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INVENTORY MODELS WITH STOCK- AND PRICE-DEPENDENT DEMAND FOR DETERIORATING ITEMS BASED ON LIMITED SHELF SPACE

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Abstract: This paper deals with the problem of determining the optimal selling price and order quantity simultaneously under EOQ model for deteriorating items. It is assumed that the demand rate depends not only on the on-display stock level but also the selling price per unit, as well as the amount of shelf/display space is limited. We formulate two types of mathematical models to manifest the extended EOQ models for maximizing profits and derive the algorithms to find the optimal solution. Numerical examples are presented to illustrate the models developed and sensitivity analysis is reported.

Keywords: Inventory control, pricing, stock-dependent demand, deterioration.

1. INTRODUCTION

In the classical inventory models, the demand rate is regularly assumed to be either constant or time-dependent but independent of the stock levels. However, practically an increase in shelf space for an item induces more consumers to buy it. This occurs owing to its visibility, popularity or variety. Conversely, low stocks of certain goods might raise the perception that they are not fresh. Therefore, it is observed that the demand rate may be influenced by the stock levels for some certain types of inventory. In years, marketing researchers and practitioners have recognized the phenomenon that the demand for some items could be based on the inventory level on display. Levin *et al.* (1972) pointed out that large piles of consumer goods displayed in a supermarket would attract the customer to buy more. Silver and Peterson (1985) noted that sales at the retail level tend to be proportional to stock displayed. Baker and Urban (1988) established an EOQ model for a power-form inventory-level-dependent demand pattern. Padmanabhan and Vrat (1990) developed a multi-item inventory model of deteriorating items with stock-dependent demand under resource constraints and solved by a non-linear goal programming method. Datta and Pal (1990) presented an inventory model in which the demand rate is dependent on the instantaneous inventory level until a given inventory level is achieved, after which the demand rate becomes constant. Urban (1992) relaxed the unnecessary zero ending-inventory at the end of each order cycle as imposed in Datta and Pal (1990). Pal *et al.* (1993) extended the model of Baker and Urban (1988) for perishable products that deteriorate at a constant rate. Bar-Lev *et al.* (1994) developed an extension of the inventory-level-dependent demand-type EOQ model with random yield. Giri *et al.* (1996) generalized Urban's model for constant deteriorating items. Urban and Baker (1997) further deliberated the EOQ model in which the demand is a multivariate function of price, time, and level of inventory. Giri and Chaudhuri (1998) expanded the EOQ model to allow for a nonlinear holding cost. Roy and Maiti (1998) developed multi-item inventory models of deteriorating items with stock-dependent demand in a fuzzy environment. Urban (1998) generalized and integrated existing inventory-control models, product assortment models, and shelf-space allocation models. Datta and Paul (2001) analyzed a multi-period EOQ model with stock-dependent, and price-sensitive demand rate. Kar *et al.* (2001) proposed an inventory model for deteriorating items sold from two shops, under single management dealing with limitations on investment and total floor-space area. Other papers related to this area are Pal *et al.* (1993), Gerchak and Wang (1994), Padmanabhan and Vrat (1995), Ray and Chaudhuri (1997), Ray *et al.* (1998), Hwang and Hahn (2000), Chang (2004), and others.

As shown in Levin *et al.* (1972), "large piles of consumer goods displayed in a supermarket will lead customers to buy more. Yet, too many goods piled up in everyone's way leave a negative impression on buyers and employees alike." Hence, in this present paper, we first consider a maximum inventory level in the model to reflect the facts that most retail outlets have limited shelf space and to avoid a negative impression on customer because of excessively piled up in everyone's way. Since the demand rate not only is influenced by stock level, but also is associated with selling price, we also take into account the selling price and then establish an EOQ model in which the demand rate is a function of the on-display stock level and the selling price. In Section 2, we provide the fundamental assumptions for the proposed EOQ model and the notations used throughout this paper. In Section 3, we set up a mathematical model. The properties of the optimal solution are discussed as well as its solution algorithm and numerical examples are presented. In Section 4, an optimal ordering policy with selling price predetermined is investigated. Theorems 1 and 2 are provided to show the characteristics of the optimal solution. An easy-to-use algorithm is developed to determine the optimal cycle time, economic order quantity and ordering point. Finally, we draw the conclusions and address possibly future work in Section 5.

2. ASSUMPTIONS AND NOTATIONS

A single-item deterministic inventory model for deteriorating items with price- and stock-dependent demand rate is presented under the following assumptions and notations.

