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EVALUATING FUZZY INEQUALITIES AND SOLVING FULLY FUZZIFIED LINEAR FRACTIONAL PROGRAMS[1]

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Abstract: In our earlier articles, we proposed two methods for solving the fully fuzzified linear fractional programming (FFLFP) problems. In this paper, we introduce a different approach of evaluating fuzzy inequalities between two triangular fuzzy numbers and solving FFLFP problems. First, using the Charnes-Cooper method, we transform the linear fractional programming problem into a linear one. Second, the problem of maximizing a function with triangular fuzzy value is transformed into a problem of deterministic multiple objective linear programming. Illustrative numerical examples are given to clarify the developed theory and the proposed algorithm.

Keywords: Fuzzy programming, triangular fuzzy number, fractional programming, centroid of triangle.

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1. INTRODUCTION

There have been significant developments in the theory and applications of fractional programming in the last decades. For more information about fractional programming problems, the reader may consult the bibliography with 491 entries presented in Stancu-Minasian [11], covering mainly the years from 1997 to 2005, which gives a clear idea of the amount of work invested in this field.

Fractional programming is important to our daily life, because various optimization problems from engineering, social life, and economy consider the minimization of a ratio between physical and/or economical functions, for example cost/time, cost/volume, cost/profit, or other quantities that measure the efficiency of a system. Fractional programming has grown rapidly in the last four decades, and it continues to find new applications in different areas of social life and economy.

Nowadays fuzzy information is widely involved in a real decision making problem and consequently, fuzzy programming continues to be developed following various directions (see for instance [1, 3, 8, 9, 10, 14]). Buckley and Feuring, in [1], considered the fully fuzzified linear programming problem (FFLP) by establishing all the coefficients and variables of a linear program as being fuzzy quantities. They transform the fully fuzzified programming problem into a multiple objective deterministic problem which, treated in the general case, is non-linear. Then, the problem is transformed into a multiple objective fuzzy problem, which helps the authors to explore the entire set of the Pareto-optimal solutions. Solving the multiple objective fuzzy problem, performed by using a genetic algorithm, leads to feasible solutions of the initial problem.

Allowing all parameters to be fuzzy, we obtain, what we have called, the fully fuzzifed linear fractional programming (FFLFP) problem. In our previous papers [8, 9], we proposed a method to solve the fully fuzzified linear fractional programming problem, where all the variables and parameters are represented by triangular fuzzy numbers. The proposed approach is the first one that resolves fuzzy inequalities by using Kerre's method [6], and then transforms the original fuzzy problem into a deterministic multiple objective linear fractional programming problem with quadratic constraints.

We also propose the idea of transforming the fully fuzzified linear fractional problem into a multiple objective deterministic problem, by establishing the coefficients and the variables of the problem as triangular fuzzy numbers aggregated with fuzzy operators obtained through applying the Zadeh's extension principle and its approximate version. The Charnes-Cooper transformation (see [2]) to obtain a fully fuzzified linear programming problem is applied before aggregating fuzzy quantities.

In the present paper, the case of a fully fuzzified linear fractional programming problem is considered. The FFLFP model is presented in Section 2. The aggregation and comparison of triangular fuzzy numbers are presented in Section 3. Fuzzy inequalities are evaluated by using centroid of triangles. The obtained general form of the deterministic equivalent problem is described in Section 4. In Section 5, in order to illustrate our method, we consider two numerical examples. Short concluding remarks are made in Section 6.

2. THE FFLFP MODEL

Let us consider the fully fuzzified linear fractional programming problem

$$\max\left(\overline{Z}=\frac{\sum_{j=1}^{n}\overline{C_jX_j}+\overline{C_0}}{\sum_{j=1}^{n}\overline{D_jX_j}+\overline{D_0}}\right) \quad (1)$$

subject to

$$\begin{cases}\overline{M_i}=\sum_{j=1}^{n}\overline{A_{ij}X_j}-\overline{B_i}\le\overline{0}, & i=1,\dots,m\\ \overline{X_j}\ge\overline{0}, & j=1,\dots,n\end{cases} \quad (2)$$

where

- $(\overline{C_j})_{j=1,\dots,n}$, $\overline{C_0}$ and $(\overline{D_j})_{j=1,\dots,n}$, $\overline{D_0}$ represent the coefficients of the linear fractional objective function,
- $(\overline{A_{ij}})_{i=1,\dots,m}^{j=1,\dots,n}$ and $(\overline{B_i})_{i=1,\dots,m}$ represent the coefficients and the right hand side of the linear constraints respectively,
- $(\overline{X_j})_{j=1,\dots,n}$ represents the decision variables, and
- $\overline{0}$ represents the triangular fuzzy number $(0,0,0)$.

