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A PRODUCTION-INVENTORY MODEL FOR A DETERIORATING ITEM WITH SHORTAGE AND TIME-DEPENDENT DEMAND

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Abstract: In the present article, a production-inventory model is developed over a finite planning horizon where the demand varies linearly with time. The machine production rate is assumed to be finite and constant. Shortages in inventory are allowed and are completely backlogged. The associated constrained minimization problem is numerically solved. Sensitivity analysis is also presented for the given model.

Keywords: Production inventory model, time-dependent demand, deteriorating item.

MSC: 90B05.

1. INTRODUCTION

The classical EOQ *(Economic Order Quantity)* model assumes that the demand rate is constant. However, in the real market, the demand for any product cannot be constant. Reaserchers have paid much attention to inventory modelling with time-dependent demand. Silver and Meal [1] developed a heuristic approach to determine EOQ in the general case of a deterministic time-varying demand pattern. Donaldson [2] discussed the classical no-shortage inventory policy for the case of a linear, time-dependent demand. His treatment was fully analytical and much computational effort was needed in order to get the optimal solution. Silver [3], using *Silver-Meal heuristic*, obtained an appropriate solution procedure for the case of a positive linear trend in demand to reduce the computational effort needed in Donaldson [2]. Subsequent contributions in this type of modelling came from researchers such as Ritchie ([4],[5],[6]), Kicks and Donaldson [7], Buchanon [8], Mitra, Cox and Jesse [9], Ritchie and Tsado [10], Goyal [11], Goyal, Kusy and Soni [12], and others.

All these works assume no shortages in inventory. However, shortages are unavoidable in many inventory systems due to various uncertainties. It is also important, from the managerial point of view, to reduce average total cost. Deb and Chaudhuri [13] were the first to modify the procedure of Silver [3] by allowing inventory shortages which are completely backordered. The problem was reconsidered by Murdeshwar [14], Dave [15] and Goyal [16]. All these models deal with a replenishment policy that allows shortages in all cycles except the last one. Each of the cycles during which shortages are permitted starts with replenishment and ends with shortage. Hariga [17] called it the DAC (an abbreviation for Deb and Chaudhuri) replenishment policy. It has also been termed as the IFS (inventory followed by shortage) policy. Goyal, Morin and Nebebe [18] suggested a new replenishment policy in which shortages are permitted in every cycle. In this policy, each cycle starts with a shortage until replenishment is made followed by a period of positive inventory. Hariga [17] called it the GMN (an abbreviation for Goyal, Morin and Nebebe) replenishment policy; it is also called the SFI (shortage followed by inventory) policy. None of these researchers took into account the physical decay or deterioration of goods over time.

Researchers then started working on inventory models with time-varying demands for items which undergo decay or deterioration. The effect of deterioration is an important feature of inventory systems. Food items, photographic films, chemicals, electronic goods, pharmaceuticals, etc. are some examples of deteriorating items. Various types of order-level inventory models for deteriorating items with no shortages were considered by Dave and Patel [19], Bahari-Kashani [20], Chung and Ting [21], and others. Some models for deteriorating items with trended demand and shortages were developed by Gowsami and Chaudhuri [22], Hariga [23], Giri, Goswami and Chaudhuri [24], Jalan, Giri and Chaudhuri [25], Teng [26], Lin, Tan and Lee [27], etc. Some of the recent works for deteriorating items are of Fujiwara [28], Hariga and Benkherouf [29], Wee [30], Chakrabarti and Chaudhuri [31], Jalan and Chaudhuri ([32], [33]), Chakraborti, Giri and Chaudhuri ([34], [35], [36]) etc.

All of the above models are purely inventory replenishment models. The classical economic production lot-size (EPLS) model in inventory literature assumes that the demand rate and the production rate are predetermined and inflexible. However, it is usually observed in the real market that the demands for products such as fashionable clothes, electronic goods, etc. increase rapidly after gaining consumer acceptance. Therefore, consideration of time-varying demand in the EPLS model is quite appropriate. Hong, Sandrapaty and Hayya [37] developed an inventory model for a linearly increasing demand with a finite production rate. Goswami and Chaudhuri [38] discussed a lot-size model with a linearly increasing demand and finite production rate, considering shortages. These two models assume that the production rate is uniform. However, the production rate may go up or down with the demand rate. The above mentioned situation is normally seen with highly demandable goods. Khouja [39] and Khouja and Mehrez [40] presented the EPLS model taking the production rate to be a decision variable. They considered a constant demand, a single production cycle and no shortage. Zhou ([41],

[42]) developed the EPLS models taking linear trend in demand with shortages over a finite planning schedule. In these models, he assumed that the production rate is adjusted at the beginning of each production cycle to cope with an increasing demand, and the cost of adjusting the production rate depends linearly on the magnitude of change in the production rate. Giri and Chaudhuri [43] discussed a production lot-size model with shortages and time-dependent demand. The model is developed over an infinite planning horizon where the unit production cost is taken to be a function of the production rate, the demand varies linearly with time, and the shortages in inventory are permitted and fully back-ordered. Significant contributions to the study of the EPLS models came from researchers such as Yan and Cheng [44], Balkhi [45] and Balkhi, Goyal and Giri [46].

