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AN EOQ INVENTORY MODEL FOR ITEMS WITH RAMP TYPE DEMAND, THREE-PARAMETER WEIBULL DISTRIBUTION DETERIORATION AND STARTING WITH SHORTAGE

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Abstract: In this present paper an inventory model is developed with ramp type demand, starting with shortage and three – parameter Weibull distribution deterioration. A brief analysis of the cost involved is carried out by an example.

Keywords: EOQ, Weibull distribution deterioration, Shortage, ramp type demand.

AMS Subject Classification: 90B05

1. INTRODUCTION

A number of inventory models were developed by researchers assuming the demand of the items to be constant, linearly increasing or decreasing demand or exponentially increasing or decreasing with time. Later it was experienced that above demand patterns do not precisely depict the demand of certain items such as newly launched fashion goods and cosmetics, garments, automobiles etc; for which the demand increases with time as they are launched into the market and after some time it becomes constant. In order to consider the demand of such items, the concept of ramp type demand is introduced. Ramp type demand function depicts a demand, which increases up to a certain time after which it stabilizes and becomes constant.

In recent years, there is a spate of interest in studying the inventory models for deteriorating items. Ghare and Schrader [5] were the earliest researchers who introduced the aspect of deterioration in the inventory models; they developed an inventory model for exponentially decay in which inventory is not only depleted by demand alone but also by direct spoilage, physical depletion or deterioration. After Ghare and Schrader's work, a number of researchers worked on inventory model for deteriorating items, assuming the rate of deterioration to be constant and dependent on time. Among these researchers Covert and Philip [2], Misra [13], Elsayed and Teresi [4], Jalan et al [8] used two-parameter Weibull distribution and Philip [15], Chakrabarty et al [1] used three-parameter Weibull distribution to represent the time to deterioration.

The work of the researchers who used ramp-type demand as demand function and various form of deterioration, for developing the economic order quantity models is summarized below:

Reference	Objective(s)	Constraints	Contributions	Limitations
Mandal & Pal [12], 1998	Finding EOQ	Ramp type demand, Const. rate of deterioration, Shortage not allowed	An approximate Sol^n for EOQ is obtained	Approximate Sol^n, Constant rate of deterioration
Kun-Shan & Ouyang [10], 2000	Finding EOQ	Ramp type demand, Const. rate of deterioration, Shortage allowed	An exact Sol^n for EOQ is obtained	Constant rate of deterioration
Jalan, Giri & Chaudhuri [8], 2001	Finding EOQ	Ramp type demand, Weibull deterioration, Shortage allowed	EOQ given by Numerical Technique	EOQ can't be obtained analytically
Kun-Shan [11], 2001	Finding EOQ	Ramp type demand, Weibull deterioration, Shortage allowed, Partial backlogging	EOQ obtained for 3 numerical examples.	Method explained by numerical examples.

The above table shows that only a few researchers developed EOQ models by taking ramp type demand, deterioration (constant / Weibull distribution) and shortage (allowed / not allowed). Among these researchers only [10] obtained an exact solution for EOQ.

It is commonly observed that there are some items, which do not start deteriorating as soon as they are received; instead, deterioration starts after some time, as they are actually included in the stock. For such items three-parameter Weibull distribution can be used to represent the time to deterioration. The motivation behind developing an inventory model in the present article is to prepare a more general inventory model, which includes three-parameter Weibull distribution deterioration, incorporating ramp type demand and starting with shortage. An exact solution of the developed model is obtained. Numerical example is presented to illustrate the effectiveness of the model.

2. ASSUMPTIONS AND NOTATIONS

The model is developed under the following assumptions and notations.

2.1 Assumptions

- The inventory system is considered over an infinite time horizon.
- Shortages in inventory are allowed and are completely backlogged.
- Rate of replenishment is assumed to be infinite.
- Lead-time is practically assumed to be zero.
- The instantaneous rate function Z(*t*) for two-parameter Weibull distribution is given by

$$Z(t) = \alpha \beta t^{\beta - 1} \tag{1}$$

where α (0 < α <<1) is the scale parameter, β (> 0) is the shape parameter; *t* (*t*>0) is the time of deterioration. From (1) and figure 2.1 it is clear that the two parameter Weibull distribution is appropriate for an item with decreasing rate of deterioration only if the initial rate of deterioration is extremely high and with increasing rate of deterioration only if the initial rate of deterioration is approximately zero.