It is assumed that the denominator in (1) is strictly positive for any $\overline{X_j}$ in the feasible region. Moreover, in this paper, we also assume that the nominator in (1) is strictly positive. The meaning of "strictly positive" will be explained later. The notation $\overline{Y}$ means that Y represents a fuzzy quantity. As it will be shown in the next section, if $\overline{C_j}$, $\overline{B_i}$, $\overline{D_j}$, $\overline{D_0}$, $\overline{B_i}$, $\overline{X_j}$ and $\overline{A_{ij}}$ are triangular fuzzy numbers for each $\overline{X_j}$ $i=1,\dots,m$ and $j=1,\dots,n$ then, using an approximate version of the extension principle for evaluating multiplication and division ([9]), $\overline{Z}$ and $\overline{M_i}$ are also triangular fuzzy numbers for each $j=1,\dots,m$. The inequalities in constraints (2) will be evaluated by using the *x*-coordinate of the centroid of triangle that describes the triangular fuzzy number on the left hand side of each inequality.

In Section 4, we propose a method for solving Problem (1)-(2) when all initial fuzzy quantities are described by triangular fuzzy numbers (trapezoidal fuzzy numbers, or even more general fuzzy numbers, can be considered as well). According to Buckley and Feuring [1], solving Problem (1)-(2) assumes defining the meaning of "maximum" of a fuzzy number, i.e. $\max\overline{Z}$, evaluating the fuzzy inequality $\overline{M_i}\le\overline{0}$.

3. TRIANGULAR FUZZY NUMBERS - AGGREGATION AND COMPARISON

The purpose of this section is to recall some concepts needed in the sequel, and to introduce new rules for describing fuzzy inequalities and equalities.

Definition 1. *A triangular fuzzy number Y is a triplet* $(y^1, y^2, y^3) \in R^3$.

The extension principle was formulated by Zadeh (see, for example [16]) in order to extend the known models implying fuzzy elements in the case of fuzzy entities. We will use this principle to add and subtract triangular fuzzy numbers, and its approximate version to multiply and divide triangular fuzzy numbers. We recall here the corresponding definition introduced in [9].

Definition 2. *Being given two triangular fuzzy numbers,* $\overline{A} = (a^1, a^2, a^3)$ *and* $\overline{B} = (b^1, b^2, b^3)$, $a^1, a^2, a^3, b^1, b^2, b^3 \in R$, *we define addition, subtraction, multiplication and division as:*

(i) $\overline{A} + \overline{B} = (a^1 + b^1, a^2 + b^2, a^3 + b^3)$,

(ii) $\overline{A} - \overline{B} = (a^1 - b^3, a^2 - b^2, a^3 - b^1)$,

(iii) $\overline{A} \cdot \overline{B} = (a^1 b^1, a^2 b^2, a^3 b^3)$,

(iv) $\dfrac{\overline{A}}{\overline{B}} = (\dfrac{a^1}{b^3}, \dfrac{a^2}{b^2}, \dfrac{a^3}{b^1})$.

One definition for the inequality between two fuzzy numbers was introduced by Kerre in 1982. The main concept of comparison of fuzzy numbers is based on the comparison of areas determined by membership functions [15]. We used this concept in our previous papers [8, 9]. Here, instead of comparing areas of triangles (see Definitions 3, Proposition 1), we compare *x*-coordinates of centroids of triangles, and obtain simpler deterministic problem, equivalent to Problem (1)-(2).

Generally, the centroid of a shape *X* is the intersection of all straight lines that divide *X* into two parts of equal moment about the line. Informally, it is the "average" of all points of *X*. The centroid of a triangle is defined as follows.

