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BANDWIDTH ALLOCATION AND PRICING PROBLEM FOR A DUOPOLY MARKET

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Abstract: This research discusses the Internet service provider (ISP) bandwidth allocation and pricing problems for a duopoly bandwidth market with two competitive ISPs. According to the contracts between Internet subscribers and ISPs, Internet subscribers can enjoy their services up to their contracted bandwidth limits. However, in reality, many subscribers may experience the facts that their on-line requests are denied or their connection speeds are far below their contracted speed limits. One of the reasons is that ISPs accept too many subscribers as their subscribers. To avoid this problem, ISPs can set limits for their subscribers to enhance their service qualities. This paper develops constrained nonlinear programming to deal with this problem for two competitive ISPs. The condition for reaching the equilibrium between the two competitive firms is derived. The market equilibrium price and bandwidth resource allocations are derived as closed form solutions.

Keywords: Duopoly market; equilibrium price; service quality; bandwidth allocation.

MSC: 91B24

1. INTRODUCTION

Two of the most popular technologies that offer speedy access to surf the Internet are the DSL (Digital Subscriber Line) broadband and the cable Modem. DSL service provides Internet access over a single dedicated telephone line, while Cable broadband service provides Internet access over a cable television line.

The advantages of DSL incorporate cheaper service than cable Internet, no dial up, no busy signals and a unique connection. Several different types of DSL exist. ADSL is one of the most popular broadband media among them, for both commercial and residential internet users. ADSL is a mod/demod technology providing high-speed web service through a traditional telephone line. ADSL can also be used to provide home office workers with company Internet services or new style interactive multimedia applications, including web games and VOID services.

Regarding the Cable Modem, it uses the high bandwidth cable system laid by the cable TV service provider to enable users to watch TV programs, use telephone services and surf the Internet simultaneously. The greatest advantage of Cable Modem is a huge bandwidth with its technological specifications, making the data transmission speed ascend promptly, and consequently, the dream of transferring more data on a given bandwidth is realized. In addition, Cable offers a much wider service area than DSL.

The bandwidth requirements are distinctive according to the needs of the Internet users. Operating in such an environment, DSL and Cable providers usually offer a variety of bandwidth commodities to serve their customers, with each Internet commodity having a different sales price and a unique bandwidth restriction.

To offer different Internet commodities to customers, both DSL and Cable providers face the problems of allocating a bandwidth resource to distinct Internet commodities. According to the subscription contracts, Internet users are allowed to access Internet service within the contracted speed. However, in reality, they are usually in a situation that their on-line requests are denied, or their connection speeds are far below their contracted speed limits. The quality of the Internet services may be influenced by the number of on-line users. It is natural to assume that the quality of the Internet services will become poorer if too many users access the Internet at the same time. To improve their corporation images, both DSL and Cable providers have to provide their Internet subscribers with the agreed service qualities. One of the possible approaches for overcoming this problem is to control the number of subscribers. To reach this goal, both DSL and Cable providers can use the pricing strategies to control the number of subscribers.

It is possible to surf the Internet by either DSL or Cable Modem. For competition purposes, DSL and Cable providers may offer similar Internet commodities to complete the same custom pool. According to this and the previously mentioned fact, both DSL and Cable providers may seriously consider their pricing strategies to complete in the Internet market. This paper considers a duopolistic market which consists of a DSL provider and a Cable provider. The two Internet service providers offer two similar Internet commodities.

To our knowledge, few works deal with the ISP competition problem with service quality guarantees. Other research related to this article includes the works on the equilibrium price in duopoly market. Palma and Leruth (1993) dealt with a duopoly model for two firms which sell homogeneous goods. They showed the existence of a symmetric Nash equilibrium price. Harrington (1995) investigated the price-setting behavior in a duopoly market where the degree of product differentiation is uncertain. They showed that in markets with highly substitutable products, price dispersion can be enhanced, since price adjustment is used to obtain more information about the context of product differentiation. Choi (1996) dealt with a price competition problem with duopoly manufacturers and duopoly common retailers. This paper proves that product differentiation is helpful to manufacturers, while store differentiation is helpful to retailers. Peha and Tewari (1998) examined the result of competition between two profit-seeking telecommunications carriers. In their work, each firm has a limited capacity that can be used to offer two different services. They showed conditions for an equilibrium where no carrier has any incentive to change its prices and outputs. Elberfeld and Wolfstetter (1999) investigated a repeated game with simultaneous entry and pricing. They provided a symmetric equilibrium solution. Mason (2000) dealt with an Internet pricing problem. This paper showed that conditions for reaching equilibrium for two competitive firms exist. Foros and Hansen (2001) dealt with a competition problem between two ISPs providing the quality of interconnection. Min et al. (2002) dealt with a pricing setting problem for two competitive firms that produce substitutable products. They derived equilibrium solutions based on a two-stage game framework. Basar and Srikant (2002) investigated a leader-follower game problem for a network with a single ISP and a larger number of users. In this problem, the service provider sets the sales price and the customers react to the price, as well as to the network congestion. Bapna et al. (2005) investigated a pricing setting problem for digital products. This study contains a service provider that uses a price setting mechanism to maximize its allocation efficiency. Symeonidis (2003) compared Bertrand and Cournot equilibrium in a differentiated duopoly with substitute goods and product R&D. This paper provides a condition that quantity competition is more beneficial than price competition for both consumers and firms. Dai et al. (2005) developed pricing strategies for a revenue management problem with multiple firms providing the same service for a common pool of customers. They showed that Nash equilibrium exists in a two-firm pricing game when the demands of both firms are deterministic.

