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CONTROL OF STAGE BY STAGE CHANGING LINEAR DYNAMIC SYSTEMS

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Abstract: In this paper, the control problems of linear dynamic systems stage by stage changing and the optimal control with the criteria of quality set for the whole range of time intervals are considered. The necessary and sufficient conditions of total controllability are also stated. The constructive solving method of a control problem is offered, as well as the definitions of conditions for the existence of programmed control and motions. The explicit form of control action for a control problem is constructed. The method for solving optimal control problem is offered, and the solution of optimal control of a specific target is brought.

Keywords: Control problem, optimal control, dynamic system, stage by stage changing.

MSC: 37N35, 93C05, 49J15, 93C15

1. INTRODUCTION

Problems of control, and an optimal control of objects on the set of operational states have an important role, theoretical and applied, for instance, in the case of the change of dynamic model's parameters due to the operation of system. These problems are typical for the energy-efficient and thermal devices, the machinery with electricity cables and other mechanisms of industrial-technological purpose. In particular, the dynamic model of controllable object with the change of operational state confined, when time intervals are strictly limited, is considered in [1, 2].

Both in the ordinary control problems and the problems of dynamic systems changing stage by stage, the challenges of phased controllability, control, and optimal control emerge [3-6]. The control and optimal control problems of linear dynamic systems and stage by stage changing linear dynamic systems are stated and investigated in [2] and [7].

The control and the optimal control problems of linear dynamic systems changing stage by stage are stated and investigated in this paper.

2. PROBLEMS' DEFINITIONS

Consider a control process, dynamics of which is described by the phased changing linear differential equations.

$$\dot{x} = \begin{cases} A_1(t)x + B_1(t)u, & for\ t \in [t_0, t_1) \\ A_2(t)x + B_2(t)u, & for\ t \in [t_1, t_2) \\ \vdots & \\ A_m(t)x + B_m(t)u, & for\ t \in [t_{m-1}, T] \end{cases} \tag{2.1}$$

where $x(t) \in R^n$ is the phase vector of system, $A_k(t)$, $B_k(t)$ $(k = 1, \ldots, m)$ are matrices $(n \times n)$ and $(n \times r)$ of parameters of the system (the models of the object) respectively, $u(t)$ is column-vector $(r \times 1)$ of control action. In general, we will assume that the entries of matrices of functions $A_k(t)$, $B_k(t)$ and the entries of column-vector $u(t)$ are measurable bounded functions.

There is given initial

$$x(t_0) = x_0 \tag{2.2}$$

and final

$$x(T) = x_T \tag{2.3}$$

states of the system (2.1).

It is assumed that at given interim moments of time

$$0 \le t_0 < t_1 < \ldots < t_{m-1} < t_m = T$$

the end of previous period motion is the beginning of the next period motion, i.e. at t_k moments of time $x(t_k - 0) = x(t_k + 0) = x(t_k)$, for $(k = 1, ..., m-1)$.

Consider the following problems.

Problem 1. Find conditions of the existence of programmed control action $u = u(t)$ transferring the motion of system (2.1) from the initial state (2.2) to the final state (2.3) at the time interval $[t_0, T]$, and construct them as well.

For the selection of optimal solutions at the time interval $[t_0, T]$, the criteria of quality æ$[u]$ which can have a sense of norm for some normed space, is given.

For the system (2.1) with states (2.2), (2.3) and the criteria of quality æ$[u]$, the problem of optimal control can be defined as follows.

Problem 2. Find the optimal control action by moving the system (2.1) from the initial state(2.2) to the final state (2.3) and having the lowest possible value of criteria of quality æ$\left[u^0\right]$.

3. SOLUTION OF PROBLEMS

To solve the defined problems, write the solution of system (2.1) for the time interval $\left[t_{k-1}, t_k\right]$ as follows:

$$x(t) = X_k[t, t_{k-1}]x(t_{k-1}) + \int_{t_{k-1}}^{t} H_k[t, \tau]u(\tau)d\tau \,, \tag{3.1}$$

where $H_k[t, \tau] = X_k[t, \tau]B_k(\tau)$ and $X_k[t, \tau]$ is the normed fundamental matrix of solution of the homogeneous part of equation k of the system (2.1) for a time interval $[t_{k-1}, t_k]$

$$\dot{x} = A_k(t)x + B_k(t)u$$

Assuming that the required control actions for $t = t_k$ are known, from (3.1) for $t = t_k$ moments of time, we have

$$x(t_k) = X_k[t_k, t_{k-1}]x(t_{k-1}) + \int_{t_{k-1}}^{t_k} H_k[t_k, \tau]u(\tau)d\tau \tag{3.2}$$

