

A NOTE ON THE P -CENTER PROBLEM

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Abstract: The p -center problem is to locate p facilities in a network so as to minimize the longest distance between a demand point and its nearest facility. In this paper, we give a construction on a graph G which produces an infinite ascending chain $G = G_0 \leq G_1 \leq G_2 \leq \dots$ of graphs containing G such that given any optimal solution X for the p -center problem on G , X is an optimal solution for the p -center problem on G_i for any $i \geq 1$.

Keywords: Location theory, p -center problem.

MSC: 90B80, 05CXX.

1. INTRODUCTION

Network location problems are concerned with finding the right locations to place one or more facilities in a network of demand points, i.e., customers represented by nodes in the network, that optimize a certain objective function related to the distance between the facilities and the demand points. Usually, the facilities to be located are desirable, i.e., customers prefer to have the facilities located as close to them as possible. For example, services such as police and fire stations, hospitals, schools, and shopping centers are typical desirable facilities.

The p -center problem is to locate p facilities in a network so as to minimize the longest distance between a set of n demand points and the p facilities. This problem is central to the field of location theory and logistics, and has been subject to extensive research. For references in p -center problem see for example [1-11].

We model the network as a graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ is the vertex set with $|V| = n$ and E is the edge set with $|E| = m$. We assume that the demand points coincide with the vertices, and restrict the location of the facilities to the vertices. Each vertex v_i has a weight w_i and the edges of graph have positive weights. Let $d(u, v)$ is the length of shortest weighted path between vertices u and v . In the P -center problem we want to find a subset $X \subseteq V$ of cardinality p such that the maximum weighted distances from X to all vertices is minimized. In other words, we want to find a subset $X \subseteq V$ of cardinality p such that $F(X) = \max_{i=1, \dots, n} w_i d(X, v_i)$ is minimized.

We call a graph G triangle-free if G does not contain any triangle as an induced subgraph. Triangle-free graphs are a class of well-studied graphs and play an important role in graph theory. Many of graph theory parameters deal with triangle-free graphs. To see some results on triangle-free graphs we refer the reader to look for example [12]. Yet determining location problems in triangle-free graphs is open.

One of the questions regarding location problems is how to nontrivially extend a graph G to a larger graph with the same optimal solution for the p -center problem. We will present a nontrivial construction. We give a construction on a graph G which produces an infinite chain $G = G_0 \leq G_1 \leq G_2 \leq \dots$ of graphs containing G such that for a given optimal solution X for the p -center problem on G , X is an optimal solution for the p -center problem on G_i for any $i \geq 1$. If G has n vertices and m edges, our construction produces a graph $M(G)$ with $2n$ vertices and $2m$ edges. Furthermore if G is triangle-free, then $M(G)$ is triangle-free. Note that by $G_i \leq G_j$ we mean that G_i is a subgraph of G_j . This construction produces bigger and bigger graphs (instances) which have the same optimal solution as the original graph G . This allows us to extend a graph with the given optimal solution for p -center problem to a bigger graph without any further calculation to find a solution for p -center problem.

All graphs handled in this paper are connected and undirected. We recall that the open neighborhood of a vertex v in a graph G is denoted by $N(v)$ or $N_G(v)$ to refer to G , thus $N(v) = \{u \in V \mid uv \in E\}$. Also, note that by "optimal solution", we mean a best possible solution for our problem. So if X and Y are two optimal solutions for the p -center problem and F is the objective function, then $F(X) = F(Y)$.

2. MAIN RESULT

We first give a construction as follows. Let $G = (V, E)$ be a weighted graph with a vertex set $V = \{v_1, v_2, \dots, v_n\}$. For $i = 1, 2, \dots, n$, let w_i be the weight of v_i . Also, for $e \in E$, let $a(e)$ be the weight of e . Our construction produces an M -graph $M(G)$ from G with $V(M(G)) = V \cup U$ where $U = \{u_1, \dots, u_n\}$ and

$$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i), i = 1, \dots, n\}$$

The weights of new vertices and new edges of $M(G)$ are the following. For $i = 1, 2, \dots, n$ the weight of v_i is equal to w_i , and for a new edge $e = v_i v_j$ the weight of e is equal to the weight of $v_i v_j$. We define the k -th M -graph of G , recursively by $M_0(G) = G$ and $M^{k+1}(G) = M(M^k(G))$ for $k \geq 1$. We shall prove the following.

Theorem 1. Assume that G is a weighted graph where all vertices have the same weight and all edges have the same weight. Let X be an optimal solution for the p -center problem on G . For any positive integer k , X is an optimal solution for the p -center problem on $M^k(G)$.

Proof. Let $G = (V, E)$ be a weighted graph with a vertex set $V = \{v_1, v_2, \dots, v_n\}$ where all vertices have the same weight w , and all edges have the same weight a . It is sufficient to prove the theorem for $k = 1$ since the result follows by induction. Let $M(G)$ be the M -graph obtained from G with $V(M(G)) = V \cup U$ where

$$U = \{u_1, \dots, u_n\} \text{ and } E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i), i = 1, \dots, n\}.$$

Thus for $i = 1, 2, \dots, n$, the weight of the new vertex v_i is w , and for a new edge e the weight is a . For a vertex $z \in V(M(G)) \setminus V(G)$ we let $v_z \in V(G)$ be the vertex which is $N_{M(G)}(v_z) = N_{M(G)}(z)$.