The strategy in duopoly environment has been widely studied in many industries. The constraints on most of the studies include the budget constraint, quantity constraint, etc. As mentioned previously, ISPs need to take their service quality into account. If ISPs are unable or have no plan to expand their equipment over a future time interval, endlessly accepting customers as their subscribers, they will run the risk of poor Internet access quality. To avoid such a situation, both DSL and Cable providers should seriously set their pricing strategies, in order to improve their service qualities.

This research discusses the bandwidth allocation and pricing problems by taking the service qualities into account. The model hypothesizes that there is only a DSL service provider and a Cable service provider in the market; the two service providers offer two similar Internet service commodities, and both of the rivaling parties confront limited total bandwidth and service level. By establishing a mathematical model and Lagrange function, reaction functions of the two service providers can be obtained. Through the derived reaction functions and some condition, this paper obtains closed form formulas for the market equilibrium prices and bandwidth resource allocations for the two competitive ISPs.

2. MODEL ASSUMPTIONS, DESCRIPTION AND FORMULATION

This section of the paper will develop a mathematical model for the problem. We assume that a DSL Internet service provider and a Cable Internet service provider exist. We use index $m=1$ to denote the DSL service provider and index $m=2$ to denote the Cable service provider. The total bandwidth units available for provider *m* are B_m. Both providers are assumed to offer two similar Internet commodities. The sales price for commodity i of provider *m* is a decision variable and is denoted by $p_{m,i}$. Customers are able to purchase their commodities either from the DSL service provider or the Cable service provider. The demand curves for the two service providers are given as functions of prices. For convenience, we let $m'=2$ if $m'=1$, $m'=1$ if $m=2$, $i'=2$ if $i=1$ and $i'=1$ if $i=2$. Demand for commodity i, $i \in \{1,2\}$ of provider *m* is assumed to follow the function of

$$q_{m,i} = a_{m,i} - v_{m,i}\, p_{m,i} + h_i\,(p_{m',i} - p_{m,i})\beta_m\,(p_{m,i'} - p_{m,i}) \tag{1}$$

where the symbol h_i represents the substitution coefficient of similar commodities offered by different service providers and is assumed to be $h_i \leqq 1$, and β_m represents the substitution coefficient of different commodities offered by service provider *m* and is assumed to be $\beta_m \leqq 1$. The demand function illustrates that the demands for bandwidth commodities are dependent not only on the sales price of rivaling service provider for the similar commodities, but on the sales price of its different commodity as well.

For provider *m*, we denote $b_{m,i}$ as the number of bandwidth units allocated for commodity *i*. Service provider *m* is confronted with the limitation of total bandwidth units. That is, the total bandwidth units that the service provider *m* allocates to all types of bandwidth commodities shall never exceed the total bandwidth unit B_m.

The fraction of the number of online users to the total number of subscribers (on-line users plus off-line users) of commodity *i* of service provider *m* at any time is assumed to be an identical random variable $r_{m,i}$. The random variable $r_{m,i}$ takes value between 0 and 1 and is assumed to follow a normal distribution with mean $\mu_{m,i}$ and deviation $\sigma_{m,i}$.

The amount of bandwidth resource required by an online subscriber of commodity *i* of service provider m is assumed to be $u_{m,i}$ bandwidth units. Suppose the total number of subscribers for commodity *i* of service provider *m* is $q_{m,i}$, and the fraction of the number of on-line subscribers to the total number of subscribers is $\bar{r}_{m,i}$. Consequently, the bandwidth units required for commodity *i* is $\bar{r}_{m,i}\ q_{m,i}\, u_{m,i}$ is the number of subscribers of commodity *i* of a service provider *m* that makes online requests. In order to keep the ISP's service quality, we assume that ISP *m* will let the probability, i.e., the ratio of the number of bandwidth units allocated for commodity *i* exceeds the number of bandwidth resource needed by subscribers of commodity *i*, be no less than the prescribed service level $0 < \alpha_{m,i} \leq 1$.

This paper aims to set the equilibrium commodity price $p_{m,i}$ and determine the bandwidth allocation decisions $b_{m,i}$. We have illustrated the model descriptions and assumptions. We will now formulate the model. Firstly, we summarize our notation and decision variables as follows.

