

## TIME MINIMIZING TRANSPORTATION PROBLEM WITH FRACTIONAL BOTTLENECK OBJECTIVE FUNCTION

M. JAIN

*Department of Mathematics,  
G.L.N.A. Institute of Technology, Mathura, India  
madhuridayalbagh@gmail.com*

P.K. SAKSENA

*Faculty of Engineering,  
Dayalbagh Educational Institute, Dayalbagh, Agra, India  
premkumarsaksena@gmail.com*

Received: August 2010 / Accepted: January 2012

**Abstract:** This paper is aimed at studying the Time Minimizing Transportation Problem with Fractional Bottleneck Objective Function (TMTP-FBOF). TMTP-FBOF is related to a Lexicographic Fractional Time Minimizing Transportation Problem (LFTMTP), which will be solved by a lexicographic primal code. An algorithm is also developed to determine an initial efficient basic solution to this TMTP-FBOF. The developed TMTP-FBOF Algorithm is supported by a real life example of Military Transportation Problem of Indian Army.

**Keywords:** Time transportation, lexicographic, optimization, fractional programming.

MSC: 90C08, 90C05.

### 1. INTRODUCTION

Transportation Problem with a bottleneck objective function is generally known as time minimizing transportation problem or bottleneck transportation problem, where a feasible transportation schedule is to be found, which minimizes the maximum of transportation time needed between a supply point and a demand point such that the distribution between the two points is positive. Seshan and Tikekar [4] presented a time

minimizing transportation problem to determine the set  $S_{hk}$  of all non basic cells which when introduced into the basis either eliminate a given basic cell  $(h,k)$  from the basis or reduce their amount. Achary and Seshan [1] discuss a time minimizing transportation problem based on a more general and realistic assumption that the time  $t_{ij}(x_{ij})$  required for transporting  $x_{ij}$  units from the  $i^{th}$  source to the  $j^{th}$  destination depends on the actual amount transported. Sonia and Puri [6] considered a two level hierarchical balanced time minimizing transportation problem. To obtain the global optimal feasible solution of the non-convex optimization problem, related balanced time minimizing transportation problems were defined.

Transportation problems with fractional objective function are widely used as performance measures in many real life situations e.g., in the analysis of financial aspects of transportation enterprises and undertaking, and in transportation management situations, where an individual, or a group of a community is faced with the problem of maintaining good ratios between some crucial parameters concerned with the transportation of commodities from certain sources to various destinations. In transportation problems, examples of fractional objectives include optimization of total actual transportation cost/total standard transportation cost, total return/total investment, risk assets/capital, total tax/total public expenditure on commodity, and etc.

Gupta et al. [3] studied a paradox in Linear Fractional Transportation Problems with mixed constraints, and established a sufficient condition for the existence of a paradox. A Paradoxical range of flow was also obtained for any flow in which the corresponding objective function value was less than that of the original Linear Fractional Transportation Problem with mixed constraints. Corban [2] extended the concept of multi-dimensional transportation problem with fractional linear objective function and derived the optimality conditions, for global optimum in terms of simplex multipliers. Sharma and Swarup [5] presented a transportation technique for time minimization in fractional functional programming problem with an objective function.

This paper deals with a Time Minimizing Transportation Problem with Fractional Bottleneck Objective Function (TMTP-FBOF). TMTP-FBOF is related to a Lexicographic Fractional Time Minimizing Transportation Problem (LFTMTP) which will be solved by a primal algorithm. The partial flows which constitute a feasible transportation schedule can be partitioned according to the transportation time involved. The main idea of the transformation is the introduction of a vector of partial flows in which the first component represents the partial flow which requires the highest transportation time, and the last component represents the partial flow which requires the lowest transportation time. By means of primal algorithm, the vector of partial flows is minimized in a lexicographic sense on the feasible set, i.e. the optimal flow vector has the property that no other feasible transportation schedule exists such that the flow vector is lexicographically smaller than the optimal one. The optimal value of the flow vector immediately indicates the minimal bottleneck transportation time and the minimal flow which requires the optimal bottleneck transportation time. A primal algorithm and its underlying methodology are also presented, used to solve such fractional decision priority problems.

## 2. MATHEMATICAL FORMULATION

Given an actual transportation time matrix  $T^a$  and a standard transportation time matrix  $T^s$ , where  $T^a = [t_{ij}^a]$  and  $T^s = [t_{ij}^s]$ , for transporting the goods from  $i^{\text{th}}$  supply point ( $i = 1, 2, \dots, M$ ) to  $j^{\text{th}}$  demand point ( $j = 1, 2, \dots, N$ ), the problem is to find a feasible distribution (of the supplies) which minimizes the maximum fractional bottleneck transportation time between a supply point and a demand point such that the distribution between the two points is positive. The mathematical formulation of the Time Minimizing Transportation Problem with Fractional Bottleneck Objective Function (TMTP-FBOF) is:

$$\min t = \max_{(i,j)} \left\{ \frac{t_{ij}^a}{t_{ij}^s} \mid x_{ij} > 0 \right\} \quad (1)$$

subject to

$$\sum_{j=1}^N x_{ij} = a_i \quad (i = 1, 2, \dots, M) \quad (2)$$

$$\sum_{i=1}^M x_{ij} = b_j \quad (j = 1, 2, \dots, N) \quad (3)$$

$$x_{ij} \geq 0 \quad (i = 1, 2, \dots, M; j = 1, 2, \dots, N) \quad (4)$$

where

$a_i$  = amount of the commodity available at the  $i^{\text{th}}$  supply point

$b_j$  = requirement of the commodity at the  $j^{\text{th}}$  demand point

$x_{ij}$  = amount of the commodity transported from the  $i^{\text{th}}$  supply point to the  $j^{\text{th}}$  demand point

$t_{ij}^a$  = actual transportation time from the  $i^{\text{th}}$  supply point to the  $j^{\text{th}}$  demand point

$t_{ij}^s$  = standard transportation time from the  $i^{\text{th}}$  supply point to the  $j^{\text{th}}$  demand point

$\frac{t_{ij}^a}{t_{ij}^s}$  = proportional contribution to the value of the fractional time objective function for

shipping one unit of commodity from the  $i^{\text{th}}$  supply point to the  $j^{\text{th}}$  demand point

$t$  = fractional bottleneck transportation time.