In the present paper, we discuss the EPLS model for a deteriorating item over a finite planning horizon with a linear trend in demand and shortages. The machine production rate is assumed to be finite. The production-inventory system in each cycle consists of four stages. The initial stock in each cycle is zero and shortages begin to accumulate at the very beginning of the cycle. Production starts after a certain time and the accumulated shortages are fully supplied after adjusting current demands and deterioration and then inventory becomes zero. As production continues, inventory begins to build up continuously after adjusting demands and deterioration. Production stops at a certain time. The accumulated inventory is sufficient enough to adjust demands and deterioration for the rest of the cycle. The cycle ends with zero inventory. The reasons for selecting this type of production-inventory cycle are as follows:

At the initial stage, shortages may occur due to several reasons, such as delay in machine maintenance, shortage of raw materials, shortage of labour, etc. Subsequently, shortages continue to accumulate for some time. When these problems are removed, production starts. In the second stage, shortages are gradually cleared after adjusting demands. As production continues, the inventory builds up. It is necessary to stop production after some time due to reasons such as limitation of warehouse space, maintenance of machines, etc. At the final stage, there is no production and the accumulated inventory is gradually depleted and ultimately becomes zero due to demands and deterioration.

In the present paper, we assume a uniform production rate which is actually the CDPR (critical design production rate) of the manufacturing machine. Production starts and stops after some time to make room for machine maintenance. The demand rate in this model is assumed to be linearly time varying, and we consider the cases of both increasing and decreasing demands. The assumption is quite appropriate from a realistic point of view because the demand for some items such as electronic goods, fashionable clothes, luxury goods, etc. increases steadily after consumer acceptance, while the demand for the obsolete items decreases steadily. It is assumed that a constant fraction $\theta,\ (0<\theta<1)$ of the on-hand inventory deteriorates per unit time. The optimal number of cycles that minimizes the average system cost over a finite time horizon is determined. The results are illustrated with numerical examples. The sensitivity of the optimal solution to changes in different parameters is also examined.

2. ASSUMPTIONS AND NOTATIONS

The following assumptions and notations have been used in developing the model.

(i) The time-dependent demand rate is $f(t) = a + bt$, $a > 0$, $b \neq 0$. Here a is the initial rate of demand, b is the rate with which the demand rate changes.

(ii) Shortages are allowed and are completely backlogged.

(iii) The time horizon H is finite.

(iv) The time horizon is divided into a finite number of replenishment cycles, e.g. n, each of which is taken to be of equal duration for the sake of simplicity.

(v) The production rate P is finite and constant.

(vi) The inventory holding cost C_h per unit per unit time, the shortage cost C_s per unit per unit time, the set up cost A_s per cycle and the production cost C_p are known and constant.

(vii) A constant fraction θ, $(0 < \theta < 1)$, of the on-hand inventory deteriorates per unit time.

3. FORMULATION AND SOLUTION OF THE MODEL

The initial stock of the i-th cycle (i=1,2, - - - -n) is zero. Shortages begin to accumulate over $[T_{i-1}, t_{i1}]$. Production starts at time t_{i1}. The accumulated shortages are fully supplied during $[t_{i1}, t_{i2}]$ after adjusting current demands. The inventory becomes zero at t_{i2}. As production continues, inventory begins to pile up continuously after adjusting demands and deterioration. Production stops at time t_{i3}. The accumulated inventory is sufficient enough to adjust demands and deterioration over the interval $[t_{i3}, T_i]$. Subsequently, the cycle ends with zero inventory. It then repeats itself. Here t_{i1}, t_{i2}, t_{i3} and T_i are connected by the following relations (see Appendix I).

$$\begin{cases} t_{i1} = u_i t_{i2} + (1-u_i)T_{i-1} \\ t_{i2} = rT_i + (1-r)T_{i-1} \\ t_{i3} = v_i T_i + (1-v_i)t_{i2} \\ T_i = \dfrac{H}{n} i \end{cases} \qquad (A)$$

Here $0 < r < 1,\ 0 < u_i < 1,\ 0 < v_i < 1,\ i = 1, 2, 3, \ldots, n.$

For every machine, there exists a critical design production rate which is taken as the production rate in the proposed model. Here we assumed that the time of shortage is a fixed proportion of each cycle; but the times at which production starts and stops in each cycle taken to be variables. $t_{i3} - t_{i1}$ denote the time during which the machine is in operation.