Notation:

$a_{m,i}$ = the intercept of demand function for commodity *i* of service provider *m*,
h_i = the coefficient of substitution for commodity *i* between service providers,
β_m = the coefficient of substitution between commodity *i* and commodity *i'* of service provider *m*,
$u_{m,i}$ = unit bandwidth consumption for commodity *i* of service provider *m*,
$\alpha_{m,i}$ = guaranteed service level of commodity *i* of service provider *m*,
$r_{m,i}$ = the fraction of the number of online users to the total number of subscribers of commodity *i* of provider *m* at any time,
$\mu_{m,i}$ = mean of the random variable $r_{m,i}$,
$\sigma_{m,i}$ = deviation of the random variable $r_{m,i}$,
B_m = total available bandwidth units for service provider *m*,
$p_{m,i}$ = decision variables, representing the sales price of commodity *i* of service provider *m*,
$b_{m,i}$ = decision variables, representing the number of bandwidth units allocated for commodity *i* of service provider *m*.

Revenue function

Let R_m be the revenue for firm *m*. Suppose service provider *m* sets the sales price of commodity *i* at $p_{m,i}$. The total revenue for service provider *m* can then be expressed as follows.

$$R_m = \sum_{i=1}^{2} p_{m,i} q_{m,i} \tag{2}$$

Resource and service constraints

Since the sum of the bandwidth units allocated for all commodities cannot exceed the available bandwidth unit B_m , we have the following bandwidth resource constraint for service provider *m*.

$$\sum_{i=1}^{2} b_{m,i} - B_m \le 0. \tag{3}$$

Suppose $b_{m,i}$ is allocated for commodity *i*. Since the number of subscribers of commodity *i* of service provider *m* is expected to be q_{mi} , the bandwidth units required by these subscriber is the number of $q_{mi} u_{mi} r_{m,i}$. Because the probability that the number of bandwidth units allocated for commodity *i* exceeds the number of bandwidth resource needed by subscribers of commodity *i* is required to be no less than a prescribed service level $0 < \alpha_{m,i} \le 1$, we have

$$\Pr\text{ob}(b_{m,i} \geq q_{mi} u_{mi} r_{m,i}) \geq \alpha_{m,i}, \qquad \forall m, i. \tag{4}$$

or, equivalently

$$\Pr\text{ob}\left(r_{m,i} \leq \frac{b_{m,i}}{q_{m,i} u_{m,i}}\right) \geq \alpha_{m,i}, \qquad \forall m, i. \tag{5}$$

As $r_{m,i}$ is assumed to be a normal random variable with mean $\mu_{m,i}$ and deviation $\sigma_{m,i}$, the above inequality can be rewritten as follows

$$\left(\frac{b_{m,i}}{q_{m,i} u_{m,i}} - \mu_{m,i}\right) / \sigma_{m,i} \geq Z_{m,i} \qquad \forall m, i \tag{6}$$

where $Z_{m,i}$ is called the z-statistic (Stone, 1996). The value of $Z_{m,i}$ can be evaluated using the Excel function NORMSINV as $Z_{m,i} = NORMSINV(\alpha_{m,i})$. The purpose of this paper is to establish the conditions for attaining the market equilibrium price $p_{m,i}$ and bandwidth allocation $b_{m,i}$. Let p_m and b_m be 2-dimensional vectors with elements $p_{m,i}$ and $b_{m,i}$ respectively, at *i*-th entry. Accordingly, the formulation of service provider *m* can be established as follows.

$$\max_{p_m, b_m} R_m = \sum_{i=1}^{2} p_{m,i} q_{m,i} \tag{7}$$

subject to inequality (3) and

$$b_{m,i} \geq q_{m,i} u_{m,i} (\sigma_{m,i} Z_{m,i} + \mu_{m,i}), \qquad \forall i \tag{8}$$

where inequality (8) is a rewritten form of inequality (6).

3. ANALYSIS

For the purpose of shortening our formulation, we will denote the following functions.

$$c_{m,i} = u_{m,i} + h_i + \beta_m, \tag{9}$$

$$g_{m,i} = u_{m,i} (\sigma_{m,i} Z_{m,i} + \mu_{m,i}). \tag{10}$$

Afterwards, we can rewrite revenue function R_m as follows.

$$R_m = -\sum_{i=1}^{2} c_{m,i} p_{m,i}^2 + 2\beta_m p_{m,1} p_{m,2} + \sum_{i=1}^{2} (\alpha_{m,i} + h_i p_{m',i}) p_{m,i}. \tag{11}$$

In addition, using (1) and (10), we can rewrite inequality (8) as follows.

$$-g_{m,i}(c_{m,i}+\beta_m)p_{m,i}+g_{m,i}(\alpha_{m,i}+h_i p_{m',i})+\beta_m g_{m,i}\sum_{i=1}^{2} p_{mi}-b_{m,i}\le 0. \quad (12)$$