Assumption:  $a_i$  and  $b_j$  are given non-negative numbers not simultaneously zero and total demand requirement equal to the total supply.

### 3. LEXICOGRAPHIC FRACTIONAL TIME MINIMIZING TRANSPORTATION PROBLEM

A vector-valued fractional objective function can be related to the nonlinear bottleneck objective function (1). Now this vector-valued fractional objective function is to be minimized in a lexicographic order. Setting  $M' = \{1, 2, \dots, M\}$ ,  $N' = \{1, 2, \dots, N\}$ ,  $J' = \{(i, j) \mid i \in M', j \in N'\}$ , the above TMTP-FBOF may be related to the following Lexicographic Fractional Time Minimizing Transportation Problem (LFTMTP) and  $\alpha_{ij} \in \mathbb{R}^h$ ,  $\beta_{ij} \in \mathbb{R}^h$ :

$$\text{lexmin} \left[ \mathfrak{F} = \frac{\sum_{(i,j) \in J'} \alpha_{ij} x_{ij}}{\sum_{(i,j) \in J'} \beta_{ij} x_{ij}} \left| \begin{array}{l} \sum_{j \in N'} x_{ij} = a_i \text{ for all } i \in M' \\ \sum_{i \in M'} x_{ij} = b_j \text{ for all } j \in N' \\ x_{ij} \geq 0, \text{ for all } (i, j) \in J' \end{array} \right. \right] \quad (5)$$

**Remark 1:** Let  $\mathbb{R}$  denote the set of the real numbers, and  $\mathbb{R}^0$  the set of the non-negative real numbers. With regard to lexicographic vector inequalities, the following convention will be applied: For  $a, b \in \mathbb{R}^h$ , the strict lexicographic inequality  $a \underset{\sim}{>} b$  holds if and only if  $a_{\bar{c}} > b_{\bar{c}}$  holds for  $\bar{c} = \min\{c \mid c = 1, 2, \dots, h, a_c \neq b_c\}$ , and the weak lexicographic inequality  $a \underset{\sim}{\geq} b$  holds if and only if  $a \underset{\sim}{>} b$  or  $a = b$ .

### 4. SOLUTION METHODOLOGY

In this section, the necessary propositions are presented on the basis of which TMTP-FBOF Algorithm is developed for determining an initial efficient basic solution to Time Minimizing Transportation Problem with Fractional Bottleneck Objective Function.

Proposition 1 holds for formulation of Lexicographic Fractional Time Minimizing Transportation Problem (LFTMTP).

**Proposition 1.** *If  $(i, j) \in \xi_c^a$ ,  $c = 1, 2, \dots, g$ , then vectors*

$$\alpha_{ij} = [e_c] \quad (6)$$

where  $[e_c]$  is an  $(g \times 1)$  unit vector. Moreover, if  $(i, j) \in \xi_d^s$ ,  $d = g + 1, g + 2, \dots, h$ , then vectors

$$\beta_{ij} = [e_d] \quad (7)$$

where  $[e_d]$  is an  $(h \times 1)$  unit vector.

**Proof:** Let the fractional bottleneck transportation time matrix be  $T = \frac{T^a}{T^s}$ , where actual transportation time matrix  $T^a = [t_{ij}^a]$  and standard transportation time matrix  $T^s = [t_{ij}^s]$ . Partition the set  $\xi^a = M \times N$  into subsets  $\xi_c^a$ , ( $c = 1, 2, \dots, g$ ) for  $T^a$ . Each of the subsets  $\xi_c^a$  consists of all  $(i, j) \in \xi^a$  for which actual transportation time ( $t_{ij}^a$ ) has the same numerical value. The subset  $\xi_1^a$  contains all  $(i, j) \in \xi^a$  with  $t_{ij}^a$  being the highest value, subset  $\xi_2^a$  contains all  $(i, j) \in \xi^a$  with  $t_{ij}^a$  being the next lower highest value, and so on. Finally subset  $\xi_g^a$  contains all  $(i, j) \in \xi^a$  with  $t_{ij}^a$  being the lowest value. By assigning an  $(g \times 1)$  unit vector  $[e_c]$  to each value of  $x_{ij}$  with  $(i, j) \in \xi_c^a$ , we obtain the vectors  $\alpha_{ij}$ .

Now partition the set  $\xi^s = M \times N$  into subsets  $\xi_d^s$ , ( $d = g + 1, g + 2, \dots, h$ ) for  $T^s$ . Each of the subsets  $\xi_d^s$  consists of all  $(i, j) \in \xi^s$  for which standard transportation time ( $t_{ij}^s$ ) has the same numerical value. The subset  $\xi_1^s$  contains all  $(i, j) \in \xi^s$  with  $t_{ij}^s$  being the highest value, subset  $\xi_2^s$  contains all  $(i, j) \in \xi^s$  with  $t_{ij}^s$  being the next lower highest value, and so on. Finally subset  $\xi_h^s$  contains all  $(i, j) \in \xi^s$  with  $t_{ij}^s$  being the lowest value. By assigning an  $(h \times 1)$  unit vector  $[e_d]$  to each value of  $x_{ij}$  with  $(i, j) \in \xi_d^s$ , we obtain the vectors  $\beta_{ij}$ .