The instantaneous inventory level $I(t)$ at any time $t \in (T_{i-1}, T_i)$ is governed by the following differential equations:

$$\frac{dI(t)}{dt} = -f(t) = -a - bt, \qquad T_{i-1} \le t \le t_{i1}, \quad with\ I(T_{i-1}) = 0; \tag{1}$$

$$\frac{dI(t)}{dt} = P - f(t) = P - a - bt, \qquad t_{i1} \le t \le t_{i2}, \quad with\ I(t_{i2}) = 0; \tag{2}$$

$$\frac{dI(t)}{dt} + \theta I(t) = P - f(t) = P - a - bt, \quad t_{i2} \le t \le t_{i3}, with\ I(t_{i2}) = 0 \tag{3}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -f(t) = -a - bt, \qquad t_{i3} \le t \le T_i, \quad with\ I(T_i) = 0. \tag{4}$$

The solution of equation (1) is

$$I(t) = -a(t - T_{i-1}) - \frac{b}{2}(t^2 - T_{i-1}^2), \qquad T_{i-1} \le t \le t_{i1}. \tag{5}$$

The solution of equation (2) is

$$I(t) - I(t_{i1}) = P(t - t_{i1}) - a(t - t_{i1}) - \frac{b}{2}(t^2 - t_{i1}^2), \quad t_{i1} \le t \le t_{i2}.$$

Substituting the value of $I(t_{i1})$ from equation (5), the above relation becomes

$$I(t) = P(t - t_{i1}) - a(t - T_{i-1}) - \frac{b}{2}(t^2 - T_{i-1}^2), \qquad t_{i1} \le t \le t_{i2}. \tag{6}$$

The solution of equation (3)

$$I(t) = \frac{(P-a)}{\theta}\{1 - e^{\theta(t_{i2}-t)}\} - \frac{b}{\theta}\{t - t_{i2}e^{\theta(t_{i2}-t)}\} + \frac{b}{\theta^2}\{1 - e^{\theta(t_{i2}-t)}\}, \quad t_{i2} \le t \le t_{i3}. \tag{7}$$

The solution of equation (4) is

$$e^{\theta t}I(t) - e^{\theta t_{i3}}I(t_{i3}) = -\frac{a}{\theta}(e^{\theta t} - e^{\theta t_{i3}}) + \frac{b}{\theta^2}(e^{\theta t} - e^{\theta t_{i3}}) - \frac{b}{\theta}(te^{\theta t} - t_{i3}e^{\theta t_{i3}}), \quad t_{i3} \le t \le T_i.$$

Putting the value of $I(t_{i3})$ from equation (7) in the above relation and then simplifying, we get

$$I(t)=\{\frac{P}{\theta}(e^{\theta t_{i3}}-e^{\theta t_{i2}})+\frac{a}{\theta}e^{\theta t_{i2}}+\frac{b}{\theta^2}(\theta t_{i2}-1)e^{\theta t_{i2}}\}e^{-\theta t}$$
$$-\{\frac{a}{\theta}+\frac{b}{\theta^2}(\theta t-1)\},\ t_{i3}\le t\le T_i. \tag{8}$$

From the relations in (A), we get

$$\begin{cases} t_{i1}=\frac{H}{n}(ru_i+i-1) \\ t_{i2}=\frac{H}{n}(r+i-1) \\ t_{i3}=\frac{H}{n}\{i-(1-r)(1-v_i)\} \\ T_i=\frac{H}{n}i \end{cases} \tag{B}$$

Putting $I(t_{i2})=0$ and using relation (B), we have from equation (6),

$$u_i=1-\frac{a}{P}-\frac{bH}{2nP}\{r+2(i-1)\}. \tag{9}$$

Putting $I(T_i)=0$ in equation (8), we get

$$0=\frac{P}{\theta}\{e^{\theta(t_{i3}-T_i)}-e^{\theta(t_{i2}-T_i)}\}-\frac{a}{\theta}\{1-e^{\theta(t_{i2}-T_i)}\}$$
$$+\frac{b}{\theta^2}\{(\theta t_{i2}-1)e^{\theta(t_{i2}-T_i)}-(\theta T_i-1)\}. \tag{10}$$

Putting the values of t_{i2}, t_{i3} and T_i from the relations in (B), in the above equation, we get (see Appendix-II)

$$v_i=\frac{n}{\theta H(1-r)}ln[1+\frac{1}{P}\{a+\frac{bH}{n}i-\frac{b}{\theta}\}e^{\frac{\theta H}{n}(1-r)}$$
$$-\frac{1}{P}\{a+\frac{bH}{n}(r+i-1)-\frac{b}{\theta}\}]. \tag{11}$$