Theorem 3.1 *R_m is concave function of p_m*

Proof: Let H_m be the Hessian matrix of R_m . Consequently, the result follows if H_m is negative definite. First, we have $\frac{\partial^2 R_m}{\partial p_{mi}^2}=-2$. Second, we have $\frac{\partial^2 R_m}{\partial p_{m,i}\, p_{md}}=2\beta_m$ and

$$\left\|H_m\right\|=\frac{\partial^2 R_m}{\partial p_{m1}^2}\frac{\partial^2 R_m}{\partial p_{m2}^2}-\left(\frac{\partial^2 R_m}{\partial p_{m1}^2\partial p_{m2}^2}\right)^2=4(c_{m,1}c_{m,2}-\beta_m^2)>0 \quad (13)$$

Thus, H_m is negative definite and we have completed the proof.

The Lagrangian function for service provider *m* can be formulated as follows.

$$\begin{aligned}L_m &= -\sum_{i=1}^{2} c_{m,i}\, p_{m,i}^2 + 2\beta_m p_{m,1} p_{m,2} + \\ &\sum_{i=1}^{2}(\alpha_{m,i}+h_i p_{m',i})p_{m,i}-\eta_m\left(\sum_{i=1}^{2} b_{m,i}-B_m+x_m^2\right) \\ &-\sum_{i=1}^{2}\lambda_{m,i}\begin{pmatrix}-g_{m,i}(c_{m,i}+\beta_m)p_{m,i}+g_{m,i}(\alpha_{m,i}+h_i p_{m',i}) \\ +\beta_m g_{m,i}\sum_{i=1}^{2} p_{mi}-b_{m,i}+y_{m,i}^2\end{pmatrix}.\end{aligned} \quad (14)$$

where $\lambda_{m,i}$, and η_m Lagrangian multipliers, and y_m and $x_{m,i}$ are slack variables. Now, we are ready to develop the reaction function for service provider *m*. Taking the partial derivatives of L_m with respect to $p_{m,i}$, $b_{m,i}$, $\lambda_{m,i}$, η_m , y_m and $x_{m,i}$, we obtain the following KKT conditions for service provider *m*.

$$\begin{aligned}\frac{\partial L_m}{\partial p_{m,i}} &= -2c_{m,i}\, p_{m,i}+2\beta_m p_{m,i'}+h_i p_{m',i}+\alpha_{m,i}+ \\ &\lambda_{m,i} g_{m,i} c_{m,i}-\lambda_{m,i'}\beta_m g_{m,i'}=0,\end{aligned} \quad (15)$$

$$\frac{\partial L_m}{\partial b_{m,i}}=-\eta_m+\lambda_{m,i}=0. \quad (16)$$

$$\frac{\partial L_m}{\partial \eta_m}=-\sum_{i=1}^{2} b_{m,i}+B_m-x_m^2=0, \quad (17)$$

$$\frac{\partial L_m}{\partial \lambda_{m,i}} = g_{m,i}(c_{mi} + \beta_m) p_{mi} - g_{mi}(\alpha_{mi} + h_i p_{m'i}) - \beta_m g_{mi} \sum_{i=1}^{2} p_{mi} + b_{mi} + y_{mi}^2 = 0, \tag{18}$$

$$\frac{\partial L_m}{\partial x_m} = -2\eta_m x_m = 0, \tag{19}$$

$$\frac{\partial L_m}{\partial y_{m,i}} = -2\lambda_{m,i} y_{m,i} = 0. \tag{20}$$

Let $p_{m,i}^*$, $b_{m,i}^*$, $\lambda_{m,i}$, η_m , y_m and $x_{m,i}$ be a solution point to the above equation system of Eq. (15) to Eq. (20). In Theorem 3.1, we have shown that R_m is concave function of p_m . Thus, if the values of $p_{m,i}^*$, $b_{m,i}^*$, $\lambda_{m,i}$, η_m , y_m and $x_{m,i}$ are no less than zero, the functions of $p_{m,i}^*$ and $b_{m,i}^*$ are service provider *m*'s reaction functions. For the purpose of simplifying our expression, we will denote the following function.

$$A_m = B_m - 0.5 \sum_{i=1}^{2} g_{mi}(\alpha_{mi} + h_i p_{m'i}). \tag{21}$$

Theorem 3.2 *Suppose* $A_m > 0$. *Then, the service provider m's reaction functions are given by* $p_{m,i}^*$ *and* $b_{m,i}^*$ *where*

$$p_{m,i}^* = \frac{\beta_m h_{i'} p_{m'i'} + h_i c_{mi'} p_{m'i} + \alpha_{mi} c_{mi'} + \beta_m \alpha_{mi'}}{2(c_{m1} c_{m2} - \beta_m^2)}, \tag{22}$$

$$b_{mi}^* = 0.5 g_{mi}(\alpha_{mi} + h_i p_{m'i}). \tag{23}$$

Proof: See Appendix A.