Definition 3. *Let us consider three points,* $A(x_1, y_1), B(x_2, y_2)$ *and* $C(x_3, y_3)$ *in a coordinate system xOy. The centroid of the triangle ABC has coordinates*

$$(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}).$$

Proposition 1. *The inequality*

$$(m_1, m_2, m_3) \le (0,0,0) \quad (3)$$

holds if and only if

$$m_1 + m_2 + m_3 \le 0 \quad (4)$$

Corollary 1. *A triangular fuzzy number (m_1, m_2, m_3) is equal to (0, 0, 0) if and only if*

$$m_1 + m_2 + m_3 = 0 \quad (5)$$

(i.e. the centroid of the corresponding triangle lies on Oy-axis).

Proposition 1 describes the fuzzy inequality (3) taking into consideration if the centroid of the triangle, which describes the membership function of the triangular fuzzy number (m_1, m_2, m_3), is on the left or on the right of y-axis. This statement offers an equivalent description of Kerre's inequality (see for instance [5, 7]).

4. SOLVING METHOD FOR FFLFP

Based on the concepts discussed in the previous section, in this paragraph we describe an alternative method of building a deterministic problem, that is the equivalent of Problem (1)-(2).

First, we transform Problem (1)-(2) into a fully fuzzified linear programming problem using the Charnes-Cooper transformation ($\sum_{j=1}^{n}\overline{D_j X_j}+\overline{D_0}=\frac{1}{\overline{T}}$ and $\overline{X_j}\,\overline{T}=\overline{Y_j}$, $i=1,...,n$) and obtain the following problem

$$\max\left(\sum_{j=1}^{n}\overline{C_j Y_j}+\overline{C_0 T}\right) \tag{6}$$

subject to

$$\begin{cases}\sum_{j=1}^{n}\overline{A_{ij}Y_j}-\overline{B_i T}\le\overline{0}, \quad i=1,...,m,\\ \sum_{j=1}^{n}\overline{D_j Y_j}+\overline{D_0 T}=\overline{1},\\ \overline{T}\ge\overline{0},\\ \overline{Y_j}\ge\overline{0}, \quad j=1,...,n.\end{cases} \tag{7}$$

After aggregating the fuzzy quantities according to Definition 2, we change the maximization of the objective function, which is described by a triangular fuzzy number, with the maximization of the three components of the fuzzy number, and obtain the following deterministic multiple objective linear program with fuzzy constraints

$$\max\left(\sum_{j=1}^{n}c_j^1 y_j^1+c_0^1 t^1,\sum_{j=1}^{n}c_j^2 y_j^2+c_0^2 t^2,\sum_{j=1}^{n}c_j^3 y_j^3+c_0^3 t^3\right) \tag{8}$$

subject to

$$\begin{cases}\left(\sum_{j=1}^{n}a_{ij}^1 y_j^1-b_i^3 t^3,\sum_{j=1}^{n}a_{ij}^2 y_j^2-b_i^2 t^2,\sum_{j=1}^{n}a_{ij}^3 y_j^3-b_i^1 t^1\right)\le\overline{0}, \quad i=1,...,m,\\ \left(\sum_{j=1}^{n}d_j^1 y_j^1+d_0^1 t^1,\sum_{j=1}^{n}d_j^2 y_j^2+d_0^2 t^2,\sum_{j=1}^{n}d_j^3 y_j^3+d_0^3 t^3\right)=\overline{1},\\ 0\le t^1\le t^2\le t^3,\\ 0\le y_j^1\le y_j^2\le y_j^3, \quad j=1,...,n\end{cases} \tag{9}$$

By applying (3) to each inequality from (9), and by applying (5) to the equality from (9), we obtain the constraints (11) of the following deterministic multiple objective linear programming Problem (MOLP):

$$\max\left(\sum_{j=1}^{n} c_j^1 y_j^1 + c_0^1 t^1, \sum_{j=1}^{n} c_j^2 y_j^2 + c_0^2 t^2, \sum_{j=1}^{n} c_j^3 y_j^3 + c_0^3 t^3\right) \tag{10}$$

subject to

$$\begin{cases} \sum_{j=1}^{n}(a_{ij}^1 y_j^1 + a_{ij}^2 y_j^2 + a_{ij}^3 y_j^3) - b_i^1 t^1 - b_i^2 t^2 - b_i^3 t^3 \le 0, \quad i = 1,\dots,m, \\ \sum_{j=1}^{n}(d_j^1 y_j^1 + d_j^2 y_j^2 + d_j^3 y_j^3) + d_0^1 t^1 + d_0^2 t^2 + d_0^3 t^3 = 3, \\ 0 \le t^1 \le t^2 \le t^3, \\ 0 \le y_j^1 \le y_j^2 \le y_j^3, \quad j = 1,\dots,n. \end{cases} \tag{11}$$

By solving Problem (10)-(11), solutions $(y_j^1, y_j^2, y_j^3)_{j=1,\dots,n}$ and (t^1, t^2, t^3) are obtained, namely the triangular fuzzy numbers $(\overline{Y_j})_{j=1,\dots,n}$ and $\overline{T}$. Then, the optimal solution of Problem (1)-(2) is $\overline{X_j} = \frac{\overline{Y_j}}{\overline{T}}$, $j = 1,\dots,n$.