The shortage during the time interval $[T_{i-1},\ t_{i1}]$ is

$$Sh_{i1}=\int_{T_{i-1}}^{t_{i1}}[-I(t)]dt$$
$$=\frac{a}{2}(t_{i1}-T_{i-1})^2+\frac{b}{6}t_{i1}^3-\frac{b}{2}t_{i1}T_{i-1}^{\,2}+\frac{b}{3}T_{i-1}^{\,3}. \tag{12}$$

The shortage during the time interval $[t_{i1},\ t_{i2}]$ is

$$Sh_{i2} = \int_{t_{i1}}^{t_{i2}} [-I(t)]\, dt$$
$$= \int_{t_{i1}}^{t_{i2}} [a(t - T_{i-1}) + \frac{b}{2}(t^2 - T_{i-1}^2) - P(t - t_{i1})]dt$$
$$= \frac{a}{2}(t_{i2} - T_{i-1})^2 - \frac{a}{2}(t_{i1} - T_{i-1})^2 + \frac{b}{6}(t_{i2}^3 - t_{i1}^3) \qquad (13)$$
$$- \frac{b}{2}T_{i-1}^{\ 2}(t_{i2} - t_{i1}) - \frac{P}{2}(t_{i2} - t_{i1})^2.$$

The total shortage in the i-th cycle is (see Appendix-III)

$$Sh_i = Sh_{i1} + Sh_{i2}$$
$$= \frac{H^2}{6n^2}[\frac{2bH}{n}(i-1)^3 + 3ar^2 - 3Pr^2(1-u_i)^2 \qquad (14)$$
$$- \frac{3bH}{n}(i-1)^2(r+i-1) + \frac{bH}{n}(r+i-1)^3].$$

Using (7), the inventory during the time interval $[t_{i2},\ t_{i3}]$ is

$$Iv_{i1} = \frac{(P-a)}{\theta}(t_{i3} - t_{i2}) + \frac{(P-a)}{\theta^2}e^{\theta(t_{i2}-t_{i3})} - \frac{(P-a)}{\theta^2}$$
$$- \frac{b}{2\theta}(t_{i3}^2 - t_{i2}^2) - \frac{b}{\theta^2}t_{i2}e^{\theta(t_{i2}-t_{i3})} + \frac{b}{\theta^2}t_{i3} + \frac{b}{\theta^3}e^{\theta(t_{i2}-t_{i3})} - \frac{b}{\theta^3} \qquad (15)$$

Using (8), the inventory during $[t_{i3}, T_i]$ is

$$Iv_{i2} = [\frac{P}{\theta}(e^{\theta t_{i3}} - e^{\theta t_{i2}}) + \frac{a}{\theta}e^{\theta t_{i2}} + \frac{b}{\theta^2}e^{\theta t_{i2}}(\theta t_{i2} - 1)]\frac{1}{\theta}(e^{-\theta t_{i3}} - e^{-\theta T_i})$$
$$- \frac{a}{\theta}(T_i - t_{i3}) - \frac{b}{2\theta}(T_i^{\ 2} - t_{i3}^2) + \frac{b}{\theta^2}(T_i - t_{i3}) \qquad (16)$$

The total inventory in the i-th cycle is (see Appendix IV)

$$Iv_i = Iv_{i1} + Iv_{i2}$$
$$= \frac{P}{\theta^2}[\frac{H\theta}{r}v_i(1-r) + e^{\frac{H\theta}{n}(r-1)} - e^{\frac{H\theta}{n}(r-1)(1-v_i)}]$$
$$+ \frac{a}{\theta^2}[\frac{H\theta}{n}(r-1) + 1 - e^{\frac{H\theta}{n}(r-1)}] + \frac{b}{\theta^3}[\frac{H^2\theta^2}{2n^2}(r-1)(r+2i-1) \qquad (17)$$
$$+ (\frac{H\theta}{n}i - 1) - \{\frac{H\theta}{n}(r+i-1) - 1\}e^{\frac{H\theta}{n}(r-1)}]$$

The amount of deteriorated item in the i-th cycle is $\theta\, Iv_i$.
If CUT be the average cost during the time horizon (0, H), then