1. Shortages are not allowed to avoid lost sales.

2. The maximum allowable number of displayed stocks is B to avoid a negative impression and due to limited shelf/display space.

3. Replenishment rate is infinite and lead time is zero.

4. The fixed purchasing cost K per order is known and constant.

5. Both the purchase cost c per unit and the holding cost h per unit per unit time are known and constant. The constant selling price p per unit is a decision variable within the replenishment cycle, where $p > c$.

6. The constant deterioration rate θ ($0 \le \theta < 1$) is only applied to on-hand inventory. There are two possible cases for the cost of a deteriorated item s: (1) if there is a salvage value, that value is negative or zero; and (2) if there is a disposal cost, that value is positive. Note that $c > s$ (or $-s$).

7. All replenishment cycles are identical. Consequently, only a typical planning cycle with T length is considered (*i.e.*, the planning horizon is [0, T]).

8. The demand rate $R(I(t), p)$ is deterministic and given by the following expression:

$$R(I(t), p) = \alpha(p) + \beta\ I(t),$$

where $I(t)$ is the inventory level at time t, β is a non-negative constant, and $\alpha(p)$ is a non-negative function of p with $\alpha'(p) = d\,\alpha(p)/d\,p < 0$.

9. As stated in Urban (1992), "it may be desirable to order large quantities, resulting in stock remaining at the end of the cycle, due to the potential profits resulting from the increased demand." Consequently, the initial and ending inventory levels y are not restricted to be zero (*i.e.*, $y \ge 0$). The order quantity Q enters into inventory at time t = 0. Consequently, $I(0) = Q + y$. During the time interval [0, T], the inventory is depleted by the combination of demand and deterioration. At time T, the inventory level falls to y, *i.e.*, $I(T) = y$. The initial and ending inventory level y can be called ordering point.

The mathematical problem here is to determine the optimal values of T, p and y such that the average net profit in a replenishment cycle is maximized.

3. MATHEMATICAL MODEL AND ANALYSIS

At time t = 0, the inventory level $I(t)$ reaches the top $\bar{I}$ (with $\bar{I} \le B$) due to ordering the economic order quantity Q. The inventory level then gradually depletes to y at the end of the cycle time $t = T$ mainly for demand and partly for deterioration. A graphical representation of this inventory system is depicted in Figure 1. The differential equation expressing the inventory level at time t can be written as follows:

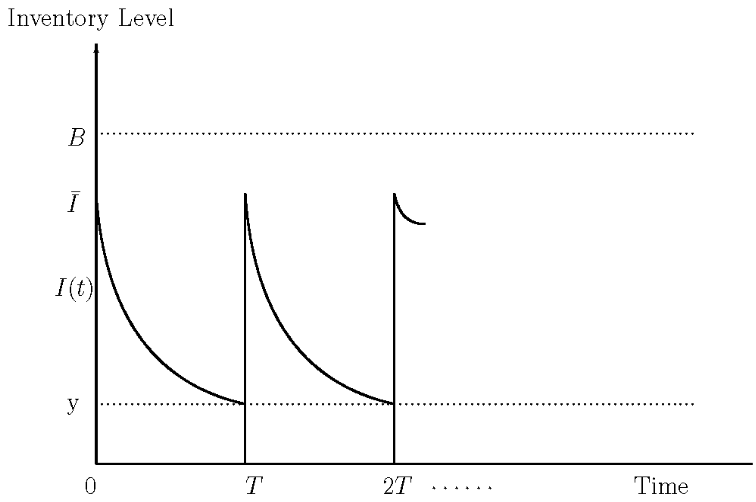

Figure 1. Graphical Representation of Inventory System

$$I'(t) + \theta\, I(t) = -R(I(t), p)\ ,\ 0 \le t \le T\ , \tag{1}$$

with the boundary condition $I(T) = y$. Accordingly, the solution of Equation (1) is given by

$$I(t) = \ y e^{(\theta+\beta)(T-t)}\ +\ \frac{\alpha(p)}{\theta+\beta}\left(e^{(\theta+\beta)(T-t)} - 1\right), \quad 0 \le t \le T\ . \tag{2}$$