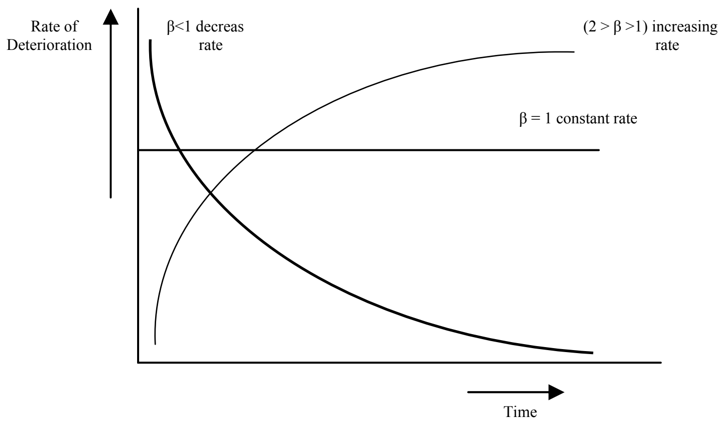

Figure 2.1 Rate of deterioration-time relationship for a two-parameter Weibull distribution

However, these limitations can be removed by using three-parameter Weibull distribution to represent the time to deterioration. The density function *f*(*t*) for this distribution is given by

$$f(t) = \alpha\beta(t - \gamma)^{\beta - 1} e^{-\alpha(t-\gamma)^{\beta}} \tag{2}$$

where α, β, t are defined as earlier and $\gamma (t \geq \gamma)$ is the location parameter.

The instantaneous rate of deterioration of the non-deteriorated inventory at time *t*, Z(*t*) can be obtained by using the relation

$$Z(t)=\frac{f(t)}{1-F(t)} \tag{3}$$

Where F(t) is the cumulative distribution function for the three-parameter Weibull distribution and is given by

$$F(t)=1-e^{-\alpha(t-\gamma)^{\beta}} \tag{4}$$

substituting the values of $f(t)$ and $F(t)$ from (2) and (4) in (3) and simplifying, we obtain

$$Z(t)=\alpha\beta(t-\gamma)^{\beta-1} \tag{5}$$

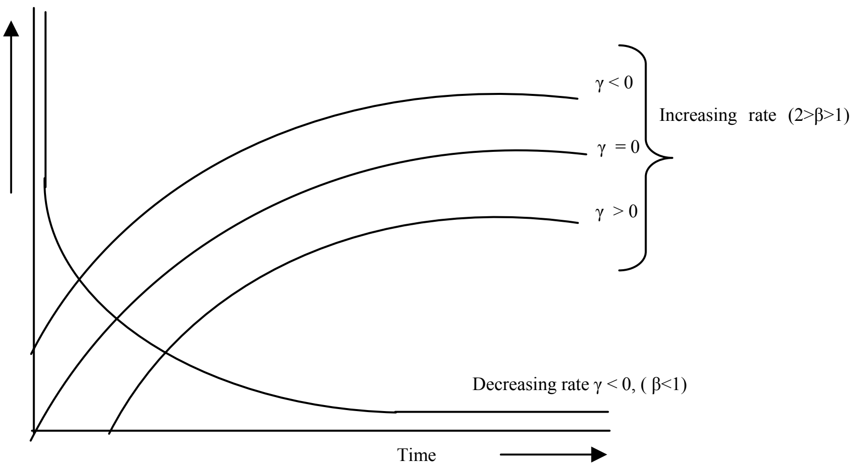

Figure 2.2 Rate of deterioration-time relationship for three-parameter Weibull distribution

From figure 2.2 it is clear that the three-parameter Weibull distribution is suitable for items with any initial value of the deterioration and also for the items, which start deteriorating only after a certain period of time.