5. COMPUTATION RESULTS

In order to illustrate the method for solving fully fuzzified linear fractional problems, let us consider the deterministic linear fractional problem (12)-(13) which was also considered in [8].

$$\max\left(z = \frac{x_1 - x_2 + 1}{x_1 + x_2 + 2}\right) \tag{12}$$

subject to

$$\begin{cases} x_1 + x_2 \le 2, \\ x_1 - x_2 \le 1, \\ x_1, x_2 \ge 0. \end{cases} \tag{13}$$

The optimal solution to this problem is $x_1 = 1, \quad x_2 = 0$, and the optimal value is $z = 0.6667$.

Now, we attach to this problem a fully fuzzified problem

$$\max\left(z = \frac{\overline{c_1 x_1} + \overline{c_2 x_2} + \overline{c_0}}{\overline{d_1 x_1} + \overline{d_2 x_2} + \overline{d_0}}\right) \tag{14}$$

subject to

$$\begin{cases} \overline{a_{11}x_1} + \overline{a_{12}x_2} - \overline{b_1} \le \overline{0}, \\ \overline{a_{21}x_1} + \overline{a_{22}x_2} - \overline{b_2} \le \overline{0}, \\ \overline{x_1}, \overline{x_2} \ge \overline{0}. \end{cases} \tag{15}$$

considering each real number coefficient m as being triangular fuzzy number $\overline{m} = (m^1, m^2, m^3)$.

Now we transform Problem (14)-(15) into a fully fuzzified linear programming problem using Charnes-Cooper transformation, and obtain:

$$\max(\overline{c_1 y_1} + \overline{c_2 y_2} + \overline{c_0 t}) \tag{16}$$

subject to

$$\begin{cases} \overline{a_{11}y_1} + \overline{a_{12}y_2} - \overline{b_1 t} \le \overline{0}, \\ \overline{a_{21}y_1} + \overline{a_{22}y_2} - \overline{b_2 t} \le \overline{0}, \\ \overline{d_1 y_1} + \overline{d_2 y_2} + \overline{d_0} = \overline{1} \\ \overline{y_1}, \overline{y_2}, \overline{t} \ge \overline{0}. \end{cases} \tag{17}$$

Example 1. Using a symmetrical definition and spread 2 for triangular fuzzy numbers $m = (m-1, m, m+1)$, coefficients values for fully fuzzified problem, that we will solve, are

$$\overline{c} = \left[(0,1,2),(-2,-1,0),(0,1,2)\right] \quad \overline{d} = \left[(0,1,2),(0,1,2),(1,2,3)\right]$$

$$\overline{a} = \begin{bmatrix} (0,1,2) & (0,1,2) \\ (0,1,2) & (-2,-1,0) \end{bmatrix} \qquad \overline{b} = \begin{bmatrix} (1,2,3) \\ (0,1,2) \end{bmatrix}$$

According to the method described in Section 4, in order to obtain the solution to this problem, we solve the following multiple objective linear problem keeping fuzzy constraints:

$$\max(f_1(y,t) = -2y_2^1, f_2(y,t) = y_1^2 - y_2^2 + t^2, f_3(y,t) = 2y_1^3 + 2t^3) \tag{17}$$

subject to

$$\begin{cases} (-3t^3, y_1^2 + y_2^2 - 2t^2, 2y_1^3 + 2y_2^3 - t^1) \le \overline{0}, \\ (-2y_2^3 - 2t^3, y_1^2 - y_2^2 - t^2, 2y_1^3) \le \overline{0}, \\ (t^1 - 1, y_1^2 + y_2^2 + 2t^2 - 1, 2y_1^3 + 2y_2^3 + 3t^3 - 1) = \overline{0} \\ \overline{y_1}, \overline{y_2}, \overline{t} \ge \overline{0}. \end{cases} \tag{18}$$