Applying (2), we obtain that the total profit *TP* over the period [0, *T*] is denoted by

$$TP = \ (p-c)\int_0^T R(I(t), p)\, dt - K - \ [h + \theta\,(c+s)]\ \ \int_0^T\ I(t)\, dt$$

$$= (p-c)\alpha(p)\, T - K + [(p-c)\beta - h - \theta\,(c+s)]\ \times$$

$$\left[\ \int_0^{\mathrm{T}}\ y e^{(\theta+\beta)(T-t)} + \frac{\alpha(p)}{\theta+\beta}\,(e^{(\theta+\beta)(T-t)} - 1)\, dt\ \right]$$

$$= \ (p-c)\alpha(p)\, T - K + \ [(p-c)\beta - h - \theta\,(c+s)] \times$$

$$\left[\ \frac{1}{\theta+\beta}\left(y + \frac{\alpha(p)}{\theta+\beta}\right)\left(e^{(\theta+\beta)T} - 1\right) - \frac{\alpha(p)}{\theta+\beta}T\ \right]. \tag{3}$$

Hence, the average profit per unit time is
AP = TP / T

$$= (p-c)\alpha(p) + \{-K + [(p-c)\beta - h - \theta(c+s)] \times$$

$$\left[\frac{1}{\theta+\beta}\left(y + \frac{\alpha(p)}{\theta+\beta}\right)\left(e^{(\theta+\beta)T} - 1\right) - \frac{\alpha(p)}{\theta+\beta}T \right]\} / T. \tag{4}$$

Necessary conditions for an optimal solution

Taking the first derivative of AP as defined in (4) with respect to T, we have

$$\partial AP / \partial T$$

$$= \frac{1}{T^2}\{K + [(p-c)\beta - h - \theta(c+s)] \times$$

$$\left(\frac{1}{\theta+\beta}\right)\left(y + \frac{\alpha(p)}{\theta+\beta}\right)[(\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T} + 1]\}. \tag{5}$$

From Appendix 1, we show that $[(\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T} + 1]$ is greater than zero. $[(p-c)\beta]$ is the benefit received from a unit of inventory and $[h + \theta(c+s)]$ is the total cost (*i.e.*, holding and deterioration costs) per unit inventory. Let $\Delta_1 = (p-c)\beta$ and $\Delta_2 = h + \theta(c+s)$, based on the values of Δ_1 and Δ_2, two distinct cases for finding the optimal T^* are discussed as follows:

Case 3.1 $\Delta_1 \geq \Delta_2$ (Building up inventory is profitable)

"$\Delta_1 \geq \Delta_2$" implies that the benefit received from a unit of inventory is larger than the total cost (*i.e.,* holding and deterioration costs) due to a unit of inventory. That is, it is profitable to build up inventory. Using Appendix 1, $\partial AP / \partial T > 0$, if $\Delta_1 \geq \Delta_2$. Namely, AP is an increasing function of T with $I(t) \leq B$. Therefore, we should pile up inventory to the maximum allowable number B of stocks displayed in a supermarket without leaving a negative impression on customers. So, $I(0) = B$. From $I(0) = B$, we know

$$T = \frac{1}{\theta+\beta}\ln\left(\frac{B(\theta+\beta)+\alpha(p)}{y(\theta+\beta)+\alpha(p)}\right), \tag{6}$$

which implies that T is a function of p and y.

Substituting (6) into (4), we know that AP is a function of y and p.

The necessary conditions of AP to be maximized are $\partial AP / \partial y = 0$ and $\partial AP / \partial p = 0$. Hence, we have the following two conditions:

$$\frac{-K(\theta+\beta)^2}{\Delta_1 - \Delta_2} = (\theta+\beta)(y\text{-}B) + [\alpha(p) + y(\theta+\beta)]\ln\left(\frac{\alpha(p)+B(\theta+\beta)}{\alpha(p)+y(\theta+\beta)}\right), \tag{7}$$

and

$$\frac{\alpha(p)\theta+[\theta(p+s)+h]\alpha'(p)}{\theta+\beta}\,T^2+\frac{(e^{(\theta+\beta)T}-1)}{\theta+\beta}\{\beta\,(y+\frac{\alpha(p)}{\theta+\beta})$$

$$+\,(\Delta_1-\Delta_2)\frac{\alpha'(p)}{\theta+\beta}\}T$$

$$=-\{K+(\Delta_1-\Delta_2)\,(\frac{1}{\theta+\beta})\,(y+\frac{\alpha(p)}{\theta+\beta})\times$$

$$[(\theta+\beta)Te^{(\theta+\beta)T}-e^{(\theta+\beta)T}+1]\}\frac{\partial T}{\partial p}, \tag{8}$$

where T is defined as (6) and

$$\frac{\partial T}{\partial p}=\frac{(y-B)\alpha'(p)}{[\alpha(p)+(\theta+\beta)B][\alpha(p)+(\theta+\beta)y]}. \tag{9}$$

From (7) and (8), the optimal values of p^* and y^* are obtained. Substituting p^* and y^* into (6), the optimal value T^* is solved.