$$CUT = \frac{1}{H}[(C_h + \theta C_p)\sum_{i=1}^{n} Iv_i + C_s \sum_{i=1}^{n} Sh_i + nAs]$$
$$= \frac{(C_h + \theta C_p)}{H}[\sum_{i=1}^{n}\{\frac{P}{\theta^2}\{v_i \frac{H\theta}{n}(1-r) + e^{\frac{H\theta}{n}(r-1)} - e^{\frac{H\theta}{n}(r-1)(1-v_i)}\}\}$$
$$+ \frac{a}{\theta^2}\{H\theta(r-1) + n - ne^{\frac{H\theta}{n}(r-1)}\}$$
$$+ \frac{b}{\theta^3}\sum_{i=1}^{n}\{\frac{H^2\theta^2}{2n^2}(r-1)(r+2i-1) + (\frac{H\theta}{n}i - 1) \tag{18}$$
$$- \frac{H\theta}{n}(r+i-1)e^{\frac{H\theta}{n}(r-1)} + e^{\frac{H\theta}{n}(r-1)}\}]$$
$$+ \frac{C_s H}{6n^2}\sum_{i=1}^{n}\{\frac{2bH}{n}(i-1)^3 + 3ar^2 - 3pr^2(1-u_i) + \frac{bH}{n}(r+i-1)^3$$
$$- \frac{3bH(i-1)^2(r+i-1)}{n}\} + \frac{nAs}{H}.$$

Now our aim is to minimize CUT, $0 < r < 1$, $0 < u_i < 1$, $0 < v_i < 1$ for i= 1, 2, 3,. . . *n*.

This is a constrained minimization problem and can be solved by various optimization techniques (such as Box Complex Algorithm, Penalty Search Method, etc.) or by various software packages (such as Mathematica). For a fixed n, we can find the minimum value of CUT and optimum value of r. Afterwards, we find the minimum of set of minimum values of CUT for different values of n. Finally, the corresponding values of n and r constitute the optimum solutions of n and r.

4. NUMERICAL EXAMPLE

Let us take the parameter values of the inventory system for an increasing demand as a =50, b =3, P =110, H =6, C_h = 4.5, C_s = 10, C_p = 12, A_s = 80, $\theta = 0.03$ in appropriate units. Using Mathematica, the optimum solution is found to be $r^* = 0.333684$, $n^* = 5$, total shortages $S = \sum_{i=1}^{n} Sh_i = 10.8199$, total inventory= $I = \sum_{i=1}^{n} Iv_i = 43.8785$, $CUT = 120.241$ which are shown in the Table 1. Here $n*$ denote the optimum number of inventory cycles within a fixed finite horizon that minimizes the average inventory cost and r^* indicates the optimum time during which shortage occurs within each inventory cycle. For a decreasing demand, we have taken $b = -3$ and the remaining parameters are the same. For decreasing demand, the optimum solution is $r^* = 0.315917$, $n^* = 4$, $S = 11.7198$, $I = 52.3408$, $CUT = 115.262$ which are shown in Table 2. For constant demand, we have taken $b = 0$ and the remaining parameters are the same. For constant demand, the optimum solution is $r^* = 0.327282$, $n^* = 5$, $S = 10.5167$, $I = 44.463$, $CUT = 120.210$ which are shown in the Table 3.

Table 1: Optimal solution for increasing demand ($b > 0$)

n	r	S	I	CUT
1	0.343000	61.7991	211.68	287.793
2	0.343764	28.4931	108.37	161.935
3	0.338148	18.4598	72.7569	129.694
4	0.335355	13.6447	54.743	120.416
5	0.333684	10.8199	43.8785	120.241
6	0.332573	8.9634	36.6112	124.594
7	0.339856	8.5162	30.6495	131.536
8	0.331406	6.6816	27.4825	140.089

Table 2: Optimal solution for decreasing demand ($b < 0$)

n	r	S	I	CUT
1	0.285522	40.8463	212.5483	253.575
2	0.305262	22.3914	105.374	149.339
3	0.31231	15.3907	69.9541	122.314
4	0.315917	11.7198	52.3408	115.262
5	0.318107	9.46143	43.0704	116.325
6	0.319577	7.93229	34.8095	121.412
7	0.320633	6.82849	29.8095	128.86
8	0.321427	5.99423	26.068	137.772

Table 3: Optimal solution for constant demand ($b = 0$)

n	r	S	I	CUT
1	0.327997	52.8114	222.372	281.473
2	0.327592	26.3414	111.165	160.612
3	0.327427	17.5432	74.107	129.266
4	0.327339	13.1503	55.580	120.27
5	0.327284	10.5167	44.463	120.210
6	0.327247	8.7620	37.053	124.616
7	0.327220	7.5090	31.759	131.573
8	0.327200	6.6569	27.789	140.126

The time-inventory relationship is pictorially shown in Figure 1.

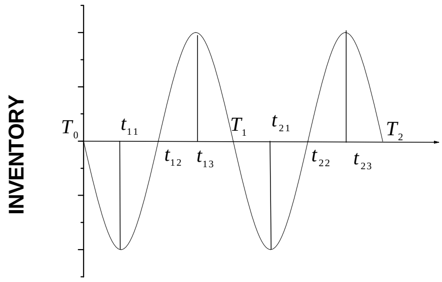

Figure 1

5. SENSITIVITY ANALYSIS

We now study the effects of changes in the values of the system parameters C_S, C_h, C_p, θ, A_s, a, b, P on the optimal solution. The sensitivity analysis is performed by changing each of the parameters by 50%, 20%, -20%, -50%, taking one parameter at a time and keeping the remaining parameters unchanged. The analysis is based on the results obtained in Table 1.