Evaluating the fuzzy constraints by using the method described in Section 3, we obtain the following equivalent system of constraints

$$\begin{cases} -3t^3 + y_1^2 + y_2^2 - 2t^2 + 2y_1^3 + 2y_2^3 - t^1 \leq 0, \\ -2y_2^3 - 2t^3 + y_1^2 - y_2^2 - t^2 + 2y_1^3 \leq 0, \\ t^1 + y_1^2 + y_2^2 + 2t^2 + 2y_1^3 + 2y_2^3 + 3t^3 = 3, \\ 0 \leq y_1^1 \leq y_1^2 \leq y_1^3, \\ 0 \leq y_2^1 \leq y_2^2 \leq y_2^3, \\ 0 \leq t^1 \leq t^2 \leq t^3. \end{cases} \tag{19}$$

In order to obtain a synthesis function of the three objective functions from (17), and to apply it to the results presented in [12], we use the importance coefficients $\pi_1 = 0.1, \pi_2 = 0.8,$ and $\pi_3 = 0.1$ respectively. The optimum of the synthesis function $\pi_1 f_1 + \pi_2 f_2 + \pi_{3 f_3}$ is reached in

$$(\overline{y^*}, \overline{t^*}) = \left[\overline{y_1} = (0, 0.333, 0.333), \overline{y_2} = (0,0,0), \bar{t} = (0.333, 0.333, 0.333)\right].$$

It follows that the solution of problem (14)-(15) is

$$\overline{x^*} = \left[\overline{x_1} = (0,1,1), \overline{x_2} = (0,0,0)\right]$$

that approximates very closely the pair of real numbers (1, 0) representing the solution (x_1, x_2) of Problem (12)-(13). Also, triangular fuzzy number $\overline{z^*} = (0, 0.55, 1.09)$ approximates very closely the real number 0.6667, that represents the optimal value of Problem (12)-(13).

Example 2. Using a non-symmetrical definition and spread 3 for triangular fuzzy numbers $m = (m-1, m, m+2)$, coefficients values for fully fuzzified problem (16), that we will solve, are

$$\bar{c} = \left[(0,1,3), (-3,-1,0), (0,1,3)\right], \quad \bar{d} = \left[(0,1,3), (0,1,3), (1,2,4)\right],$$

$$\bar{a} = \begin{bmatrix} (0,1,3) & (0,1,3) \\ (0,1,3) & (-3,-1,0) \end{bmatrix}, \qquad \bar{b} = \begin{bmatrix} (1,2,4) \\ (0,1,3) \end{bmatrix}.$$

The obtained solution is

$$(\overline{y^*}, \overline{t^*}) = \left[\overline{y_1} = (0, 0.273, 0.273), \overline{y_2} = (0,0,0), \bar{t} = (0.273, 0.273, 0.273)\right].$$

It follows that the solution of Problem (14)-(15) is

$$\overline{x^*} = \left[\overline{x_1} = (0,1,1), \overline{x_2} = (0,0,0)\right]$$

which approximates very closely the pair of real numbers (1, 0) representing the solution (x_1, x_2) of problem (12)-(13). Also, triangular fuzzy number $\overline{z^*} = (0, 0.55, 1.09)$ approximates very closely the real number 0.6667, that represents the optimal value of problem (12)-(13).

6. CONCLUDING REMARKS

In this paper we proposed a new method to evaluate fuzzy equalities and inequalities, and we solved fully fuzzified linear fractional programming problems where all parameters and variables were triangular fuzzy numbers.

We have transformed the fractional problem into a linear one, by using Charnes-Cooper method. After that, the problem of maximizing a triangular fuzzy number was transformed into a deterministic multiple objective linear programming problem with quadratic constraints. We have applied the extension principle of Zadeh, and its approximate version to aggregate fuzzy numbers. In order to evaluate each fuzzy constraint, the x-coordinate of the centroid of triangle was used to avoid a disjunctive system of deterministic constraints.

Given examples illustrate the fact that the developed method can be successfully applied in solving fuzzy programming problems. The division of variable triangular fuzzy numbers (which involves approximation) was avoided, but the division of solution triangular fuzzy numbers was used once to obtain solution for fractional programming problem from the solution of a linear programming problem.

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