Since $AP(y, p)$ is a complicated function, it is not possible to show analytically the validity of the sufficient conditions. However, according to the following mention, we know that the optimal solution can be obtained by numerical examples. Because building up is profitable and AP is a continuous function of y and p over the compact set $[0, B]\times[0, L]$, where L is a sufficient large number, so AP has a maximum value. It is clear that AP is not maximum at $y = 0$ (or B) and $p = 0$ (or L). Therefore, the optimal solution is an inner point and must satisfy (7) and (8). If the solution from (7) and (8) is unique, then it is the optimal solution. Otherwise, we have to substitute them into (4) and find the one with the largest values.

Case 3.2. $\Delta_1 < \Delta_2$ (Building up inventory is not profitable)

First taking the partial derivative of AP with respect to y, we obtain

$$\partial AP/\partial\,y\;=\;\frac{1}{T}[(\Delta_1-\Delta_2)\frac{1}{\theta+\beta}(e^{(\theta+\beta)T}-1)]<0. \tag{10}$$

Next, we get $y^* = 0$. Substituting $y^* = 0$ into (4), we have AP is a function of p and T.

So, the necessary conditions of AP to be maximized are $\partial AP/T = 0$ and $\partial AP/p = 0$. Then, we get the following two conditions:

$$\frac{-K(\theta+\beta)^2}{\alpha(p)(\Delta_1-\Delta_2)}=(\theta+\beta)Te^{(\theta+\beta)T}-e^{(\theta+\beta)T}+1, \tag{11}$$

and

$$[\,\alpha(p)\theta+(p\theta+h+\theta\,\mathrm{s})\alpha'(p)\,]T$$

$$=-\frac{(e^{(\theta+\beta)T}-1)}{\theta+\beta}[\,\beta\alpha(p)+(\Delta_1-\Delta_2)\,\alpha'(p)\,]. \tag{12}$$

From (11) and (12), we can obtain the values for T and p. Substituting $y^* = 0$, T and p into (2) and check whether $I(0) < B$ or not. If $I(0) < B$, then the optimal values $T^* = T$, $p^* = p$ and $Q^* = I(0)$. If $I(0) \geq B$, then set $I(0) = B$ and obtain

$$T = \frac{1}{\theta+\beta} \ln\left(\frac{B(\theta+\beta)+\alpha(p)}{\alpha(p)}\right), \tag{13}$$

which is a function of p. Substituting $y^* = 0$ and (13) into (4), we have AP is only depend on p. Then, the necessary conditions of AP to be maximized is $dAP / dp = 0$. Hence,

$$\frac{\alpha(p)\theta+[\theta(p+s)+h]\alpha'(p)}{\theta+\beta} T^2 + \frac{(e^{(\theta+\beta)T}-1)}{\theta+\beta}[\beta \frac{\alpha(p)}{\theta+\beta} +(\Delta_1-\Delta_2)\frac{\alpha'(p)}{\theta+\beta}]\, T$$

$$= -\ \{K + (\Delta_1-\Delta_2)\frac{\alpha(p)}{(\theta+\beta)^2}[(\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T}+1]\}\frac{dT}{dp}, \tag{14}$$

where T is defined as (13) and

$$\frac{dT}{dp} = \frac{-B\alpha'(p)}{[\alpha(p)+(\theta+\beta)B]\alpha(p)}. \tag{15}$$

The optimal value p^* is determined by (14). Substituting p^* into (13), the optimal value T^* is solved.

Algorithm :

The algorithm for determining an optimal selling price p*, optimal ordering point y*, optimal cycle time T*, and optimal economic order quantity Q* is summarized as follows:

Step 1. Solving (7) and (8), we get the values for p and y.

Step 2. If $\Delta_1 \geq \Delta_2$, then $p^* = p$, $y^* = y$, $Q^* = B - y^*$, and the optimal value T^* can be obtained by substituting p and y into (6).

Step 3. If $\Delta_1 < \Delta_2$, then re-set $y^* = 0$. By solving (11) and (12), we get the values for T and p. Substituting $y^* = 0$, p and T into (2) to find $I(0)$. If $I(0) < B$, then the optimal values $T^* = T$, $p^* = p$ and $Q^* = I(0)$, and stop. Otherwise, go to Step 4.