The following propositions are used to reduce the dimension of the vectors  $\alpha_{ij}$  and  $\beta_{ij}$  of Lexicographic Fractional Time Minimizing Transportation Problem.

**Proposition 2.** Let  $\bar{\psi}_i$  and  $\bar{\gamma}_j$  be the permutations of column  $j$ ,  $j = 1, 2, \dots, N$  and row  $i$ ,  $i = 1, 2, \dots, M$  respectively of every actual transportation time  $t_{ij}^a$ , then  $\bar{r}_i = t_{i, \bar{\psi}_i[\bar{\sigma}^v]}^a$  and  $\bar{s}_j = t_{\bar{\gamma}_j[\bar{\sigma}^v], j}^a$  are the lower bounds for every row  $i$  and column  $j$  respectively.

**Proof:** For every row  $i$  of every actual transportation time  $t_{ij}^a$ , let  $\bar{\psi}_i$  be any permutation of  $\{1, 2, \dots, N\}$  such that

$$t_{i, \bar{\psi}_i[1]}^a \leq t_{i, \bar{\psi}_i[2]}^a \leq \dots \leq t_{i, \bar{\psi}_i[N]}^a.$$

Let  $\bar{D}_{i\bar{\sigma}} = \sum_{j=1}^{\bar{\sigma}} b_{\bar{\psi}_i[j]}$  and  $\bar{D}_{i\bar{\sigma}^v} = \min_{\bar{\sigma}} \left\{ \bar{D}_{i\bar{\sigma}} \mid \bar{D}_{i\bar{\sigma}} \geq a_i \right\}$ .

Then  $\bar{r}_i = t_{i, \bar{\psi}_i[\bar{\sigma}^v]}^a$  is a lower bound for  $t_{ij}^a$ , since the  $i^{th}$  supply constraints cannot be satisfied using only cells with time less than  $\bar{r}_i$ . Similarly, for each column  $j$  of every  $t_{ij}^a$  let  $\bar{\gamma}_j$  be a permutation of  $\{1, 2, \dots, M\}$  such that

$$t_{\bar{y}_j[1],j}^a \leq t_{\bar{y}_j[2],j}^a \leq \dots \leq t_{\bar{y}_j[M],j}^a.$$

Let  $\bar{D}_{\bar{o}j} = \sum_{i=1}^{\bar{o}} a_{\bar{y}_j[i]}$  and  $\bar{D}_{\bar{o}^v j} = \min\{\bar{D}_{\bar{o}j} \mid \bar{D}_{\bar{o}j} \geq b_j\}$ .

Then  $\bar{s}_j = t_{\bar{y}_j[\bar{o}^v],j}^a$  is a lower bound for  $t_{ij}^a$ .

**Proposition 3.** For every standard transportation time  $t_{ij}^s$ , let  $\bar{\psi}_i$  be the permutation of  $\{1, 2, \dots, N\}$  and  $\bar{\gamma}_j$  be the permutation of  $\{1, 2, \dots, M\}$ , then  $\bar{r}_i = t_{i, \bar{\psi}_i[\bar{o}^v]}^s$  is a lower bound for every row  $i$  and  $\bar{s}_j = t_{\bar{\gamma}_j[\bar{o}^v], j}^s$  is a lower bound for every column  $j$ .

*Proof.* For every row  $i$ ,  $i = \{1, 2, \dots, M\}$  of every standard transportation time  $t_{ij}^s$ , let  $\bar{\psi}_i$  be any permutation of column  $j$ ,  $j = \{1, 2, \dots, N\}$  such that

$$t_{i, \bar{\psi}_i[1]}^s \leq t_{i, \bar{\psi}_i[2]}^s \leq \dots \leq t_{i, \bar{\psi}_i[N]}^s.$$

Let  $\bar{D}_{i\bar{o}} = \sum_{j=1}^{\bar{o}} b_{\bar{\psi}_i[j]}$  and  $\bar{D}_{i\bar{o}^v} = \min\{\bar{D}_{i\bar{o}} \mid \bar{D}_{i\bar{o}} \geq a_i\}$ .

Then  $\bar{r}_i = t_{i, \bar{\psi}_i[\bar{o}^v]}^s$  is a lower bound for  $t_{ij}^s$ , since the  $i^{\text{th}}$  supply constraints cannot be satisfied using only cells with time less than  $\bar{r}_i$ . Similarly, for each column  $j$ ,  $j = \{1, 2, \dots, N\}$  of every  $t_{ij}^s$  let  $\bar{\gamma}_j$  be a permutation of row  $i$ ,  $i = \{1, 2, \dots, M\}$  such that

$$t_{\bar{\gamma}_j[1],j}^s \leq t_{\bar{\gamma}_j[2],j}^s \leq \dots \leq t_{\bar{\gamma}_j[M],j}^s.$$

Let  $\bar{D}_{\bar{o}j} = \sum_{i=1}^{\bar{o}} a_{\bar{\gamma}_j[i]}$  and  $\bar{D}_{\bar{o}^v j} = \min\{\bar{D}_{\bar{o}j} \mid \bar{D}_{\bar{o}j} \geq b_j\}$ .

Then  $\bar{s}_j = t_{\bar{\gamma}_j[\bar{o}^v], j}^s$  is a lower bound for  $t_{ij}^s$ .

## 5. TMTP-FBOF ALGORITHM

The TMTP-FBOF Algorithm for solving the Time Minimizing Transportation Problem with Fractional Bottleneck Objective Function generates a finite sequence of basic feasible solutions to Lexicographic Fractional Time Minimizing Transportation Problem (LFTMTP). The optimal basic solution to LFTMTP provides a feasible transportation schedule that minimizes the fractional bottleneck transportation time as well as its total distribution. The steps of the TMTP-FBOF Algorithm are:

Step 1: Determine the row threshold  $\bar{r}_i$  and column threshold  $\bar{s}_j$  for actual transportation time matrix  $T^a$ , where  $T^a = [t_{ij}^a]$ , using Proposition 2 and

calculate the best lower bound  $t_l^a = \max (\bar{r}_1, \bar{r}_2, \dots, \bar{r}_M; \bar{s}_1, \bar{s}_2, \dots, \bar{s}_N)$  for the actual transportation time  $t_{ij}^a$ .