It is assumed here that the rate of deterioration at any time t > 0 follows the three-parameter Weibull distribution $Z(t)=\alpha\beta(t-\gamma)^{\beta-1}$ where α , β, t are defined as earlier and γ (0< γ <1) is the location parameter. The reason behind imposing the condition (0 < γ <1) on location parameter lies in the fact that we are developing the inventory model for which items start deteriorating after a short period of time, as they are included into stock.

- The demand function $R\,(t)$ is taken to be a ramp type function of time:

  $R(t) = Ae^{b[t-(t-\mu)H(t-\mu)]}$

Where H (t – μ) is the well known Heaviside's function defined as :

$$H(t-\mu) = \begin{cases} 1, & t \geq \mu \\ 0, & t < \mu \end{cases}$$

A = initial demand rate, b = a constant governing the exponential demand rate

- Inventory model is developed to start with shortages and only for $\mu < t_1$.

2.2 Notations

T= the fixed length of each ordering cycle
S= the maximum inventory level for each ordering cycle
C_h= the inventory holding cost per unit per unit time
C_s= the shortage cost per unit per unit time
C_d= the cost of deterioration for single unit
I (t)= the on hand inventory at time t over [0, T]
t_1= Procurement time

3. DEVELOPMENT OF THE MODEL

The inventory system developed is depicted by the following figure

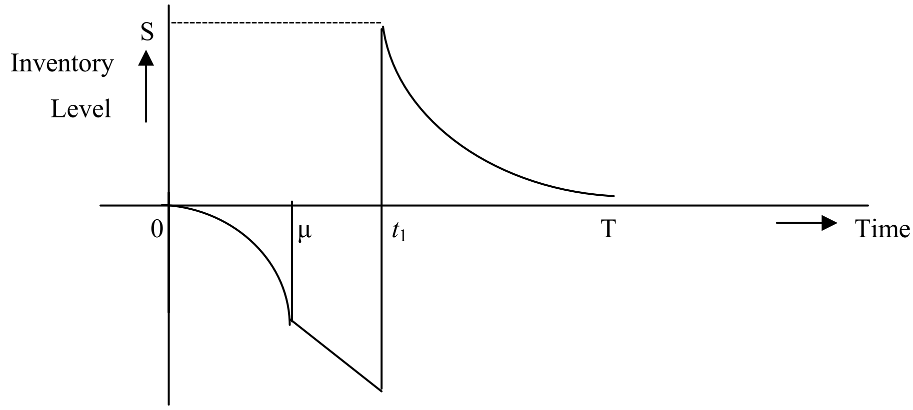

Figure 3.1 Inventory level-time relationship

The inventory system starts with zero inventory at t = 0. Shortages are allowed to accumulate up to time t_1. At time t_1 inventory is replenished. The quantity received at t_1 is partly used to meet the shortages which accumulated from time 0 to t_1, leaving a balance of S items at time t_1. As time passes, the inventory level S declines only due to demand during the period [t_1, γ], and mainly due to demand and partly due to deterioration of items during the period [γ, T]. At time T the inventory level gradually falls to zero.

The inventory level of the system at any time over the period [0, T] can be described the following differential equations:

$$\frac{dI(t)}{dt} = -Ae^{bt},\ 0 \le t \le \mu \tag{6}$$

$$\frac{dI(t)}{dt} = -Ae^{b\mu},\ \mu \le t \le t_1 \tag{7}$$

$$\frac{dI(t)}{dt} = -Ae^{b\mu},\ t_1 \le t \le (t_1+\gamma) \tag{8}$$

$$\frac{dI(t)}{dt} + Z(t)\,I(t) = -Ae^{b\mu}$$

or

$$\frac{dI(t)}{dt} + \alpha\beta(t-\gamma)^{\beta-1} I(t) = -Ae^{b\mu},\ (t_1+\gamma) \le t \le T \tag{9}$$