Step 4. If the simultaneous solutions T and p in (11) and (12) make $I(0) > B$, then the optimal value p^* is determined by (14), T^* is obtained by substituting p^* into (13), and $Q^* = I(0)$ by substituting p^* and T^* into (2).

Numerical examples

To illustrate the proposed model, we provide the numerical examples here. For simplicity, we set the function $\alpha(p) = xp^{-r}$, where x and r are non-negative constants. That is, we assume that demand is a constant elasticity function of the price.

Example 3.1 Let K = \$10 per cycle, x = 1000 units per unit time, h = \$0.5 per unit per unit time, s = \$0 per unit, r = 2.5 and θ = 0.05. Following through the proposed algorithm, the optimal solution can be obtained. Since (4) and (6)-(9) are nonlinear, they are extremely difficult to solve. We use *Maple* 9.5 software to solve them. The computational results for the optimal values of p, y, T, Q and AP with respect to different values of β , B, c are shown in Table 3.1.

Table 3.1 Computational results for the case of $\Delta_1 \geq \Delta_2$

β	B	c	y^*	Q^*	p^*	T^*	AP^*
0.15	100	1.5	29.7671	70.2329	6.036963	2.995380	53.8080
0.20			27.5915	72.4085	5.057843	2.228339	65.6087
0.25			21.6955	78.3045	4.401015	1.874682	74.6548
0.30			12.9392	87.0608	3.916335	1.700138	81.5477
0.35			1.5681	98.4319	3.542865	1.626419	86.6871
0.20	100	1.5	27.5915	72.4085	5.057843	2.228339	65.6087
	110		25.7399	84.2601	4.916473	2.437927	66.5322
	130		19.8247	110.1753	4.727722	2.927107	67.8135
	150		12.1859	137.8141	4.618470	3.478172	68.6228
	170		3.9578	166.0422	4.552949	4.059602	69.1629
0.20	100	1.1	47.2880	52.7120	5.192483	1.538303	79.0717
		1.3	38.7618	61.2382	5.099564	1.811917	72.2547
		1.5	27.5915	72.4085	5.057843	2.228339	65.6087
		1.7	14.7100	85.2900	5.094514	2.827599	59.3269
		1.9	2.3596	97.6404	5.209061	3.598091	53.5902

Based on the computational results as shown in Table 3.1, we obtain the following managerial phenomena when building up inventory is profitable:

(1) A higher value of β causes higher values of Q^* and AP^*, but lower values of y^*, p^* and T^*. It reveals that the increase of demand rate will result in the increases of optimal economic order quantity and average profit, but the decreases of optimal ordering point, selling price and cycle time.

(2) A higher value of B causes higher values of Q^*, T^* and AP^*, but lower values of $y^*$and p^*. It implies that the increase of shelf space will result in the increases of optimal economic order quantity, cycle time and average profit, but the decreases of optimal ordering point and selling price.

(3) A higher value of c causes higher values of Q^* and T^*, but lower values of y^* and AP^*. It implies that the increase of purchase cost will result in the increases of optimal economic order quantity and cycle time, but the decreases of optimal ordering point and average profit.

Example 3.2 Let K = \$10 per cycle, x = 1000 units per unit time, h = \$0.2 per unit per unit time, c = \$1.0 per unit, s = \$0 per unit, r = 2.8, θ = 0.05 and B = 300. From Step 3 of the proposed algorithm, we obtain the optimal ordering point y^* = 0. Using *Maple* 9.5 software, we solve (2), (4), (11) and (12). The computational results for the optimal values of p, Q, T and AP with respect to different values of β are shown in Table 3.2.

Table 3.2 Computational results for the case of $\Delta_1 < \Delta_2$

β	Q^*	p^*	T^*	AP^*
0.10	162.6161	1.685130	0.666568	129.4149
0.12	169.2624	1.689956	0.693068	130.4691
0.15	181.2873	1.698773	0.741555	132.1537
0.17	191.2537	1.706166	0.782279	133.3611
0.20	211.0556	1.721085	0.864684	135.3406

Table 3.2 shows that a higher value of β causes in higher values of Q^*, p^*, T^* and AP^*. It indicates that the increase of demand rate will result in the increases of optimal economic order quantity, selling price, cycle time and average profit, when building up inventory is not profitable.

4. AN OPTIMAL ORDERING POLICY MODEL WITH SELLING PRICE PREDETERMINED

In the previous section, only the necessary condition was outlined for determining optimal values of p, T, Q and y. The existence and uniqueness of the optimal solution remained unexplored. In addition, most firms have no pricing power in today's business competition. As a result, most firms are not able to change price. In order to reflect this important fact, in this section, we study a special case that the selling price is predetermined. In this special case, we are able to show that the optimal solution to the relevant problem exists uniquely. Theorems 1 and 2 are provided to present the characteristics of the optimal solution. An easy-to-use algorithm is proposed to determine the optimal cycle time, ordering point and order quantity.