Substituting the expression for $x(t_{k-1})$ into (3.1), obtained from writing (3.2) for the previous period $[t_{k-2}, t_{k-1}]$, we get

$$\begin{aligned} x(t) &= X_k[t, t_{k-1}]X_{k-1}[t_{k-1}, t_{k-2}]x(t_{k-2}) + \\ &+ X_k[t, t_{k-1}]\int_{t_{k-2}}^{t_{k-1}} H_{k-1}[t_{k-1}, \tau]u(\tau)d\tau + \int_{t_{k-1}}^{t} H_k[t, \tau]u(\tau)d\tau \end{aligned} \tag{3.3}$$

If we continue this process for the previous time intervals, we will get the formula, describing the motion of system (2.1) for $t \in [t_{k-1}, t_k]$ moment of time

$$\begin{aligned} x(t) &= V(t, t_0)x(t_0) + \sum_{j=1}^{k-1} V(t, t_j)\int_{t_{j-1}}^{t_j} H_j[t_j, \tau]u(\tau)d\tau + \\ &+ \int_{t_{k-1}}^{t} H_k[t, \tau]u(\tau)d\tau \end{aligned} \tag{3.4}$$

where the following notations are used

$$\begin{gathered} V(t, t_j) = X_k[t, t_{k-1}]V(t_{k-1}, t_j), \quad V(t_k, t_j) = \prod_{i=0}^{k-j-1} X_{k-i}[t_{k-i}, t_{k-i-1}], \\ (k = 1, \ldots, m;\ j = 0, \ldots, k-1) \end{gathered} \tag{3.5}$$

According to the introduced notations, when $j = k-1$, $V(t_k, t_{k-1}) = X_k[t_k, t_{k-1}]$. When $j = k$, $V(t_k, t_k) = E$, then for $t = t_k$ (3.4) can be written in the following form

$$x(t_k) = V(t_k, t_0)x(t_0) + \sum_{j=1}^{k} V(t_k, t_j) \int_{t_{j-1}}^{t_j} H_j[t_j, t]u(t)dt$$

for $k = m$

$$x(T) = V(T, t_0)x(t_0) + \sum_{j=1}^{m} V(T, t_j) \int_{t_{j-1}}^{t_j} H_j[t_j, t]u(t)dt . \qquad (3.6)$$

So, having an initial state $x(t_0)$ of system (2.1) and giving the control action $u(t)$, the phase state $x(t)$ of system (2.1) can be determined for any point of time t from $[t_{k-1}, t_k)$ using equation (3.4).

Now instead of $H_k[t_k, t]$, we introduce $\overline{H}_k[t_k, t]$ function in the following form

$$\overline{H}_1[t_1, t] = \begin{cases} H_1[t_1, t], & for\ t_0 \le t < t_1 \\ 0, & for\ t_1 \le t \le T \end{cases}$$

$$\overline{H}_k[t_k, t] = \begin{cases} 0, & for\ t_0 \le t < t_{k-1} \\ H_k[t_k, t], & for\ t_{k-1} \le t < t_k \\ 0, & for\ t_k \le t \le T \end{cases} \qquad k = 2, ..., m-1 \qquad (3.7)$$

$$\overline{H}_m[t_m, t] = \begin{cases} 0, & for\ t_0 \le t < t_{m-1} \\ H_m[t_m, t], & for\ t_{m-1} \le t \le T \end{cases}$$

Using functions introduced in (3.7), the relation (3.6) can be rewritten as

$$\int_{t_0}^{T} \sum_{j=1}^{m} V(T, t_j)\overline{H}_j[t_j, t]u(t)dt = x(T) - V(T, t_0)x(t_0) = C . \qquad (3.8)$$

Note, that there are n integral relations in (3.8).

Thereby, the system (2.1) is totally controllable if and only if for any $C = x(T) - V(T, t_0)x(t_0)$ vector from R^n. There can be found control $u = u(t, C)$ satisfying condition (3.8).