The solutions of the differential equations (6), (7), (8) and (9) with the boundary conditions I(0) = 0, I(T) = 0 and I(t_1) = S are

$$I(t) = \frac{A}{b}(1-e^{bt});\ 0 \le t \le \mu \tag{10}$$

$$I(t) = e^{b\mu}(\mu - t) + \frac{A}{b}(1-e^{b\mu});\ \mu \le t \le t_1 \tag{11}$$

$$I(t) = S + Ae^{b\mu}(t_1 - t);\ t_1 \le t \le (t_1+\gamma) \tag{12}$$

$$I(t) = Ae^{b\mu-\alpha(t-\gamma)^{\beta}}\left[(T-t) + \left(\frac{\alpha}{\beta+1}\right)\left\{(T-\gamma)^{\beta+1} - (t-\gamma)^{\beta+1}\right\}\right];(t_1+\gamma) \le t \le T \tag{13}$$

In equations (12) and (13) values of I(t) at $t = (t_1+\gamma)$ should coincide, which implies that

$$S = Ae^{b\mu-\alpha t_1^{\beta}}\left[\left\{T-(t_1+\gamma)\right\} + \left(\frac{\alpha}{\beta+1}\right)\left\{(T-\gamma)^{\beta+1} - t_1^{\beta+1}\right\}\right] + A\gamma e^{b\mu} \tag{14}$$

Now the total number of items deteriorate during [t_1, T] is

χ_d = S – Demand during [t_1, T]

or, $A\,e^{b\mu-\alpha t_1^{\beta}}\left[\{T-(t_1+\gamma)\}+\left(\dfrac{\alpha}{\beta+1}\right)\left\{(T-\gamma)^{\beta+1}-t_1^{\beta+1}\right\}\right]+A\gamma\,e^{b\mu}-Ae^{b\mu}(T-t_1)$

By using (14) and R(t) = $Ae^{b[t-(t-\mu)H(t-\mu)]}$, we have

$$\chi_d = A\,e^{b\mu-\alpha t_1^{\beta}}\left[\{T-(t_1+\gamma)\}+\left(\frac{\alpha}{\beta+1}\right)\left\{(T-\gamma)^{\beta+1}-t_1^{\beta+1}\right\}\right] - A\,e^{b\mu}\{T-(t_1+\gamma)\} \tag{15}$$

The shortage which accumulate during [0, t_1] is

$$\begin{aligned}\chi_s &= -\left[\int_0^{\mu}\frac{A}{b}(1-e^{bt})\,dt+\int_{\mu}^{t_1}\left\{Ae^{b\mu}(\mu-t)+\frac{A}{b}(1-e^{b\mu})\right\}dt\right]\\ &= -\frac{A}{b}\mu+\frac{A}{2}e^{b\mu}(t_1-\mu)^2-\frac{A}{b}(1-e^{b\mu})\left\{\frac{1}{b}+(t_1-\mu)\right\}\end{aligned} \tag{16}$$

The inventory holded over the period [t_1, T] is

$$\begin{aligned}\chi_h &= \int_{t_1}^{t_1+\gamma}\left[Ae^{b\mu-\alpha t_1^{\beta}}\left[(T-(t_1+\gamma))+\left(\frac{\alpha}{\beta+1}\right)((T-\gamma)^{\beta+1}-t_1^{\beta+1})\right]+Ae^{b\mu}(t_1+\gamma-t)\right]dt\\ &+\int_{t_1+\gamma}^{T}Ae^{b\mu-\alpha(t-\gamma)^{\beta}}\left[(T-t)+\left(\frac{\alpha}{\beta+1}\right)((T-\gamma)^{\beta+1}-(t-\gamma)^{\beta+1})\right]dt\\ &= A\gamma\,e^{b\mu-\alpha t_1^{\beta}}\left[\{T-(t_1+\gamma)\}+\left(\frac{\alpha}{\beta+1}\right)((T-\gamma)^{\beta+1}-t_1^{\beta+1})\right]+\frac{A}{2}\gamma^2e^{b\mu}\\ &+Ae^{b\mu}\left[(T-(t_1+\gamma))\left\{T+\left(\frac{\alpha}{\beta+1}\right)(T-\gamma)^{\beta+1}-\frac{(T+t_1+\gamma)}{2}\right\}-\left(\frac{\alpha}{\beta+1}\right)T((T-\gamma)^{\beta+1}-t_1^{\beta+1})\right.\\ &\left.+\left(\frac{\alpha}{\beta+2}\right)(T^{\beta+2}-(t_1+\gamma)^{\beta+2})-\frac{\alpha}{(\beta+1)(\beta+2)}((T-\gamma)^{\beta+2}-t_1^{\beta+2})\right]\end{aligned} \tag{17}$$