Necessary conditions for an optimal solution

Since p is predetermined, $\alpha(p)$ is reduced to α . Equation (4) can be rewritten as follows:

$$AP = (p-c)\alpha + \{-K + (\Delta_1 - \Delta_2)\times \left[\frac{1}{\theta+\beta}\left(y+\frac{\alpha}{\theta+\beta}\right)\left(e^{(\theta+\beta)T}-1\right) - \frac{\alpha}{\theta+\beta}T\right]\}/T. \tag{16}$$

Evidently, AP is a function of T and y. The model now is to determine the optimal values of T and y such that AP in (16) is maximized.

Taking the first derivative of AP with respect to T, we have

$$\partial AP / \partial T = \frac{1}{T^2}\{K + (\Delta_1 - \Delta_2)\left(\frac{1}{\theta+\beta}\right)\left(y+\frac{\alpha}{\theta+\beta}\right)[(\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T} + 1]\}. \tag{17}$$

By applying analogous argument with Equation (5), there are two distinct cases for finding the optimal T^* are discussed as follows:

Case 4.1 $\Delta_1 \geq \Delta_2$ (Building up inventory is profitable)

Using Appendix 1, $\partial AP / \partial T > 0$ if $\Delta_1 \geq \Delta_2$. Namely, AP is an increasing function of T with $I(t) \leq B$. Consequently, $I(0) = B$. From $I(0) = B$, we know

$$T = \frac{1}{\theta+\beta} \ln\left(\frac{B(\theta+\beta)+\alpha}{y(\theta+\beta)+\alpha}\right), \tag{18}$$

which indicates that T is a function of y.

Substituting (18) into (16), we know that AP is only a function of y. The first-order condition for finding the optimal y^* is $dAP / dy = 0$, which leads to

$$\frac{-K(\theta+\beta)^2}{\Delta_1-\Delta_2} = (\theta+\beta)\,(y - B) + [\alpha + y(\theta+\beta)] \ln\left(\frac{\alpha + B(\theta+\beta)}{\alpha + y(\theta+\beta)}\right). \tag{19}$$

To examine whether (19) has a unique solution, we set

$$H(y) = (\theta+\beta)\,(y - B) + [\alpha + y(\theta+\beta)] \ln\left(\frac{\alpha + B(\theta+\beta)}{\alpha + y(\theta+\beta)}\right). \tag{20}$$

Taking the first derivative of $H(y)$ with respect to y, we get

$$H'(y) = (\theta+\beta)\ \ln\left(\frac{B(\theta+\beta)+\alpha}{y(\theta+\beta)+\alpha}\right) > 0. \tag{21}$$

By $H(B) = 0$ and (20), we know that $H(y)$ is negative and strictly increasing to zero at $y = B$. Consequently, we can obtain the following theorem.

Theorem 1. Under the condition $\Delta_1 \geq \Delta_2$, $I(0) = B$ and the following results state

If $H(0) \leq -K(\theta+\beta)^2/(\Delta_1-\Delta_2)$, then there exists a unique solution y^* in (19) which maximizes AP in (16).

If $H(0) > -K(\theta+\beta)^2/(\Delta_1-\Delta_2)$, then $y^* = 0$.

Proof. AP is a continuous function of y over the compact set $[0, B]$, and hence a maximum exists. The proof of part (a) immediately follows from (21) and $H(0) \leq -K(\theta+\beta)^2 / (\Delta_1-\Delta_2) < H(B) = 0$. From Appendix 2, we show that AP is a strictly concave function at y^*. Therefore, the unique optimal solution is an inner point if $H(0) < -K(\theta+\beta)^2/(\Delta_1-\Delta_2)$. Otherwise (*i.e.*, $H(0) > -K(\theta+\beta)^2 / (\Delta_1-\Delta_2)$), the optimal solution is at the boundary point $y = 0$ (Since AP is zero at $y = B$, $y = B$ is not an optimal solution). The proof of part (b) is completed.