Although the condition of $A_m > 0$ is restrictive to the problem, it exists in many business practices since an Internet service provider's available bandwidth units are usually very large.

Theorem 3.3 *Suppose* $A_m > 0$. *Then, the equilibrium sales price* $\hat{p}_{mi}$ *and the bandwidth allocation decision* $\hat{b}_{mi}$ *are given by the following}*

$$\hat{p}_{mi} = -\frac{8(c_{m'i} c_{m'i'} - \beta_{m'}^2)(\alpha_{mi} c_{mi'} + \beta_m \alpha_{mi'})}{E} - \frac{4(\beta_m h_{i'} \alpha_{m'i'} c_{m'i} + h_i c_{mi'} \beta_{m'} \alpha_{m'i'} + h_i c_{mi'} \alpha_{m'i} c_{m'i'} + \beta_m h_{i'} \beta_{m'} \alpha_{m'i})}{E} + \frac{2h_{i'}(h_{i'} c_{m'i} \alpha_{mi} - \alpha_{mi'} \beta_{m'} h_i)}{E} + \frac{h_{i'}^2 h_i \alpha_{m'i}}{E} \tag{24}$$

$$\hat{b}_{mi} = 0.5 g_{mi}\left(\alpha_{mi} + h_i \hat{p} <_{m'i}\right) \qquad (25)$$

where

$$E = -16(c_{1,1}c_{1,2} - \beta_1^2)(c_{2,1}c_{2,2} - \beta_2^2) + 8h_1h_2\beta_1\beta_2 \\ + 4(c_{1,2}c_{2,2}h_1^2 + c_{1,1}c_{2,1}h_2^2) - h_2^2h_1^2. \qquad (26)$$

Proof: *According to Theorem 3.2, we see that the solution to the equation systems* $p_{1,1} = p_{1,1}^*$ *,* $p_{1,2} = p_{1,2}^*$ *,* $p_{2,1} = p_{2,1}^*$ *and* $p_{2,2} = p_{2,2}^*$ *are equilibrium sales price. By solving them, we obtain* $p_{1,1} = \hat{p}_{1,1}$ *,* $p_{1,2} = \hat{p}_{1,2}$ *,* $p_{2,1} = \hat{p}_{2,1}$ *,* $p_{2,2} = \hat{p}_{2,2}$ *. Thus, the equilibrium sales price is given by* $\hat{p}_{mi}$ *. Replacing* p_{mi}^* *in (22) with* $\hat{p}_{mi}$ *, we see that the bandwidth allocation decision is given by (25). Therefore, we have completed the proof.*

4. NUMERICAL EXAMPLES AND SENSITIVITY ANALYSIS

This section will provide some examples to illustrate the functioning of the model in the proposed framework. Let us assume that the parameters of $a_{m,i}$, $v_{m,i}$ and $u_{m,i}$ are given in Table 1, and the parameters of $\alpha_{m,i}$, $\mu_{m,i}$ and $\sigma_{m,i}$ are given in Table 2 where $Z_{m,i}$ is determined by using the Excel function NORMSINV as $Z_{m,i}$ =NORMSINV($\alpha_{m,i}$).

Table 1: parameters

m	$\alpha_{m,1}$	$\alpha_{m,2}$	$\sigma_{m,1}$	$\sigma_{m,2}$	$\mu_{m,1}$	$\mu_{m,2}$	$Z_{m,1}$	$Z_{m,2}$	B_m
1	120000	80000	100	90	0.50	0.50	4	2	250000
2	100000	85000	90	100	0.50	0.50	4	2	300000

Table 2: parameters of $\alpha_{m,i}$, $\mu_{m,i}$, $\sigma_{m,i}$ and $Z_{m,i}$

m	$\alpha_{m,1}$	$\alpha_{m,2}$	$\sigma_{m,1}$	$\sigma_{m,2}$	$\mu_{m,1}$	$\mu_{m,2}$	$Z_{m,1}$	$Z_{m,2}$
1	0.750	0.800	0.100	0.08	0.55	0.65	0.6745	0.8416
2	0.800	0.825	0.080	0.07	0.65	0.70	0.8416	0.9346

Substituting the above values into (21), we obtain A_1 =43,931.9>0 and A_2 =90,874.6>0. Thus, by Theorem 3.3, $\hat{p}_{mi}$ in (24) and $\hat{b}_{mi}$ in (25) are the equilibrium sales price and bandwidth allocation.