On the basis of the results shown in Table 4, the following observations can be made:

(1) CUT^*, I^* both increase while S^*, r^* both decrease with the increase in the value of the parameter C_s. CUT^* has low sensitivity and I^*, S^*, r^* have moderate sensitivity to changes in C_s.

(2) CUT^*, S^*, r^* increase while I^* decreases with the increase in the value of the parameter C_h. CUT^*, I^*, S^*, r^* have moderate sensitivity to changes in C_h.

(3) CUT^*, S^*, r^* increase while I^* decreases with the increase in the value of the parameter C_p. CUT^*, I^*, S^*, r^* have low sensitivity to changes in C_p.

(4) CUT^*, S^*, r^* increase while I^* decreases with the increase in the value of the parameter θ. CUT^*, I^*, S^*, r^* have low sensitivity to changes in θ.

(5) CUT^*, I^*, S^*, r^* increase or decrease with the increase or decrease in the value of the parameter A_s. However, r^* is almost insensitive and CUT^*, I^*, S^* are moderately sensitive to changes in A_s.

(6) r^* is almost insensitive and CUT^*, I^*, S^* are moderately sensitive to changes in a.

(7) CUT^*, S^*, I^* increase while r^* decreases with the increase in the value of the parameter b. These are almost insensitive to changes in the parameter b.

(8) CUT^*, I^*, S^*, r^* increase or decrease with the increase or decrease in the value of the parameter P. However, r^* is almost insensitive and CUT^*, I^*, S^* are moderately sensitive to changes in P. The model has no feasible solution for 50% negative error in P. However, this outcome may be due to the choice of the particular parameter values in this numerical example.

Table 4: Sensitivity analysis for the optimal solution when $b > 0$

parameter	% change of parameter	No. of cycle n^*	r^* change in %	S^* change in %	I^* change in %	CUT^* change in %
C_s	50	5	-24.99	-43.625	26.874	5.631
	20	5	-11.759	-22.058	12.029	2.586
	-20	5	15.907	34.161	-15.417	-3.458
	-50	4	54.336	67.795	-27.029	-13.953
C_h	50	5	27.268	61.591	-25.627	11.840
	20	5	11.621	24.473	-11.342	5.170
	-20	4	-12.687	-4.684	42.035	-7.141
	-50	3	-35.467	-30.466	134.313	-23.269
C_p	50	5	05.654	04.912	-02.488	01.081
	20	5	00.982	01.966	-00.9866	00.438
	-20	5	-00.992	-00.775	00.3938	-00.439
	-50	5	-02.499	-05.4705	02.541	-01.109
θ	50	5	02.403	04.843	-02.448	01.071
	20	5	00.969	01.938	-00.991	00.431
	-20	5	-00.979	-01.939	01.006	-00.436
	-50	5	-02.466	-04.852	02.544	-01.098
A_s	50	3	1.338	70.610	65.801	24.496
	20	4	0.501	26.107	24.760	9.017
	-20	5	-0.054	-0.107	8.773	-8.479
	-50	6	-0.333	-17.158	-16.562	-29.646
a	50	4	-00.710	-08.015	-10.021	-15.253
	20	4	-00.054	-18.418	-7.184	-03.247
	-20	4	01.215	25.747	23.884	00.167
	-50	4	02.889	09.906	06.707	07.621
b	50	5	07.876	-02.846	-03.423	-01.189
	20	5	00.325	-00.268	-01.146	-00.378
	-20	5	-00.340	00.007	00.782	00.251
	-50	5	-00.887	00.536	01.583	00.388
P	50	5	02.766	38.560	39.483	17.454
	20	5	00.137	19.244	19.741	08.718
	-20	4	-00.255	-09.751	-12.242	-16.169

	-50	-	-	-	-	-

6. CONCLUDING REMARKS

The production inventory model developed here incorporates the following practical features:

1. It is applicable to an inventory which deteriorates over time.
2. It is concerned with a linearly time-varying demand.
3. It allows shortages in inventory.
4. It is suitable for a finite planning horizon.
5. The item in stock is manufactured at a uniform rate.