Case 4.2 $\Delta_1 < \Delta_2$ (Building up inventory is not profitable)

The necessary conditions for maximizing AP are $\partial AP/\partial T = 0$ and $\partial AP/\partial y = 0$. For the part $\partial AP/\partial T = 0$, we have

$$K + (\Delta_1 - \Delta_2)\left(\frac{1}{\theta+\beta}\right)\left(y + \frac{\alpha}{\theta+\beta}\right)\left[(\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T} + 1\right] = 0. \quad (22)$$

Taking the partial derivative of AP with respect to y, we obtain

$$\partial AP/\partial y = \frac{1}{T}\left[\ (\Delta_1 - \Delta_2)\frac{1}{\theta+\beta}\left(e^{(\theta+\beta)T} - 1\right)\ \right] < 0. \quad (23)$$

Therefore, we get $y^* = 0$. Substituting $y^* = 0$ into (22), we can get

$$0 < \frac{-K(\theta+\beta)^2}{\alpha(\Delta_1 - \Delta_2)} = (\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T} + 1. \quad (24)$$

Again, to examine whether (24) has a solution or not, we set

$$G(T) = (\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T} + 1. \quad (25)$$

Taking the first derivative of $G(T)$ with respect to T, we have

$$G'(T) = (\theta+\beta)^2 Te^{(\theta+\beta)T} > 0. \quad (26)$$

Since $G(0) = 0$, there exists a unique solution T (which is greater than 0) for (24). This is done in the following theorem.

Theorem 2. If $\Delta_1 < \Delta_2$, then the optimal ordering point $y^* = 0$, and there exists a unique solution T^* in (24) which maximizes AP in (16).

Proof. AP is a continuous function of T over the compact set $[0, T]$, and hence a maximum exists. Since AP is zero at $T = 0$, the optimal T^* is an inner point. From Appendix 3, we know that AP is a strictly concave function at T^*. Thus, the unique solution to (24) is the optimal solution that maximizes AP in (16).

Algorithm

It is apparent from Theorem 1 and 2 that the value of AP is influenced by the values of Δ_1 and Δ_2. Consequently, the algorithm for determining the optimal cycle time T^*, optimal ordering point y^* and optimal economic order quantity Q^* is summarized as follows:

Step 1. If $\Delta_1 \geq \Delta_2$ and $H(0) \leq -K(\theta+\beta)^2/(\Delta_1 - \Delta_2)$, then the optimal ordering point y^* can be determined by (19), the optimal cycle time T^* can be obtained by substituting y^* into (18), and the optimal economic order quantity $Q^* = B - y^*$.

Step 2. If $\Delta_1 \geq \Delta_2$ and $H(0) > -K(\theta+\beta)^2/(\Delta_1 - \Delta_2)$, then the optimal ordering point $y^* = 0$, and thus the optimal cycle time T^* can be obtained by substituting $y^* = 0$ into (18), and the optimal economic order quantity $Q^* = B$.

Step 3. If $\Delta_1 < \Delta_2$, then the optimal $y^* = 0$. By solving (24), we get the value for T. Substituting $y^* = 0$ and T into (2) to find $I(0)$. If $I(0) < B$, then the optimal economic order quantity $Q^* = I(0)$ and the optimal cycle time $T^* = T$. Otherwise, $Q^* = B$ and the optimal cycle time T^* can be determined by $I(0) = B$.

Numerical examples

The numerical examples are given here to demonstrate the applicability of the proposed model.

Example 4.1 Let K = \$100 per cycle, α = 100 units per unit time, h = \$1.0 per unit per unit time, c = \$1.0 per unit, s = \$0 per unit, p = \$6 per unit, θ = 0.2 and B = 250. If β = 0.05, 010, 0.15 and 0.20, then $\Delta_1 < \Delta_2$. Using the Step 3 of the proposed algorithm, we can obtain the optimal solution that is the optimal ordering point y^* = 0, T^* and Q^*. If β = 0.25 and 0.30, then $\Delta_1 \geq \Delta_2$ and $H(0) > -K\ (\theta+\beta)^2/(\Delta_1-\Delta_2)$. We can use Step 2 of the proposed algorithm and find the optimal solution that is the optimal ordering point y^* = 0, optimal economic order quantity $Q^* = B$ and T^*. If β = 0.4, 05, 0.6 and 0.7, then $\Delta_1 \geq \Delta_2$ and $H(0) \leq -K(\theta+\beta)^2 / (\Delta_1-\Delta_2)$. From Step 1 of the proposed algorithm, the optimal solutions of y^*, T^* and Q^* can be attained. The computational results for the optimal values of y, T, Q and AP with respect to different values of β are shown in Table 4.1.