The equilibrium sales prices are $\hat{p}_{1,1}$ = 597.63, $\hat{p}_{1,2}$ =444.01, $\hat{p}_{2,1}$ =553.43 and $\hat{p}_{2,2}$ =424.63. The bandwidth allocations are $\hat{b}_{1,1}$ =148,529, $\hat{b}_{1,2}$ =57,539, $\hat{b}_{2,1}$ =143,895 and $\hat{b}_{2,2}$ = 65,231.The sales revenues for service provider m =1 is 53,747,889.2. The sales revenues for service provider m = 2 is 45,847,907.6.

The following will provide some insight into the impact of different input characteristics on the equilibrium sales prices and bandwidth allocations. We refer to the data set used in the above example as the basic parameter set. We investigate the impact of the change of h_i , β_m and α_{m1} on the equilibrium values of $\hat{p}_{mi}$ and $\hat{b}_m$. In the following cases, only one parameter changes, while others remain unchanged. The computational results are described in Table 3 to Table 8.

Table 3 shows that the values of $\hat{p}_{11}$ and $\hat{p}_{21}$ slightly decrease in the value of h_1 , and the values of $\hat{b}_{11}$ and $\hat{b}_{21}$ slightly increase in the value of h_1 . This phenomenon could be explained as follows. Note from the demand function in (01) that the demand for commodity 1 of service provider $m=1$ increases in the value of h_1 when the sales price of $\hat{p}_{11}$ is lower than its rival's sales price $\hat{p}_{21}$. In this environment, as the value of h_1 increases, the impact of the difference between $\hat{p}_{21}$ and $\hat{p}_{11}$ on demand becomes extensive. In case of increasing demand, service provider m = 1 may decrease its sales price to spur more market demand. On the other hand, the rival may decrease its sales price to keep its market share. Thus, the values of $\hat{p}_{11}$ and $\hat{p}_{21}$ decrease in the value of h_1 . In addition, since both service providers may try to sell more, they provide more bandwidth units for commodity 1. Thus, the values of $\hat{p}_{11}$ and $\hat{p}_{21}$ increase in the value of h_1 .

Table 3 The impact of h_1 on equilibrium price and bandwidth allocation

h_1	$\hat{p}_{11}$	$\hat{p}_{12}$	$\hat{p}_{21}$	$\hat{p}_{22}$	$\hat{b}_{11}$	$\hat{b}_{12}$	$\hat{b}_{21}$	$\hat{b}_{22}$
0.4	597.95	444.01	553.71	424.63	148461	57539	143809	65231
0.5	597.63	444.01	553.43	424.63	148529	57539	143895	65231
0.6	597.31	444.01	553.15	424.63	148598	57539	143980	65231
0.7	596.99	444.01	552.87	424.63	148666	57539	144065	65231
0.8	596.67	444.01	552.59	424.63	148734	57539	144151	65231
0.9	596.36	444.00	552.31	424.63	148802	57539	144236	65231

Table 4 shows that the values of $\hat{p}_{12}$ and $\hat{p}_{22}$ slightly decrease in the value of h_2 , and the values of $\hat{b}_{12}$ and $\hat{b}_{22}$ slightly increase in the value of h_2 . The explanation of this Table can be illustrated in a similar way to that of Table 3.

Table 4 The impact of h_2 on equilibrium price and bandwidth allocation

h_2	$\hat{p}_{11}$	$\hat{p}_{12}$	$\hat{p}_{21}$	$\hat{p}_{22}$	$\hat{b}_{11}$	$\hat{b}_{12}$	$\hat{b}_{21}$	$\hat{b}_{22}$
0.4	597.63	444.27	553.43	424.83	148529	57508	143895	65197
0.5	597.63	444.01	553.43	424.63	148529	57539	143895	65231
0.6	597.63	443.76	553.43	424.43	148529	57569	143895	65265
0.7	597.62	443.50	553.43	424.23	148529	57599	143895	65298
0.8	597.62	443.25	553.43	424.03	148529	57630	143895	65332
0.9	597.62	442.99	553.43	423.83	148529	57660	143895	65366

Table 5 shows that the value of $\hat{p}_{11}$ slightly decreases in the value of β_1 , and the value of $\hat{p}_{12}$ slightly increases in the value of β_1 . This phenomenon could be explained as follows. It is noted that the demand for commodity 1 of service provider m =

1 decreases in the value of β_1 when the sales price of $\hat{p}_{11}$ is higher than $\hat{p}_{12}$. As the value of β_1 increases, service provider m = 1 may try to lessen the difference between $\hat{p}_{11}$ and $\hat{p}_{12}$ to avoid the decrease in demand. Thus, service provider m = 1 decreases the value of $\hat{p}_{11}$, and increases the value of $\hat{p}_{12}$ as β_1 increases.

Table 6 shows that the value of $\hat{p}_{21}$ slightly decreases in the value of β_2, and the value of $\hat{p}_{22}$ slightly increases in the value of β_2. The explanation of this Table is similar to that of Table 5.