(Neglecting higher order of α)

The total relevant cost of the system during the time interval [0, T] by using (15), (16) and (17) is

$$\chi = C_d\chi_d + C_h\chi_h + C_s\chi_s \tag{18}$$

Therefore the average total cost per unit time is

$$\begin{aligned} TC(t_1) = \frac{\chi}{T} = \frac{C_d}{T}&\left[Ae^{b\mu-\alpha\, t_1^{\beta}}\left[(T-(t_1+\gamma)) + \left(\frac{\alpha}{\beta+1}\right)((T-\gamma)^{\beta+1} - t_1^{\beta+1})\right] - Ae^{b\mu}(T-(t_1+\gamma))\right] \\ &+ \frac{C_h}{T}\left[\gamma Ae^{b\mu-\alpha t_1^{\beta}}\left[(T-(t_1+\gamma)) + \left(\frac{\alpha}{\beta+1}\right)\left((T-\gamma)^{\beta+1} - t_1^{\beta+1}\right)\right] + \gamma^2 \frac{A}{2}e^{b\mu}\right] \\ &+ \frac{C_h}{T}Ae^{b\mu}\left[(T-(t_1+\gamma))\left\{T + \left(\frac{\alpha}{\beta+1}\right)(T-\gamma)^{\beta+1} - \frac{(T+t_1+\gamma)}{2}\right\} - \left(\frac{\alpha}{\beta+1}\right)T((T-\gamma)^{\beta+1} - t_1^{\beta+1})\right. \\ &\left. + \frac{\alpha}{(\beta+2)}(T^{\beta+2} - (t_1+\gamma)^{\beta+2}) - \frac{\alpha}{(\beta+1)(\beta+2)}((T-\gamma)^{\beta+2} - t_1^{\beta+2})\right] \\ &+ \frac{C_s}{T}\left[-\frac{A}{b}\mu + \frac{A}{2}e^{b\mu}(t_1-\mu)^2 - \frac{A}{b}(1-e^{b\mu})\left\{\frac{1}{b} + (t_1-\mu)\right\}\right] \end{aligned} \tag{19}$$

In order to minimize the average total cost per unit of time, the optimal value of t_1 (denoted by t_1^*) can be obtained by solving

$$\frac{d}{dt_1}\left[TC(t_1)\right] = 0 \tag{20}$$

which also satisfies the condition

$$\left[\frac{d^2}{dt_1^2}\left[TC(t_1)\right]\right]_{t=t_1^*} > 0 \tag{21}$$

Equation (19) together with (21) gives

$$\begin{aligned} \frac{d\chi}{dt_1} &= \frac{C_d}{T}Ae^{b\mu}\left[e^{-\alpha t_1^{\beta}}\left\{-1 - \alpha\,\beta(T-t_1-\gamma)\,t_1^{\beta-1} - \alpha\,t_1^{\beta}\right\} + 1\right] \\ &+ \frac{C_s}{T}A\left[e^{b\mu}\left\{t_1 - \mu - \frac{1}{b}\right\} - \frac{1}{b}\right] + \frac{C_h}{T}Ae^{b\mu} \\ &\left[-T - \left(\frac{\alpha}{\beta+1}\right)(T-\gamma)^{\beta+1} + (t_1+\gamma) + \alpha\,T\,t_1^{\beta} - \alpha(t_1+\gamma)^{\beta+1} + \left(\frac{\alpha}{\beta+1}\right)t_1^{\beta+1}\right] \\ &+ \frac{C_h}{T}\gamma\,Ae^{b\mu}\left[e^{-\alpha t_1^{\beta}}\left\{-\alpha\,\beta(T-t_1-\gamma)\,t_1^{\beta-1} - 1 - \alpha\,t_1^{\beta}\right\}\right] = 0 \end{aligned} \tag{22}$$