- Step 2: Calculate row threshold  $\bar{r}_i$  and column threshold  $\bar{s}_j$  for standard transportation time matrix  $T^s$ , where  $T^s = [t_{ij}^s]$ , using Proposition 3 and determine the best lower bound  $t_l^s = \max (\bar{r}_1, \bar{r}_2, \dots, \bar{r}_M; \bar{s}_1, \bar{s}_2, \dots, \bar{s}_N)$  for standard transportation time  $t_{ij}^s$ .
- Step 3: Determine an initial basic feasible solution to LFTMTP by North-West Corner Rule.
- Step 4: Determine an upper bound  $t_U^a$  by selecting the highest actual transportation time from the resulting actual transportation time  $t_{ij}^a$  of the initial basic feasible solution.
- Step 5: Determine an upper bound  $t_U^s$  by selecting the highest standard transportation time from the resulting standard transportation time  $t_{ij}^s$  of the initial basic feasible solution.
- Step 6: Partition the set  $\xi^a = M \times N$  into subsets  $\xi_c^a$ , ( $c = 1, 2, \dots, g$ ) for  $T^a$ . Each of the subsets  $\xi_c^a$  consists of all  $(i, j) \in \xi^a$  for which actual transportation time ( $t_{ij}^a$ ) has the same numerical value. The subset  $\xi_1^a$  contains all  $(i, j) \in \xi^a$  with  $t_{ij}^a$  being the highest value, subset  $\xi_2^a$  contains all  $(i, j) \in \xi^a$  with  $t_{ij}^a$  being the next lower highest value, and so on. Finally subset  $\xi_g^a$  contains all  $(i, j) \in \xi^a$  with  $t_{ij}^a$  being the lowest value.
- Step 7: Partition the set  $\xi^s = M \times N$  into subsets  $\xi_d^s$ , ( $d = g+1, g+2, \dots, h$ ) for  $T^s$ . Each of the subsets  $\xi_d^s$  consists of all  $(i, j) \in \xi^s$  for which standard transportation time ( $t_{ij}^s$ ) has the same numerical value. The subset  $\xi_1^s$  contains all  $(i, j) \in \xi^s$  with  $t_{ij}^s$  being the highest value, subset  $\xi_2^s$  contains all  $(i, j) \in \xi^s$  with  $t_{ij}^s$  being the next lower highest value, and so on. Finally subset  $\xi_h^s$  contains all  $(i, j) \in \xi^s$  with  $t_{ij}^s$  being the lowest value.
- Step 8: Now the best lower bound  $t_l^a$  is greater than the actual transportation time  $t_{ij}^a$  for at least one pair of  $\xi^a$ , and the upper bound  $t_U^a$  is less than the actual transportation time  $t_{ij}^a$  for at least one pair of  $\xi^a$  which give the following subsets:

$$\xi_1^a = \{(i, j) \in \xi^a \mid t_{ij}^a > t_U^a\}, \xi_2^a = \{(i, j) \in \xi^a \mid t_{ij}^a = t_U^a\},$$

$$\xi_3^a = \{(i, j) \in \xi^a \mid t_{ij}^a = t_l^a\}, \xi_4^a = \{(i, j) \in \xi^a \mid t_{ij}^a < t_l^a\}$$

Step 9: Let the best lower bound  $t_l^s$  is greater than the standard transportation time  $t_{ij}^s$  for at least one pair of  $\xi^s$  and the upper bound  $t_U^s$  is less than the standard transportation time  $t_{ij}^s$  for at least one pair of  $\xi^s$  which give the following subsets:

$$\xi_5^s = \{(i, j) \in \xi^s \mid t_{ij}^s > t_U^s\}, \xi_6^s = \{(i, j) \in \xi^s \mid t_{ij}^s = t_U^s\},$$

$$\xi_7^s = \{(i, j) \in \xi^s \mid t_{ij}^s = t_l^s\}, \xi_8^s = \{(i, j) \in \xi^s \mid t_{ij}^s < t_l^s\}$$

Step 10: Determine the vectors  $\alpha_{ij}$  and  $\beta_{ij}$ ,  $(i, j) \in \xi_c^a$  and  $\xi_d^s$  respectively ( $c = 1, \dots, g$ ;  $d = g + 1, \dots, h$ ) such that

$$\alpha_{ij} = [e_c], \beta_{ij} = [e_d]$$

by using the Proposition 1 to obtain the fractional bottleneck transportation time matrix  $T = \begin{bmatrix} \alpha_{ij} \\ \beta_{ij} \end{bmatrix}$ .

Step 11: Designate the set of pairs of indices  $(i, j)$  of the basic variable by  $I$  and using initial basic feasible solution, determine recursively the vector-valued row multipliers  $u_i^1, u_i^2$  and the vector-valued column multipliers  $v_j^1, v_j^2$  defined such that:

$$[\alpha_{ij} - (u_i^1 + v_j^1)] = 0 \quad (8)$$

$$[\beta_{ij} - (u_i^2 + v_j^2)] = 0 \quad (9)$$

for those  $i, j$  for which  $x_{ij}$  is in the basis.