The controlability analysis of any system is important for the solution of control problem.

Let, $h_i(T, t)$ be the i-th column of matrix $\left(\sum_{j=1}^{m} V(T, t_j)\overline{H}_j[t_j, t] \right)'$; C_i be the i th component of vector C. Here and further, prime denotes an operation of transposition.

Then equation (3.8) can be written

$$\int_{t_0}^{T} h_i'(T,t)u(t)dt = C_i \qquad (i = 1,\dots,n) .$$

So, the condition of controllability of system (2.1) can be formulated in the following theorem.

Theorem 1. *The system (2.1) is totally controllable in the interval* $[t_0, T]$ *if and only if the vector-functions* $h_1(T,t),\cdots,h_n(T,t)$ *are linearly independent in that interval.*

Now, the function $u(t)$, satisfying the integral relation (3.8), has the form [5]

$$u(t) = \left(\sum_{j=1}^{m} V(T,t_j)\overline{H}_j[t_j,t] \right)' \eta + v(t) , \tag{3.9}$$

where η - constant vector (will be defined further), $v(t)$ -some vector function (can be a measurable bounded function in the time interval $[t_0,T]$), for which

$$\int_{t_0}^{T} \left(\sum_{j=1}^{m} V(T,t_j)\overline{H}_j[t_j,t] \right) v(t)dt = 0 . \tag{3.10}$$

Equation (3.10) is the condition of orthogonality of a vector-function $v(t)$ to all rowes of matrix $\left(\sum_{j=1}^{m} V(T,t_j)\overline{H}_j[t_j,t] \right)$.

Substituting (3.9) into (3.8), we get

$$Q(t_0,....,T)\eta = C , \tag{3.11}$$

where

$$Q(t_0,....,T) = \int_{t_0}^{T} \left(\sum_{j=1}^{m} V(T,t_j)\overline{H}_j[t_j,t] \right) \left(\sum_{j=1}^{m} V(T,t_j)\overline{H}_j[t_j,t] \right)' dt . \tag{3.12}$$

(3.11) is a system of n algebraic equations with C_j $(j = 1,\dots,n)$ unknowns.

Equation (3.11) can be solved if either $\det Q \neq 0$ or the matrix Q, and extended matrix $\{Q, C\}$ have the same rank.

The solution of (3.11) will be of this form

$$\eta = Q^{-1}C$$

hence, taking into account the value of vector C, (3.9) is written as

$$u(t) = \left(\sum_{j=1}^{m} V(T,t_j)\overline{H}_j[t_j,t] \right)' Q^{-1}(x(T) - V(T,t_0)x(t_0)) + v(t) . \tag{3.13}$$

So, the solution of **Problem 1** can be formulated in the following theorem.

Theorem 2. *Program control (3.13) and the corresponding solution of system (2.1), satisfying (2.2) and (2.3), exists if and only if, the matrix (3.12) isn't singular, or matrices* Q *and* $\{Q,C\}$ *have the same rank.*

Taking into account notations in (3.7), the control action $u(t)$ when $v(t)=0$ according to (3.13), is written in the following form

$$u(t)=\begin{cases}(V(T,t_1)H_1[t_1,t])'Q^{-1}(x(T)-V(T,t_0)x(t_0)), \text{ for } t\in[t_0,t_1)\\(V(T,t_2)H_2[t_2,t])'Q^{-1}(x(T)-V(T,t_0)x(t_0)), \text{ for } t\in[t_1,t_2)\\\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\(H_m[t_m,t])'Q^{-1}(x(T)-V(T,t_0)x(t_0)), \text{ for } t\in[t_{m-1},T]\end{cases} \tag{3.14}$$

Substituting (3.14) into (3.4), and, taking into account the continuity of motion in the intermediate moments of time t_k, we get the program motion of system (2.1) in $[t_0,T]$, and satisfying conditions (2.2), (2.3).

For the solution of Problem 2, note the following. For the given quality criterion æ$[u]$, the problem of optimal control with the integral conditions (3.8) can be considered as a problem of a conditional extremum from the variation calculus, where one should find minimum of functional æ$[u]$ subject to (3.8). However, as it is obvious from (3.7), the sub integral functions in (3.8) are disruptive and because of it, the theorems of variation calculus aren't valid for solving this problem [4].