The production inventory cycle is also based on some practical considerations. Each cycle starts with a zero stock. Production cannot start at the very beginning of the cycle due to some practical difficulties such as delay in machine setup, shortages of raw materials, shortages of man power, etc. As a result, shortage continues to accumulate for some time at the very beginning of the cycle. Once the practical difficulties are removed, production starts at a rate greater than the demand rate. Inventory continues to build up after clearing the backlog, meeting the current demands and adjusting stock-loss due to deterioration. It then becomes necessary to stop production after some time due to difficulties such as limitation of warehouse space, machine failure, etc. The demands and deterioration for the remaining portion of the cycle period are met from the accumulated stock.

Numerical examples clearly show that both shortages and inventory continue to decrease when the number of cycles increases within a finite time-horizon. This trend is evidenced in both the case of increasing and decreasing demands.

Appendix-I

The interval $[T_{i-1},\ t_{i1}]$ being a fraction $u_i(0 < u_i < 1)$ of the interval $[T_{i-1},\ t_{i2}]$, we get

$$t_{i1} - T_{i-1} = u_i(t_{i2} - T_{i-1})$$

$$or,\quad t_{i1} = u_i t_{i2} + (1-u_i)T_{i-1}.$$

Again, the interval $[T_{i-1},\ t_{i2}]$ being a fraction $r(0 < r < 1)$ of the interval $[T_{i-1},\ T_i]$, we similarly have

$$t_{i2} = rT_i + (1-r)T_{i-1}.$$

Also, considering $[t_{i2},\ t_{i3}]$ to be fraction of $v_i(0 < v_i < 1)$ of $[t_{i2},\ T_i]$, we similarly get

$$t_{i3} = v_i T_i + (1-v_i)t_{i2}$$

Appendix II

We have $t_{i2} - T_i = \frac{H}{n}(r-1)$

and $t_{i3} - T_i = \frac{H}{n}(r-1)(1-v_i)$.

Therefore, we have from (10),

$$0 = \frac{P}{\theta}\{e^{\frac{\theta H}{n}(r-1)(1-v_i)} - e^{\frac{\theta H}{n}(r-1)}\} - \frac{a}{\theta}\{1 - e^{\frac{\theta H}{n}(r-1)}\}$$

$$+ \frac{b}{\theta^2}[\{\frac{\theta H(r+i-1)}{n} - 1\}e^{\frac{\theta H}{n}(r-1)} - (\frac{\theta H}{n}i - 1)]$$

$$or, \quad \frac{P}{\theta}e^{\frac{\theta H}{n}(r-1)(1-v_i)} = \frac{P}{\theta}[1 - \frac{1}{P}\{a + \frac{bH}{n}(r+i-1) - \frac{b}{\theta}\}]e^{\frac{\theta H}{n}(r-1)}$$

$$+ \frac{P}{\theta}[\frac{1}{P}\{a + \frac{bHi}{n} - \frac{b}{\theta}\}]$$

$$or, \quad e^{\frac{\theta H}{n}(r-1)(1-v_i)} = [[1 - \frac{1}{P}\{a + \frac{bH}{n}(r+i-1) - \frac{b}{\theta}\}]$$

$$+ \frac{1}{P}\{a + \frac{bHi}{n} - \frac{b}{\theta}\}e^{\frac{\theta H}{n}(1-r)}]e^{\frac{\theta H}{n}(r-1)}$$

Taking logarithm on both sides, we get

$$\frac{\theta H}{n}(r-1)(1-v_i) = ln[1 - \frac{1}{P}\{a + \frac{bH}{n}(r+i-1) - \frac{b}{\theta}\}$$

$$+ \frac{1}{P}e^{\frac{\theta H}{n}(1-r)}\{a + \frac{bHi}{n} - \frac{b}{\theta}\}] + \frac{\theta H}{n}(r-1)$$

$$\frac{\theta H}{n}(1-r)v_i = ln[1 - \frac{1}{P}\{a + \frac{bH}{n}(r+i-1) - \frac{b}{\theta}\}$$

$$+ \frac{1}{P}e^{\frac{\theta H}{n}(1-r)}\{a + \frac{bHi}{n} - \frac{b}{\theta}\}]$$

$$v_i = \frac{n}{\theta H(1-r)} ln[1 - \frac{1}{P}\{a + \frac{bH}{n}(r+i-1) - \frac{b}{\theta}\} + \frac{1}{P}e^{\frac{\theta H}{n}(1-r)}\{a + \frac{bHi}{n} - \frac{b}{\theta}\}]$$

