between variable δ and the minimum total relevant cost TC^*.

$$\frac{dTC^*}{d\delta} = \{(\frac{c_s}{R} - c_L)[\frac{[1-(1-k^*)T^*(R-\delta)]e^{(R-\delta)(1-k^*)T^*} - 1}{(R-\delta)^2}]$$
$$+(c_p - \frac{c_s}{R})\frac{[1+(1-k^*)T^*\delta]e^{-\delta(1-k^*)T^*} - 1}{\delta^2}\}\alpha W^*$$

Let $x = (1-k)T$, where $0 \le x \le T - t_d$, and

$$f(x) = (\frac{c_s}{R} - c_L)\frac{[1-x(R-\delta)]e^{(R-\delta)x} - 1}{(R-\delta)^2} + (c_p - \frac{c_s}{R})\frac{(1+x\delta)e^{-\delta x} - 1}{\delta^2}$$

$$\frac{df(x)}{dx} = -\left[(\frac{c_s}{R} - c_L)e^{Rx} + (c_p - \frac{c_s}{R})\right]xe^{-\delta x}$$

If $c_p < c_s / R < c_L$, it must be $df(x)/dx > 0$ for $0 \le x \le T - t_d$. Hence, $f(x)$ is a strictly increasing function of x for $0 \le x \le T - t_d$. Because $f(0) = 0$, we can sure $f(x) > 0$ for $0 \le x \le T - t_d$. This implies $dTC^* / d\delta > 0$, we say TC^* is a strictly increasing function of δ for $\delta \ge 0$.

APPENDIX 3

Proof of Theorem 3

Considering TC^* and Q^* derivative to t_d respectively.

$$\frac{dTC^*}{dt_d} = -\frac{\alpha\theta}{\theta+\beta}[e^{(\theta+\beta)(k^*T^*-t_d)} - 1][c_p e^{\beta t_d} + \frac{c_h(e^{\beta t_d} - e^{-Rt_d})}{R+\beta}]V^*$$

and

$$\frac{dQ^*}{dt_d} = -\frac{\alpha\theta}{\theta+\beta}[e^{(\theta+\beta)(k^*T^*-t_d)} - 1]e^{\beta t_d}$$

The conditions for $dTC^* / dt_d > 0$ and $dQ^* / dt_d > 0$ are $e^{(\theta+\beta)(k^*T^*-t_d)} - 1 < 0$, *i.e.*, $k^*T^* < t_d$. It violates the definition $t_d \le k^*T^* \le T^*$. Hence, dTC^* / dt_d and dQ^* / dt_d would be smaller than zero in the interval $t_d < k^*T^* < T^*$. It also satisfies *Theorem 1(a)* which judges whether k^*T^* exists between t_d and T^* or not. In this situation, TC^* and Q^* are strictly decreasing functions of t_d for $t_d \ge 0$.

APPENDIX 4

Proof of Theorem 4

Now, we consider the relation between variable δ and optimal order quantity Q^*.

$$\frac{dQ^*}{d\delta} = \alpha \frac{[\delta(1-k^*)T^* + 1]e^{-\delta(1-k^*)T^*} - 1}{\delta^2}$$

Let $\lambda = \delta(1-k)T$, where $0 \le \lambda \le \delta(T - t_d)$, and $g(\lambda) = \alpha[(\lambda+1)e^{-\lambda} - 1] / \delta^2$. Because $dg(\lambda) / d\lambda = -\alpha e^{-\lambda} / \delta^2 < 0$, $g(\lambda)$ is a strictly decreasing function of λ for $0 \le \lambda \le \delta(T - t_d)$. Because $g(0) = 0$, we can sure $g(\lambda) < 0$ for $0 \le \lambda \le \delta(T - t_d)$. This implies $dQ^* / d\delta < 0$, we can say Q^* is a strictly decreasing function of δ for $\delta \ge 0$.

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