Step 12: Let  $\tilde{U} = (\tilde{u}_i^1, \tilde{u}_i^2, i \in M'; \tilde{v}_j^1, \tilde{v}_j^2, j \in N')$  be the solution of (8) and (9). Compute the relative criterion vectors  $\Delta_{ij}$  by using the following equation set:

$$\Delta_{ij} = [V_2 \alpha'_{ij} - V_1 \beta'_{ij}] \quad (10)$$

where

$$\alpha'_{ij} = [\alpha_{ij} - (\tilde{u}_i^1 + \tilde{v}_j^1)] \quad (11)$$

$$\beta'_{ij} = [\beta_{ij} - (\tilde{u}_i^2 + \tilde{v}_j^2)] \quad (12)$$

$$V_1 = \sum_{i \in M'} \tilde{u}_i^1 a_i + \sum_{j \in N'} \tilde{v}_j^1 b_j \quad (13)$$

$$V_2 = \sum_{i \in M'} \tilde{u}_i^2 a_i + \sum_{j \in N'} \tilde{v}_j^2 b_j \quad (14)$$

for all  $(i, j) \in J' \setminus I$

Step 13: If all  $\Delta_{ij}$  are lexicographically greater than or equal to the zero vector for all  $(i, j) \in J' \setminus I$ , the current basic feasible solution is optimal to the LFTMTP and go to Step 15. Otherwise go to Step 14. The lexicographic order is obtained by following the general convention of lexicographic method.

Step 14: Select

$$\Delta_{i_s, j_s} = \text{lexmin} \left\{ \Delta_{ij} \mid \Delta_{ij} \prec 0 \right\} \quad (15)$$

for all  $(i, j) \in J' \setminus I$  and determine the variable  $x_{i_s, j_s}$  which is to be the enter.

Now the variable  $x_{i_s, j_s}$  becomes a basic variable of the new basic feasible solution. Change the current basic feasible solution to the new basic feasible solution using the standard transportation method and go to Step 11.

Step 15: If  $\tilde{X} = (\tilde{x}_{ij})$  is optimal transportation schedule for LFTMTP denoted by

$$\text{equation (5), then } \tilde{\mathfrak{z}} = \frac{\sum_{(i,j) \in J'} \alpha_{ij} \tilde{x}_{ij}}{\sum_{(i,j) \in J'} \beta_{ij} \tilde{x}_{ij}} \text{ and } \tilde{c}/\tilde{d} \text{ is the index of the first positive}$$

component of the optimal flow vector  $\tilde{\mathfrak{z}}$  or  $\tilde{\mathfrak{z}}(\tilde{X})$ . The vector of partial flows is minimized in a lexicographic sense on the feasible set, i.e. the optimal flow vector has the property that no other feasible transportation schedule exists such that the flow vector is lexicographically smaller than the optimal one. The optimal flow vector  $\tilde{\mathfrak{z}}(\tilde{X})$  immediately specified the minimal fractional bottleneck transportation time and the minimal flow which requires the optimal fractional bottleneck transportation time. Then  $\tilde{t} = \frac{\tilde{t}_{ij}^a}{\tilde{t}_{ij}^s}$  with  $(i, j) \in \xi_{\tilde{c}}^a / \xi_{\tilde{d}}^s$  is

the minimal value of the fractional bottleneck objective function in equation (1) and is the optimal fractional bottleneck transportation time. The optimal transportation schedule  $\tilde{X} = (\tilde{x}_{ij})$  of TMTP-FBOF also minimizes the function

$$\frac{\tilde{\mathfrak{z}}_{\tilde{c}}}{\tilde{\mathfrak{z}}_{\tilde{d}}} = \sum_{(i,j) \in J'} (x_{ij}), \text{ (summing overall } (i, j) \in \xi_{\tilde{c}}^a / \xi_{\tilde{d}}^s \text{), which represents the total}$$

distribution that requires the fractional bottleneck transportation time  $\tilde{t}$ .

The TMTP-FBOF Algorithm starts with an initial basic feasible solution to LFTMTP and generates a finite sequence of basic feasible solutions until an optimal basic solution has been determined.

## 6. MILITARY TRANSPORTATION PROBLEM OF INDIAN ARMY

The TMTP-FBOF Algorithm will be illustrated with the help of the following example of Military Transportation Problem of Indian Army:

The different locations of J & K Border receive a fixed quantity of military units with arms, ammunitions, food and etc. which can be deputed from four regiments ( $i$ ) available at Army Headquarter Pathankot. Indian Army used to depute different type of regiments on four crucial locations ( $j$ ) - Kargil, NEFA, Baramula and Uri sector of J & K Border. The goal is to determine the feasible transportation schedule which minimizes the maximum Fractional Bottleneck Loading-unloading Transportation Time (Total Actual Loading-unloading Transportation Time/Total Standard Loading-unloading Transportation Time) in transporting the military units with arms, ammunitions and food etc. during emergency situations. Table 1 shows the Fractional Bottleneck Loading-unloading Transportation Time (in hours) from regiments  $i$  to locations  $j$ . Availabilities  $a_i$  are shown in the last column, while requirements  $b_j$  are shown in the last row.

Let  $x_{ij}$  be the quantity of military units with arms, ammunitions and food etc. sent from regiments  $i$  to locations  $j$ . Then it is required to

$$\min t = \max_{(i,j)} \left\{ \frac{t_{ij}^a}{t_{ij}^s} \mid x_{ij} > 0 \right\}$$