The left hand side of condition (3.8) is linear operation generated by the function $u(t)$ in $[t_0,T]$ time interval.

Hence, if functional æ$[u]$ is a norm of some linear normed space, the solution of Problem 2 should be find using the problem of moments, then, the optimal control action $u^0(t)$, $t\in[t_0,T]$ minimizing functional æ$[u]$ is the solution to Problem 2.

So, the problem 2 is reduced to the problem of moments, whose solution is known from [4].

4. EXAMPLE

As an illustration of the presented method for constructing optimal control action consider the model of the controled object, described in [2], whose dynamics of changes stage by stage has the form

$$\begin{cases}\dot{x}_1=x_2\\\dot{x}_2=b^{(1)}u\end{cases} \qquad t\in[t_0,t_1) \tag{4.1}$$

$$\begin{cases}\dot{x}_1=x_2\\\dot{x}_2=a_2^{(2)}x_2+b^{(2)}u\end{cases} \qquad t\in[t_1,t_2) \tag{4.2}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = a_1^{(3)} x_1 + a_2^{(3)} x_2 + b^{(3)} u \end{cases} \qquad t \in [t_2, t_3] \tag{4.3}$$

where $a_2^{(2)}$, $a_1^{(3)}$, $a_2^{(3)}$, $b^{(1)}$, $b^{(2)}$, $b^{(3)}$ are the parameters of object model and $t_0 < t_1 < t_2 < t_3$ are the intermediate moments of time.

Initial and final phase states are given as

$$x(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix}, \; x(t_3) = \begin{pmatrix} x_1(t_3) \\ x_2(t_3) \end{pmatrix}$$

A quality criteron of control has the form

$$æ[\, u \,] = \left(\int_{t_0}^{T} u^2(t) dt \right)^{\frac{1}{2}} . \tag{4.4}$$

Avoiding vigorous mathematical expressions, assume that

$b^{(1)} = b^{(2)} = b^{(3)} = 1$, $a_2^{(2)} = 1$, $a_1^{(3)} = -2$, $a_2^{(3)} = 3$.

The normed fundamental matrices of solution of the homogeneous parts of systems (4.1)-(4.3) have the following form respectively

$$X_1[t, t_0] = \begin{pmatrix} 1 & t - t_0 \\ 0 & 1 \end{pmatrix}, \; X_2[t, t_1] = \begin{pmatrix} 1 & e^{t-t_1} - 1 \\ 0 & e^{t-t_1} \end{pmatrix},$$

$$X_3[t, t_2] = \begin{pmatrix} 2e^{t-t_2} - e^{2(t-t_2)} & e^{2(t-t_2)} - e^{t-t_2} \\ 2e^{t-t_2} - 2e^{2(t-t_2)} & 2e^{2(t-t_2)} - e^{t-t_2} \end{pmatrix}$$

For the matrix $H_k[t_k, t] = X_k[t_k, t] B(t_k) \quad (k = 1, 2, 3)$, the following is obtained

$$H_1[t_1, t] = \begin{pmatrix} h_{11}(t_1, t) \\ h_{12}(t_1, t) \end{pmatrix} = \begin{pmatrix} t_1 - t \\ 1 \end{pmatrix}, \quad H_2[t_2, t] = \begin{pmatrix} h_{21}(t_2, t) \\ h_{22}(t_2, t) \end{pmatrix} = \begin{pmatrix} e^{t_2 - t} - 1 \\ e^{t_2 - t} \end{pmatrix}$$

$$H_3[t_3, t] = \begin{pmatrix} h_{31}(t_3, t) \\ h_{32}(t_3, t) \end{pmatrix} = \begin{pmatrix} e^{2(t_3 - t)} - e^{t_3 - t} \\ 2e^{2(t_3 - t)} - e^{t_3 - t} \end{pmatrix}$$