Table 7 shows that the value of $\hat{b}_{11}$ increases in the value of α_{11}. This phenomenon could be explained as follows. It is noted that the bandwidth required increases as the service level increases. Thus, service provider m = 1 increases the bandwidth allocation to commodity 1 when it expects to have a higher service level.

Table 8 shows that the value of $\hat{b}_{21}$ increases in the value of α_{21}. The explanation of this Table is similar to that of Table 7.

Table 5 The impact of β_1 on equilibrium price and bandwidth allocation

β_1	$\hat{p}_{11}$	$\hat{p}_{12}$	$\hat{p}_{21}$	$\hat{p}_{22}$	$\hat{b}_{11}$	$\hat{b}_{12}$	$\hat{b}_{21}$	$\hat{b}_{22}$
0.4	597.78	443.84	553.43	424.63	148529	57539	143895	65231
0.5	597.63	444.01	553.43	424.63	148529	57539	143895	65231
0.6	597.48	444.18	553.43	424.63	148529	57539	143895	65231
0.7	597.33	444.35	553.43	424.63	148529	57539	143894	65231
0.8	597.18	444.51	553.43	424.63	148529	57539	143894	65231
0.9	597.03	444.68	553.43	424.63	148529	57539	143894	65231

Table 6 The impact of β_2 on equilibrium price and bandwidth allocation

β_2	$\hat{p}_{11}$	$\hat{p}_{12}$	$\hat{p}_{21}$	$\hat{p}_{22}$	$\hat{b}_{11}$	$\hat{b}_{12}$	$\hat{b}_{21}$	$\hat{b}_{22}$
0.4	597.63	444.01	553.57	424.50	148530	57539	143895	65231
0.5	597.63	444.01	553.43	424.63	148529	57539	143895	65231
0.6	597.63	444.01	553.29	424.76	148529	57539	143895	65231
0.7	597.63	444.01	553.15	424.88	148529	57539	143895	65231
0.8	597.63	444.01	553.01	425.01	148529	57539	143895	65231
0.9	597.63	444.01	552.87	425.13	148529	57539	143895	65231

Table 7 The impact of α_{11} on equilibrium price and bandwidth allocation

α_{11}	$\hat{p}_{11}$	$\hat{p}_{12}$	$\hat{p}_{21}$	$\hat{p}_{22}$	$\hat{b}_{11}$	$\hat{b}_{12}$	$\hat{b}_{21}$	$\hat{b}_{22}$
0.65	597.63	444.01	553.43	424.63	141573	57539	143895	65231
0.70	597.63	444.01	553.43	424.63	144919	57539	143895	65231
0.75	597.63	444.01	553.43	424.63	148529	57539	143895	65231
0.80	597.63	444.01	553.43	424.63	152550	57539	143895	65231
0.85	597.63	444.01	553.43	424.63	157236	57539	143895	65231
0.90	597.63	444.01	553.43	424.63	163133	57539	143895	65231

Table 8 The impact of α_{21} on equilibrium price and bandwidth allocation

α_{21}	$\hat{p}_{11}$	$\hat{p}_{12}$	$\hat{p}_{21}$	$\hat{p}_{22}$	$\hat{b}_{11}$	$\hat{b}_{12}$	$\hat{b}_{21}$	$\hat{b}_{22}$
0.65	597.63	444.01	553.43	424.63	148529	57539	136572	65231
0.70	597.63	444.01	553.43	424.63	148529	57539	138804	65231
0.75	597.63	444.01	553.43	424.63	148529	57539	141213	65231
0.80	597.63	444.01	553.43	424.63	148529	57539	143895	65231
0.85	597.63	444.01	553.43	424.63	148529	57539	147021	65231
0.90	597.63	444.01	553.43	424.63	148529	57539	150955	65231

5. CONCLUSIONS

This research discussed the problem of bandwidth allocation and pricing in an Internet duopoly market. The bandwidth commodities' service level was taken into consideration. In order to find a solution to the problem, this research developed a Lagrange function to develop the optimal sales price and bandwidth allocation for both Internet service providers. The results were used to establish the reaction functions. Under certain conditions, this paper proposed closed form formulas for the market equilibrium price and bandwidth resource allocation.

A numerical analysis is also provided to illustrate the impact of different input parameters on the equilibrium sales prices and bandwidth allocations. The results are summarized as follows:

1. For both service providers, the equilibrium sales price of a commodity decreases, and the bandwidth allocation of a commodity increases in the degree of substitutability between commodities of the two service providers (Tables 3 and 4).
2. The equilibrium sales price of commodity-1 of service provider-1 and the bandwidth allocation of commodity-1 of service provider-2 decrease, and the equilibrium sales price of commodity-2 of service provider-1 and the bandwidth allocation of commodity-2 of service provider-2 increase in the degree of substitutability between the commodities offered by service provider-1 (Table 5).
3. The equilibrium sales price of commodity-1 of service provider-2 and the bandwidth allocation of commodity-1 of service provider-1 decrease, and the equilibrium sales price of commodity-2 of service provider-2 and the bandwidth allocation of commodity-2 of service provider-1 increase in the degree of substitutability between the commodities offered by service provider-2 (Table 6).
4. The bandwidth allocation of a commodity-*i* for a service provider increases in the intercept of demand function for that commodity for that service provider- (Tables 7 and 8).