subject to

$$\sum_{j=1}^4 x_{ij} = a_i \quad (i = 1, 2, \dots, 4)$$

$$\sum_{i=1}^4 x_{ij} = b_j \quad (j = 1, 2, \dots, 4)$$

$$x_{ij} \geq 0 \quad (i = 1, 2, \dots, 4; j = 1, 2, \dots, 4)$$

**Table 1:** Fractional Bottleneck Loading-unloading Transportation Time (in hours) for Military Transportation Problem

Regiments $i$	Locations $j$				$a_i$
	1	2	3	4	
1	$\left[ \frac{4:40}{3:20} \right]$	$\left[ \frac{4:50}{3:30} \right]$	$\left[ \frac{4:20}{3:40} \right]$	$\left[ \frac{4:45}{3:50} \right]$	7
2	$\left[ \frac{4:55}{3:30} \right]$	$\left[ \frac{4:35}{3:35} \right]$	$\left[ \frac{4:45}{3:50} \right]$	$\left[ \frac{4:00}{3:00} \right]$	1
3	$\left[ \frac{5:00}{3:30} \right]$	$\left[ \frac{4:45}{3:40} \right]$	$\left[ \frac{4:30}{3:20} \right]$	$\left[ \frac{4:50}{3:40} \right]$	8
4	$\left[ \frac{4:40}{3:45} \right]$	$\left[ \frac{4:50}{3:20} \right]$	$\left[ \frac{4:20}{3:10} \right]$	$\left[ \frac{4:45}{3:35} \right]$	4
$b_j$	5	6	3	6	

**6.1. Computational Procedure**

The lower bound for the total actual loading-unloading transportation time matrix  $T^a$  is obtained by calculating row thresholds (4:40, 4:00, 4:45, 4:40) and column thresholds (4:40, 4:45, 4:20, 4:45), which gives  $t_l^a = 4:45$ .

Similarly, for the total standard loading-unloading transportation time matrix  $T^s$ , the lower bound is  $t_l^s = 3:35$ .

The initial basic feasible solution  $X^1$  of Military Transportation Problem is:

$$x_{11} = 5, x_{12} = 2, x_{22} = 1, x_{32} = 3, x_{33} = 3, x_{34} = 2, x_{44} = 4.$$

with the resulting Total Actual Loading-unloading Transportation Time 4:50, which gives the upper bounds  $t_U^a = 4:50$  and resulting Total Standard Loading-unloading Transportation Time 3:40, which gives the upper bounds  $t_U^s = 3:40$ .

Hence  $g = 4$  and  $h = 4$ , so  $\xi^a$  and  $\xi^s$  has four subsets:

$$\begin{aligned} \xi_1^a &= \{(i, j) \in \xi^a \mid t_{ij}^a > 4:50\}, & \xi_2^a &= \{(i, j) \in \xi^a \mid t_{ij}^a = 4:50\}, \\ \xi_3^a &= \{(i, j) \in \xi^a \mid t_{ij}^a = 4:45\}, & \xi_4^a &= \{(i, j) \in \xi^a \mid t_{ij}^a < 4:45\} \end{aligned}$$

and

$$\begin{aligned} \xi_5^s &= \{(i, j) \in \xi^s \mid t_{ij}^s > 3:40\}, & \xi_6^s &= \{(i, j) \in \xi^s \mid t_{ij}^s = 3:40\}, \\ \xi_7^s &= \{(i, j) \in \xi^s \mid t_{ij}^s = 3:35\}, & \xi_8^s &= \{(i, j) \in \xi^s \mid t_{ij}^s < 3:35\} \end{aligned}$$

The related Lexicographic Fractional Time Minimizing Military Transportation Problem is:

$$\text{lexmin } \left[ \begin{array}{l} \sum_{j=1}^4 x_{ij} = a_i, \quad (i=1,2,\dots,4) \\ \sum_{i=1}^4 \sum_{j=1}^4 \alpha_{ij} x_{ij} \\ \sum_{i=1}^4 \sum_{j=1}^4 \beta_{ij} x_{ij} \\ \sum_{i=1}^4 x_{ij} = b_j, \quad (j=1,2,\dots,4) \\ x_{ij} \geq 0 \end{array} \right]$$

and the Fractional Bottleneck Loading-unloading Transportation Time Matrix  $T$  can be written as:

$$T = \begin{bmatrix} \begin{bmatrix} e_4 \\ e_8 \end{bmatrix} & \begin{bmatrix} e_2 \\ e_8 \end{bmatrix} & \begin{bmatrix} e_4 \\ e_6 \end{bmatrix} & \begin{bmatrix} e_3 \\ e_5 \end{bmatrix} \\ \begin{bmatrix} e_1 \\ e_8 \end{bmatrix} & \begin{bmatrix} e_4 \\ e_7 \end{bmatrix} & \begin{bmatrix} e_3 \\ e_5 \end{bmatrix} & \begin{bmatrix} e_4 \\ e_8 \end{bmatrix} \\ \begin{bmatrix} e_1 \\ e_8 \end{bmatrix} & \begin{bmatrix} e_3 \\ e_6 \end{bmatrix} & \begin{bmatrix} e_4 \\ e_8 \end{bmatrix} & \begin{bmatrix} e_2 \\ e_6 \end{bmatrix} \\ \begin{bmatrix} e_4 \\ e_5 \end{bmatrix} & \begin{bmatrix} e_2 \\ e_8 \end{bmatrix} & \begin{bmatrix} e_4 \\ e_8 \end{bmatrix} & \begin{bmatrix} e_3 \\ e_7 \end{bmatrix} \end{bmatrix}$$

Using the initial basic feasible solution  $X^1$ , the associated vector-valued row multipliers  $u_i^1, u_i^2$ , ( $i = 1, 2, \dots, 4$ ) and associated vector-valued column multipliers  $v_j^1, v_j^2$ , ( $j = 1, 2, \dots, 4$ ) are calculated as explained in Step 11.

An arbitrary value of zero is assigned to  $u_i^1 = 0$  and  $u_i^2 = 0$ .