Appendix-III

We have

$$t_{i2} - T_{i-1} = \frac{H}{n} r \,;$$

$$t_{i2} + T_{i-1} = \frac{H}{n}(r + 2i - 2) \,;$$

$$t_{i2} - t_{i1} = \frac{H}{n}(1 - u_i) r \,;$$

$$t_{i2} + t_{i1} = \frac{H}{n}\{r(1 + u_i) + 2i - 2\}$$

The total shortage in the i-th cycle

$$Sh_i = Sh_{i1} + Sh_{i2}$$

$$= \frac{b}{3} T_{i-1}^{3} + \frac{a}{2}(t_{i2} - T_{i-1})^2 - \frac{P}{2}(t_{i2} - t_{i1})^2 - \frac{b}{2} t_{i2} T_{i-1}^2 + \frac{b}{6} t_{i2}^{\,2}$$

$$= \frac{H^2}{6n^2}\{\frac{2bH}{n}(i-1)^3 + 3ar^2 - 3pr^2(1-u_i)^2 - 3b(i-1)^2 \frac{H}{n}(r + i - 1)$$

$$+ \frac{bH}{n}(r + i - 1)^3\}$$

Appendix IV

The total inventory in the i-th cycle is

$$Iv_i = Iv_{i1} + Iv_{i2}$$

$$= P\{\frac{1}{\theta}(t_{i3} - t_{i2}) + \frac{1}{\theta^2} e^{\theta(t_{i2} - t_{i3})} - \frac{1}{\theta^2} + \frac{1}{\theta^2}(e^{\theta t_{i3}} - e^{\theta t_{i2}})(e^{-\theta t_{i3}} - e^{-\theta T_i})\}$$

$$+ a\{-\frac{1}{\theta}(t_{i3} - t_{i2}) - \frac{1}{\theta^2} e^{\theta(t_{i2} - t_{i3})} + \frac{1}{\theta^2} + \frac{1}{\theta^2} e^{\theta t_{i2}}(e^{-\theta t_{i3}} - e^{-\theta T_i}) - \frac{1}{\theta}(T_i - t_{i3})\}$$

$$+ b\{-\frac{1}{2\theta}(t_{i3}^2 - t_{i2}^2) - \frac{t_{i2}}{\theta^2} e^{\theta(t_{i2} - t_{i3})} + \frac{1}{\theta^2} t_{i3} + \frac{1}{\theta^3} e^{\theta(t_{i2} - t_{i3})} - \frac{1}{\theta^3}$$

$$+ \frac{1}{\theta^3}(\theta t_{i2} - 1) e^{\theta t_{i2}}(e^{-\theta t_{i3}} - e^{-\theta T_i}) - \frac{1}{2\theta}(T_i^{\,2} - t_{i3}^2) + \frac{1}{\theta^2}(T_i - t_{i3})]$$

$$= P[\frac{1}{\theta}(t_{i3} - t_{i2}) + \frac{1}{\theta^2} e^{\theta(t_{i2} - T_i)} - \frac{1}{\theta^2} e^{\theta(t_{i3} - T_i)}\}$$

$$+ a\{\frac{1}{\theta}(t_{i2} - T_i) + \frac{1}{\theta^2} - \frac{1}{\theta^2} e^{\theta(t_{i2} - T_i)}\}$$

$$+b\{\frac{1}{2\theta}(t_{i2}^2-T_i^{\,2})-\frac{t_{i2}}{\theta^2}e^{\theta(t_{i2}-T_i)}+\frac{1}{\theta^2}T_i^2-\frac{1}{\theta^3}+\frac{1}{\theta^3}e^{\theta(t_{i2}-T_i)}\}$$

$$=\frac{P}{\theta^2}\{\theta(t_{i3}-t_{i2})+e^{\theta(t_{i2}-T_i)}-e^{\theta(t_{i3}-T_i)}\}+\frac{a}{\theta^2}\{\theta(t_{i2}-T_i)-e^{\theta(t_{i2}-T_i)}+1\}$$

$$+\frac{b}{\theta^3}\{\frac{\theta^2}{2}(t_{i2}^2-T_i^{\,2})+(\theta T_i-1)-(\theta t_{i2}-1)e^{\theta(t_{i2}-T_i)}\}$$

We have

$$t_{i3}-t_{i2}=\frac{H}{n}v_i(1-r)$$

$$t_{i2}-T_i=\frac{H}{n}(r-1)$$

$$t_{i3}-T_i=\frac{H}{n}(r-1)(1-v_i)$$

$$t_{i2}+T_i=\frac{H}{n}(r+2i-1)$$

So,

$$Iv_i=\frac{P}{\theta^2}[\frac{H\theta}{n}v_i(1-r)+e^{\frac{H\theta}{n}(r-1)}-e^{\frac{H\theta}{n}(r-1)(1-v_i)}]$$

$$+\frac{a}{\theta^2}[\frac{H\theta}{n}(r-1)+1-e^{\frac{H\theta}{n}(r-1)}]$$

$$+\frac{b}{\theta^3}[\frac{H^2\theta^2}{2n^2}(r-1)(r+2i-1)+(\frac{H\theta}{n}i-1)-\{\frac{H\theta}{n}(r+i-1)-1\}e^{\frac{H\theta}{n}(r-1)}]$$

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