Let $X = \{x_1, x_2, \dots, x_p\} \subseteq V$ be an optimal solution for the p -center problem on G , and let $y \in V(G) \setminus X$ be a vertex such that $d_G(y, X) = \max_{v \in V(G) \setminus X} d_G(v, X) = d$. It follows that $d_{M(G)}(y, X) = \max_{v \in V(G) \setminus X} d_{M(G)}(v, X) = d$. Let Y be an optimal solution for the p -center problem on $M(G)$. We prove that $\max_{v \in V(M(G)) \setminus Y} d_{M(G)}(v, Y) = d$. Let $z \in V(M(G)) \setminus Y$ be a vertex such that

$$d_{M(G)}(z, Y) = \max_{v \in V(M(G)) \setminus Y} d_{M(G)}(v, Y).$$

We consider the following cases.

Case 1. $Y \subseteq V(G)$ and $z \in V(G)$. Then Y is a solution for the p -center problem on G . Since X is an optimal solution for the p -center problem on G , we obtain $d_G(z, Y) \geq d_G(y, X)$. This implies that $d_{M(G)}(z, Y) \geq d_{M(G)}(y, X)$. But Y is an optimal solution for the p -center problem on $M(G)$. Thus $d_{M(G)}(z, Y) \geq d_{M(G)}(y, X)$.

Case 2. $Y \subseteq V(G)$ and $z \notin V(G)$. If $v_z \in Y$, then $d_{M(G)}(z, Y) \in \{wa, 2wa\}$.

Let $u \in V(M(G)) \setminus V(G)$ be a vertex such that $u \notin Y$ and $N_G(v_u) \cap Y \neq \emptyset$. Then $d_{M(G)}(u, Y) = wa$. This implies that $d_{M(G)}(z, Y) = wa$.

But $d_{M(G)}(y, X) = d_G(y, X) \geq wa$. We deduce that $d_{M(G)}(y, X) = d_{M(G)}(z, Y) = wa$. Thus we may assume that $u_z \notin Y$. Then $d_{M(G)}(z, Y) = d_{M(G)}(v_z, Y)$. Now the result follows from Case 1.

Case 3. $Y \not\subseteq V(G)$. If for any $u \in V(M(G)) \setminus V(G)$, $|\{u, v_u\} \cap Y| \leq 1$, then $Y' = (Y \cap V(G)) \cup \{v_u : u \in (V(M(G)) \setminus V(G)) \cap Y\}$ is a solution for the p -center problem on $M(G)$ with

$$\max_{v \in V(M(G)) \setminus Y} d_{M(G)}(v, Y) = \max_{v \in V(M(G)) \setminus Y'} d_{M(G)}(v, Y'),$$

and the result follows from Cases 1 and 2. So we assume that there is some vertex $u \in V(M(G)) \setminus V(G)$ such that $\{u, v_u\} \subseteq Y$. Let

$$A = \{u \in V(M(G)) \setminus V(G) : \{u, v_u\} \subseteq Y\}$$

and let y_0 be a vertex such that

$$d_{M(G)}(y_0, Y) = \max_{v \in V(M(G)) \setminus Y} d_{M(G)}(v, Y).$$

Without loss of generality assume that $y_0 \in V(G)$. Let $y_1 \in Y$ be a vertex with $d_{M(G)}(y_0, Y) = d_{M(G)}(y_0, y_1)$. Consider a vertex $u_1 \in A$. Let $v_{u_1}^*$ be a vertex in $V(G) \setminus Y$ such that

$$d_{M(G)}(y_0, v_{u_1}^*) \geq d_{M(G)}(y_0, Y_1)$$

and let $Y_1 = (Y \setminus \{u_1\}) \cup \{u_1^*\}$. Then $d_{M(G)}(y_0, Y) = d_{M(G)}(y_0, Y_1) = d_{M(G)}(y_0, y_1)$

so Y_1 is an optimal solution for the p – center problem on $M(G)$. Let $u_2 \in A \setminus \{u_1\}$. Then, let $v_{u_2}^*$ be a vertex in $V(G) \setminus Y_1$ such that

$$d_{M(G)}(y_0, v_{u_2}^*) \geq d_{M(G)}(y_0, Y_1).$$

Then

$$Y_2 = (Y_1 \setminus \{u_2\}) \cup \{u_2^*\}$$

is an optimal solution for the p – center problem on $M(G)$. By continuing this process, $Y_{|A|}$ is an optimal solution for the p – center problem on $M(G)$. Since $Y_{|A|} \subseteq V(G)$, the result follows from Cases 1 and 2.

Now the result follows by induction. ■

We remark that Theorem 1 does not work if the weight of vertices and edges of G are different. To check this, let G be a path P_3 with vertices v_1, v_2, v_3 , and edges v_1, v_2 and v_2, v_3 . Let $w(v_1) = w(v_3) = 1, w(v_2) = 2$, and the weight of each edge be 1. Then $X = \{v_2\}$ is an optimal solution for the 1-center problem on G , and $F(X) = 1$. Now, $V(M(G)) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$ where the weight of each edge is one and $i = 1, 2, 3, w(u_i) = w(v_i)$. It is easy to see that $Y = \{v_1\}$ is an optimal solution for the 1-center problem on $M(G)$ and $F(Y) = 2$, while $F(X) = F(\{v_2\}) = 4$.

3. CONCLUSION

Our construction produces an infinite ascending chain $G = G_0 \leq G_1 \leq G_2 \leq \dots$ of graphs containing G such that given any optimal solution X for the p – center problem on G , X is an optimal solution for the p – center problem on G_i for any $i \geq 1$. This allows us to extend a graph to a sufficiently bigger graph with no more calculation required to find an optimal solution for p – center problem. Further, if G is a graph on n vertices and m edges, then for any positive $k \geq 1$, $M^k(G)$ is a graph with $2^k n$ vertices and $2^k m$ edges. Hence, if $f(G)$ is the complexity function for the p – center problem on a graph G , then

$$\lim_{k \rightarrow \infty} \frac{f(G)}{f(M^k(G))} = 0$$

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