Since  $x_{11}$  is in the basis, so

$$\begin{array}{llll} & \alpha_{11} = u_1^1 + v_1^1 & \text{and} & \beta_{11} = u_1^2 + v_1^2 \\ \text{giving} & v_1^1 = e_4 & \text{and} & v_1^2 = e_8 \end{array}$$

As  $x_{12}$  is in the basis, therefore

$$\begin{array}{llll} & \alpha_{12} = u_1^1 + v_2^1 & \text{and} & \beta_{12} = u_1^2 + v_2^2 \\ \text{which gives} & v_2^1 = e_2 & \text{and} & v_2^2 = e_8 \end{array}$$

Similarly  $x_{22}$  is in the basis, therefore

$$u_2^1 = -e_2 + e_4 \quad \text{and} \quad u_2^2 = e_7 - e_8$$

Now  $x_{32}$  is in the basis, leading to

$$u_3^1 = -e_2 + e_3 \quad \text{and} \quad u_3^2 = e_6 - e_8$$

As  $x_{33}$  and  $x_{34}$  are in the basis,

$$\begin{array}{llll} \text{Giving} & v_3^1 = e_2 - e_3 + e_4 & \text{and} & v_3^2 = -e_6 + 2e_8 \\ & v_4^1 = 2e_2 - e_3 & \text{and} & v_4^2 = e_8 \end{array}$$

Also  $x_{44}$  is in the basis, therefore

$$u_4^1 = -2e_2 + 2e_3 \quad \text{and} \quad u_4^2 = e_7 - e_8$$

Once the vector-valued row and column multipliers are determined as above, the values of relative criterion vectors  $\Delta_{ij}$  are obtained from the equation (10).

Table 2 shows the initial basic feasible solution  $X^1$  of Military Transportation Problem. The amount  $x_{ij}$  is shown in the upper right hand side of the cell and  $\alpha_{ij}$  is displayed as numerator and  $\beta_{ij}$  as denominator in the upper left hand side of the cell. The last column contains the vector-valued multipliers  $u_i^1$  and  $u_i^2$ , while the bottom row contains  $v_j^1$  and  $v_j^2$ . For all  $(i, j) \in J' \setminus I$ , the lower side of the cell contains  $\Delta_{ij}$ , if  $\Delta_{ij}$  are lexicographically smaller than or equal to zero vectors.  $b_j$  is displayed in the top row of the table, while  $a_i$  in the first column. The flow vector  $\mathfrak{Z}(X^1) = (0, 2, 0, 0, 0, 0, 8, 4, 2, 0, 1, 3, 0, 0, 0, 0)^T$  indicates the Fractional Bottleneck Loading-unloading Transportation Time = 1.380 and Bottleneck Flow = 2. As  $X^1$  is not optimal, therefore using the standard transportation method, variable  $x_{14}$  becomes an entering basic variable.

**Table 2:** Initial basic feasible solution  $X^1$  of Military Transportation Problem

$b_j$ $a_i$	5		6		3		6		$[u_i^1][u_i^2]$
7	$\frac{e_4}{e_8}$	5	$\frac{e_2}{e_8}$	2	$\frac{e_4}{e_6}$		$\frac{e_3}{e_5}$		$[0]$ $[0]$
					[0,0,0,0,-13,0,0,-9,-5,0,-18,5,-2,0,24,18]		[0,-4,0,-7,-10,0,-9,10,-10,0,0,10,-16,0,27,9]		
1	$\frac{e_1}{e_8}$		$\frac{e_4}{e_7}$	1	$\frac{e_3}{e_5}$		$\frac{e_4}{e_8}$		$[-e_2 + e_4]$ $[e_7 - e_8]$
	[0,0,5,0,5,5,0,0,9,10,-10,7,6,-1,-7,-29]				[0,-4,0,-7,-4,0,-9,3,4,0,-19,17,4,-1,27,-11]		[0,0,0,0,-5,0,0,5,-1,0,0,12,-14,9,3,-9]		
8	$\frac{e_1}{e_8}$		$\frac{e_3}{e_6}$	3	$\frac{e_4}{e_8}$	3	$\frac{e_2}{e_6}$	2	$[-e_2 + e_3]$ $[e_6 - e_8]$
	[0,0,5,0,9,5,0,2,5,10,4,-5,6,-5,-17,-19]								
4	$\frac{e_4}{e_3}$		$\frac{e_2}{e_8}$		$\frac{e_4}{e_8}$		$\frac{e_3}{e_7}$	4	$[-2e_2 + 2e_3]$ $[e_7 - e_8]$
	[0,-4,0,-7,10,0,-9,-10,14,0,0,-3,20,9,-20,0]		[0,0,0,0,10,0,0,-10,14,0,0,-3,16,9,-27,-9]		[0,0,0,0,1,0,0,-12,9,0,-9,2,10,9,-10,0]				
$[v_j^1]$ $[v_j^2]$	$[e_4]$ $[e_8]$		$[e_2]$ $[e_8]$		$[e_2 - e_3 + e_4]$ $[-e_6 + 2e_8]$		$[2e_2 - e_3]$ $[e_8]$		$\mathfrak{Z}(X^1) =$ $[0,2,0,0,0,0,8,$ $4,2,0,1,3,0,0,$ $0,0]$

The new basic feasible solution  $X^2$  together with the values of the vector-valued multipliers and the relative criterion vectors are displayed in Table 3. The flow vector  $\mathfrak{Z}(X^2) = (0,0,0,0,0,0,8,4,0,0,1,5,0,0,2,0)^T$  indicates the Fractional Bottleneck Loading-unloading Transportation Time = 1.334 and Bottleneck Flow = 8. However,  $X^2$  does not satisfy the optimality conditions. Proceeding in the same manner described above, Table 4 shows optimal solution  $\tilde{X}^3$  together with the values of the vector-valued multipliers and relative criterion vectors. The current basic feasible solution  $X^3$  is the optimal for Lexicographic Fractional Time Minimizing Military Transportation Problem. The optimal value of the flow vector is  $\mathfrak{Z}(\tilde{X}^3) = (0,0,0,0,0,0,4,0,0,0,1,5,0,0,6,4)^T$ . Thus the optimal Fractional Bottleneck Loading-unloading Transportation Time is  $\tilde{t} = 1.334$ , and the optimal Bottleneck Flow = 4.