Equation (22) is a nonlinear equation in t_1, using iterative numerical method one can solve it for different values of various parameters. The superiority of Newton-Raphson method over the bisection method is established by Wan and Chu [17]. The validity for using the Newton-Raphson method lies in selecting a proper initial root so that an excellent convergent sequence is obtained. Hence Newton-Raphson method with proper selection of initial root is suitable to locate the exact root of (22). Using the optimal value t_1^* (obtained by Newton-Raphson method) the minimum average total cost per unit of time can be obtained from (19).

Further the total demand backordered (R_B) can be obtained as:

$$
\begin{aligned}
R_B &= \int_0^{\mu} Ae^{bt}\,dt + \int_{\mu}^{t_1^*} Ae^{b\mu}\,dt \\
&= Ae^{b\mu}\left[\frac{1}{b} + (t_1^* - \mu)\right] - \frac{A}{b}
\end{aligned}
\tag{23}
$$

Also inventory level at t_1^* from (14)

$$
S = Ae^{b\mu - \alpha t_1^{*\beta}}\left[\left\{T - (t_1^* + \gamma)\right\} + \left(\frac{\alpha}{\beta+1}\right)\left\{(T-\gamma)^{\beta+1} - t_1^{*\beta+1}\right\}\right] + A\gamma\, e^{b\mu} \tag{24}
$$

Therefore the optimal order quantity Q* (using (23) and (24)) is

$$
\begin{aligned}
Q^* &= Ae^{b\mu}\left[\frac{1}{b} + (t_1^* - \mu)\right] - \frac{A}{b} \\
&+ Ae^{b\mu - \alpha t_1^{*\beta}}\left[(T - (t_1^* + \gamma)) + \left(\frac{\alpha}{\beta+1}\right)((T-\gamma)^{\beta+1} - t_1^{*\beta+1})\right] + A\gamma\, e^{b\mu}
\end{aligned}
\tag{25}
$$

4. NUMERICAL EXAMPLE

For the numerical illustration of the developed model, the values of various parameters can be taken as follows:

A=100 units, C_d = Rs. 5 per unit, C_h = Rs. 3 per unit per year, C_s = Rs.15 per unit per year, T = 1 year, b = .08, α = .002 , Ramp in demand (μ) = .12 and β = 1.5 (assuming the rate of deterioration to be increasing with time), γ (location parameter) = {.08, .1, .12}

The demand function $R(t) = Ae^{b[t-(t-\mu)H(t-\mu)]}$ will be as follows

$$
R(t) = \begin{cases} 100\,e^{.08t}, & t < \mu \\ 100\,e^{.0096}, & t \ge \mu \end{cases}
$$

Now using above data we find the following result for different values of γ

γ	t_1^*	Q*	Average Cost Deterioration	Holding	Shortage	Average Total Cost
.08	.1675094	100.960838	.2711025	105.0465565	22.024143	127.3418020
.1	.1675021	100.9576095	.254960	105.0529283	22.0223292	127.3302175
.12	.1674958	100.9544959	.2393918	105.0588216	22.0207363	127.3189497

It can be observed from the table that as the time of start of deterioration (γ) of items (after replenishment) increases, t_1^* (period of shortage) optimal order quantity decreases. The cost of deterioration also decreases as γ increases. Since the value of t_1^* decreases (with the increase in value of γ), the shortage cost decreases; and the items are holded in the inventory for a comparatively longer period; there by increasing the holding cost. The average total cost of the inventory system decreases with increase in the value of γ .

5. CONCLUSION

The inventory model developed above is concerned with ramp type demand, starting with shortage and three-parameter Weibull distribution deterioration. Ramp type demand precisely represents the demand of a number of consumer items of present era; also the model starting with shortage is a salient feature of the developed model. We have provided an exact solution procedure for the model; a numerical example is also given in support of the theory. A possible direction for further research may be to consider the situation when $\mu > t_1$.

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