Table 4.1 Computational results with respect to different values of β

β	y^*	Q^*	T^*	AP^*
0.05	0	153.6248	1.300090	354.0565
0.10	0	182.7822	1.457292	372.0525
0.15	0	235.4000	1.717078	394.0700
0.20	0	250.0000	1.732868	420.1574
0.25	0	250.0000	1.675048	445.7724
0.30	0	250.0000	1.621860	470.8288
0.40	26.5083	223.4917	1.281150	521.2066
0.50	63.1794	186.8206	0.921989	582.1333
0.60	82.9370	167.0630	0.737114	649.2866
0.70	92.5158	157.4842	0.620301	719.6862

Table 4.1 reveals that (1) If $\Delta_1 < \Delta_2$, then the values of Q^*, T^* and AP^* increase when the value of β increases. It implies that the increase of demand rate causes the increases of optimal economic order quantity, cycle time and average profit when building up inventory is not profitable.(2) If $\Delta_1 \geq \Delta_2$, then the values of y^* and AP^* increase but the values of Q^* and T^* decrease when the value of β increase. It shows that a higher demand rate causes higher values of optimal ordering point and average profit, but lower values of economic order quantity and cycle time.

5. CONCLUSION

This article presents the inventory models for deterioration items when the demand is a function of the selling price and stock on display. We also impose a limited maximum amount of stock displayed in a supermarket without leaving a negative impression on customers. Under these conditions, a proposed model has been shown for maximizing profits. Then, the properties of the optimal solution are discussed as well as its solution algorithm and numerical examples are presented to illustrate the model. In addition, in order to reflect an important fact that most firms have no pricing power in today's business competition, we study a special case that the selling price is considered by predetermination. We then provide Theorems 1 and 2 to show the characteristics of the optimal solution and establish an easy-to-use algorithm to determine the optimal cycle time, economic order quantity and ordering point. Furthermore, we discover some intuitively reasonable managerial results. For example, if the benefit received from a unit of inventory is larger than the total cost per unit inventory, then the building up inventory is profitable and thus the beginning inventory should reach to the maximum allowable level. Otherwise, building up inventory is not profitable and the ending inventory should be zero. Finally, numerical examples are provided to demonstrate the applicability of the proposed model. The results also indicate that the effect of stock dependent selling rate on the system behavior is significant, and hence should not be ignored in developing the inventory models. The sensitivity analysis shows the influence effects of parameters on decision variables.

The proposed models can further be enriched by incorporating inflation, quantity discount, and trade credits etc. Besides, it is interested to extend the proposed model to multi-item inventory systems based on limited shelf space or to consider the demand rate which is a polynomial form of on-hand inventory dependent demand. Finally, we may extend the deterministic demand function to stochastic fluctuating demand patterns.

APPENDIX

Appendix 1. If $\Delta_1 \geq \Delta_2$, then *AP* is an increasing function of *T*.

To prove $\partial AP/\partial T > 0$, we set

$$f(x) = xe^x - e^x + 1, \text{ for } x \geq 0. \tag{A.1}$$

Then (A.1) yields $f'(x) = xe^x > 0$. So, $f(x)$ is an increasing function of x for $x \geq 0$. We get

$$f(x) > f(0) = 0. \tag{A.2}$$

Let $x = (\theta+\beta)T$. Using (A.1) and (A.2), we obtain

$$(\theta+\beta)Te^{(\theta+\beta)T} - e^{(\theta+\beta)T} + 1 > 0, \text{ for } T > 0. \tag{A.3}$$

Applying (5) and (A.3), we have $\partial AP/\partial T > 0$.

Appendix 2. If $\Delta_1 \geq \Delta_2$, then *AP* is strictly concave at y*.

From (19), we know the second-order derivative of *AP* with respect to y as:

$$\frac{\partial^2 AP}{\partial y^2}$$

$$= \frac{1}{T^2}\{(\Delta_1 - \Delta_2)(\frac{1}{\theta+\beta})[\frac{\alpha + B(\theta+\beta)}{\alpha + y(\theta+\beta)} \text{-1}](\frac{-1}{\alpha + y(\theta+\beta)})\} < 0, \tag{A.4}$$

which implies *AP* is strictly concave at y*.

Appendix 3. If $\Delta_1 < \Delta_2$, then *AP* is strictly concave at T *.

Applying (22) and $y^* = 0$, we obtain the second-order derivative:

$$\frac{\partial^2 AP}{\partial T^2} = (\Delta_1 - \Delta_2)\,\alpha T e^{(\theta+\beta)T} < 0, \tag{A.5}$$

which implies *AP* is strictly concave at T*.

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