Appendix

Setting the values of η_m , y_{m1} and y_{m2} in the equation system of (15) to (20) at $\eta_m = 0$, $y_{m1} = 0$ and $y_{m2} = 0$ and letting them be equal to zero, we have

$$\frac{\partial L_m}{\partial p_{mi}} = -2c_{mi}\,p_{mi} + 2\beta_m\,p_{mi'} + a_{mi} + h_i\,p_{m'i} + \lambda_{mi}\,g_{mi}c_{mi} - \lambda_{mi'}\,g_{mi'}\beta_m = 0 \tag{27}$$

$$\frac{\partial L_m}{\partial p_{mi}} = \lambda_{mi} = 0 \tag{28}$$

$$\frac{\partial L_m}{\partial \eta_{mi}} = -b_{m1} - b_{m2} + B_m - x_m^2 = 0 \tag{29}$$

$$\frac{\partial L_m}{\partial \lambda_{mi}} = g_{mi}(c_{mi} + \beta_m)\,p_{mi} - g_{mi'}h_i\,p_{m'i} - \beta_m g_{mi}(p_{m1} + p_{m2}) - g_{mi}a_{mi} + b_{mi} = 0 \tag{30}$$

$$\frac{\partial L_m}{\partial x_{mi}} = -2\eta_m y_m = 0 \tag{31}$$

$$\frac{\partial L_m}{\partial y_{mi}} = -2\lambda_{mi}\,y_{mi} = 0 \tag{32}$$

Solving the above equation system, we have $\lambda_{mi} = 0$ for all i and

$$p^*{}_{m,i} = \frac{\beta_m h_{i'}\,p_{m'i'} + h_i c_{mi'}\,p_{m'i} + a_{mi}c_{mi'} + \beta_m a_{mi'}}{2(c_{m1}c_{m2} - \beta_m^2)} \tag{33}$$

$$b^*{}_{mi} = \frac{1}{2}\,g_{mi}(a_{mi} + h_i\,p_{m'i}) \tag{34}$$

$$x_m = \sqrt{A_m}\ . \tag{35}$$

Since the values of $p_{mi}^*, b_{mi}^*, \lambda_{mi}, \eta_m, y_m$ and x_{mi} are no less than zero, p_{mi}^* and b_{mi}^* are optimal solutions for service provider m.

REFERENCES

[1] Basar, T., and Srikant, R., “Stackelberg games and multiple equilibrium behaviors on network”, *Journal of Optimization Theory and Applications*, 115 (3) (2002) 479-490.

[2] Bapna, R., Goes, P., and Gupta, A., “Pricing and allocation for quality-differentiated online services”, *Management Science*, 51 (7) (2005) 1141-1150.

[3] Choi, S.C., “Price competition in a duopoly common retailer channel”, *Journal of Retailing*, 72 (2) (1996) 117-134.

[4] Dai, Y., Chao, X., Fang, S.C., and Henry, L.W., “Pricing in revenue management for multiple firms competing for customers”, *International Journal of Production Economics,* 98(1) (2005) 1-16.

[5] Elberfeld, W., and Wolfstetter, E., “A dynamic model of Bertrand competition with entry”, *International Journal of Industrial Organization*, 17 (1999) 513-52.

[6] Foros, Ø., and Hansen, B., "Competition and compatibility among Internet Service Providers", *Information Economics and Policy*, 13(4) (2001) 411-425.

[7] Harrington, J.E., "Experimentation and learning in a differentiated-products duopoly", *Journal of Economic Theory*, 66 (1995) 275-288.

[8] Mason, R., "Simple competitive Internet pricing", *European Economics Review*, 44 (4-6) (2000) 1045-1056.

[9] Min, T., Kim, S.Y., Shin, C., and Hahn, M., "Competitive nonlinear pricing with product differentiation", *International Review of Economics and Finance*, 11 (2) (2002) 155-173.

[10] Palma, A.D., and Leruth, L., "Equilibrium in competing networks with differentiated products", *Transportation Science*, 27 (1) (1993) 73-80.

[11] Peha, J.M., and Tewari, S., "The results of competition between integrated-services telecommunications carriers", *Information Economic and Policy*, 10 (1998) 127-155.

[12] Stone, C.J., *A Course in Probability and Statistics*, Duxbury, USA, 1996.

[13] Symeonidis, G., "Comparing Cournot and Bertrand equilibria in a differentiated duopoly with product R & D", *International Journal of Industrial Organization*, 21 (2003) 39-55.