According to (3.5), introduce the following notations

$$V(t_3, t_0) = \begin{pmatrix} V_{11}^{(30)} & V_{12}^{(30)} \\ V_{21}^{(30)} & V_{22}^{(30)} \end{pmatrix} = \begin{pmatrix} e(2-e) & e^2 \\ 2e(1-e) & e^2 \end{pmatrix},$$

$$V(t_3,t_1)=\begin{pmatrix} V_{11}^{(31)} & V_{12}^{(31)} \\ V_{21}^{(31)} & V_{22}^{(31)} \end{pmatrix}=\begin{pmatrix} e(2-e) & 2e(e-1) \\ 2e(e-1) & e(3e-2) \end{pmatrix}$$

$$V(t_3,t_2)=\begin{pmatrix} V_{11}^{(32)} & V_{12}^{(32)} \\ V_{21}^{(32)} & V_{22}^{(32)} \end{pmatrix}=\begin{pmatrix} e(e-2) & e(e-1) \\ 2e(1-e) & e(2e-1) \end{pmatrix}$$

Now following (3.7), denote

$$\overline{h}_{11}(t_1,t)=\begin{cases} t_1-t, \text{ for } t\in[t_0,t_1) \\ 0, \text{ for } t\in[t_1,t_3] \end{cases}$$

$$\overline{h}_{12}(t_1,t)=\begin{cases} 1, \quad \text{for } t\in[t_0,t_1) \\ 0, \quad \text{for } t\in[t_1,t_3] \end{cases}$$

$$\overline{h}_{21}(t_2,t)=\begin{cases} 0, \text{ for } t\in[t_0,t_1) \\ e^{t_2-t}-1, \text{ for } t\in[t_1,t_2) \\ 0, \text{ for } t\in[t_2,t_3] \end{cases}$$

$$\overline{h}_{22}(t_2,t)=\begin{cases} 0, \text{ for } t\in[t_0,t_1) \\ e^{t_2-t}-1, \text{ for } t\in[t_1,t_2) \\ 0, \text{ for } t\in[t_2,t_3] \end{cases}$$

$$\overline{h}_{31}(t_3,t)=\begin{cases} 0, \text{ for } t\in[t_0,t_2) \\ e^{2(t_3-t)}-e^{t_3-t}, \text{ for } t\in[t_2,t_3] \end{cases}$$

$$\overline{h}_{31}(t_3,t)=\begin{cases} 0, \text{ for } t\in[t_0,t_2) \\ 2e^{2(t_3-t)}-e^{t_3-t}, \text{ for } t\in[t_2,t_3] \end{cases}$$

According to (3.8), we have integral relations

$$\int_{t_0}^{T} h_1(t)u(t)\,dt=c_1\ , \quad \int_{t_0}^{T} h_2(t)u(t)\,dt=c_2\ , \tag{4.5}$$

where

$$h_1(t)=V_{11}^{(31)}\overline{h}_{11}(t_1,t)+V_{12}^{(31)}\overline{h}_{12}(t_1,t)+V_{11}^{(32)}\overline{h}_{21}(t_2,t)+V_{12}^{(32)}\overline{h}_{22}(t_2,t)+\overline{h}_{31}(t_3,t)\ ,$$

$$h_2(t)=V_{21}^{(31)}\overline{h}_{11}(t_1,t)+V_{22}^{(31)}\overline{h}_{12}(t_1,t)+V_{21}^{(32)}\overline{h}_{21}(t_2,t)+V_{22}^{(32)}\overline{h}_{22}(t_2,t)+\overline{h}_{32}(t_3,t)\ ,$$

$$c_1=x_1(t_3)-V_{11}^{(30)}x_1(t_0)-V_{12}^{(30)}x_2(t_0)\,,\ \ c_2=x_2(t_3)-V_{21}^{(30)}x_1(t_0)-V_{22}^{(30)}x_2(t_0)$$

Assume that the following numerical values are given $t_0=0, t_1=1, t_2=2,$ $t_3=3$, and

$$x(0)=\begin{pmatrix}0\\0\end{pmatrix},\ x(3)=\begin{pmatrix}3\\2\end{pmatrix}$$

Solving problems (4.4) and (4.5) as the problems of moments, the optimal control action are obtained

$$u^0(t)=\begin{cases}0,997012-0,841126t, \text{ for } t\in[t_0,t_1)\\ -0,841126+2,71016e^{-t}, \text{ for } t\in[t_1,t_2)\\ -5,87287e^{-2t}-2,71016e^{-t}, \text{ for } t\in[t_2,t_3]\end{cases}$$

and the value of the criteron of quality is æ $\left[u^0\right]=0,55481$.

Substituting $u^0(t)$ into equation (3.4), written for systems (4.1)-(4.3), the optimal motion of object for every interval of time will be obtained.

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