### 7. CONCLUSION

The algorithm helps the Transportation System Decision Maker in determining all efficient transportation schedules with respect to the minimization of non-linear time function and the distribution that requires the fractional time. The developed algorithm solves fractional time transportation problems. The algorithm offers a more universal apparatus for a wider class of real life decision priority problems.

**Table 3:** Solution  $X^2$  of Military Transportation Problem

$b_j$ →	5		6		3		6		$[u_i^1][u_i^2]$
$a_i$ ↓									
7	$\frac{e_4}{e_8}$	5	$\frac{e_2}{e_8}$		$\frac{e_4}{e_6}$		$\frac{e_3}{e_5}$	2	[0] [0]
			[0,2,0,9,5,0,9, -5,5,0, 0,6,8, 9,-30,-18]		[0,0,0,11,0,0,9, -22,0,0,-18,11,0, 9,0,0]				
1	$\frac{e_1}{e_8}$		$\frac{e_4}{e_7}$	1	$\frac{e_3}{e_5}$		$\frac{e_4}{e_8}$	0	$[-e_3 + e_4]$ $[-e_5 + e_8]$
	[2,0,5,-9,0,5,-13, 5,0,8,-10,5,0, -10,19,-7]				[0,0,0,-7,0,0,-13, -1,0,0,-19,21,0, -1,27,-7]				
8	$\frac{e_1}{e_8}$		$\frac{e_3}{e_6}$	5	$\frac{e_4}{e_8}$	3	$\frac{e_2}{e_6}$		[0] $[-e_5 + e_6]$ $-e_7 + e_8]$
	[2,0,5,-11,0,5, -11,11,0,8,4,-11, 0,-14,11,1]						[0,2,0,-2,5,0, 0,-5,5,0,0,-16, 8,-9,3,9]		
4	$\frac{e_4}{e_5}$		$\frac{e_2}{e_8}$		$\frac{e_4}{e_8}$		$\frac{e_3}{e_7}$	4	[0] $[-e_5 + e_7]$
	[0,0,0,-22,0,0, -18,0,0,0,0,11, 0,9,11,9]		[0,2,0,-2,5,0, 0,-5,5,0,0,17, 8,18,-30,-18]		[0,0,0,0,0,0,0, -11,0,0,-9,22,0, 18,-11,-9]				
$[v_j^1]$	$[e_4]$		$[e_3]$		$[e_4]$		$[e_3]$		$\mathfrak{S}(X^2) =$ [0,0,0,0,0,0,8, 4,0,0,1,5,0,0, 2,0]
$[v_j^2]$	$[e_8]$		$[e_5 + e_7 - e_8]$		$[e_5 - e_6 + e_7]$		$[e_5]$		

**Table 4:** Optimal Solution  $\tilde{X}^3$  of Military Transportation Problem

$b_j$	5	6	3	6	$[u_i^1][u_i^2]$
$a_i$					
7	$\frac{e_4}{e_8}$ 1	$\frac{e_2}{e_8}$	$\frac{e_4}{e_6}$	$\frac{e_3}{e_5}$ 6	[0] [0]
		[0,10,0,1,5,0,9, -5,1,0,0,10,4, 9,-26,-18]	[0,0,0,11,0,0,9, -22,0,0,-18,11,0, 9,0,0]		
1	$\frac{e_1}{e_8}$	$\frac{e_4}{e_7}$ 1	$\frac{e_3}{e_5}$	$\frac{e_4}{e_8}$ 0	$[-e_3 + e_4]$ $[-e_5 + e_8]$
	[10,0,5,-1,0,1, -29,5,0,4,- 10,1,0, -2,15,1]		[0,0,0,9,0,0,-29, -1,0,0,-19,13,0, 7,19,1]		
8	$\frac{e_1}{e_8}$	$\frac{e_3}{e_6}$ 5	$\frac{e_4}{e_8}$ 3	$\frac{e_2}{e_6}$	[0] $[-e_5 + e_6]$ $-e_7 + e_8]$
	[10,0,5,- 11,0,1, -19,11,0,4,4,- 11, 0,-10,11,5]			[0,10,0,-10,5, 0,0,-5,1,0,0, -12,4,-9,7,9]	
4	$\frac{e_4}{e_5}$ 4	$\frac{e_2}{e_8}$	$\frac{e_4}{e_8}$	$\frac{e_3}{e_7}$	[0] $[e_5 - e_8]$
		[0,10,0,12,5,0, 18,-5,1,0,0,10, 4,9,-37,-27]	[0,0,0,22,0,0,18, -11,0,0,- 9,11,0,9, -22,-18]	[0,0,0,22,0,0, 18,0,0,0,0,-11, 0,-9,-11,-9]	
$[v_j^1]$	$[e_4]$	$[e_3]$	$[e_4]$	$[e_3]$	$\tilde{z}(\tilde{X}^3) =$
$[v_j^2]$	$[e_8]$	$[e_5 + e_7 - e_8]$	$[e_5 - e_6 + e_7]$	$[e_5]$	[0,0,0,0,0,0, 4,0,0,0,1,5,0,0,6,4]

**REFERENCES**

[1] Achary, K.K., and Seshan, C.R., "A time minimizing transportation problem with quantity dependent time", *European Journal of Operational Research*, 7 (1981) 290–298.

[2] Corban, A., "Programming with fractional linear objective function", *Revue Roumaine de Mathematiques Pures et Appliquees*, 18 (1973) 633–637.

[3] Gupta, A., Khanna, S., and Puri, M.C., "A paradox in linear fractional transportation problems with mixed constraints", *Optimization*, 27 (4) (1993) 375–387.

[4] Seshan, C.R., and Tikekar, V.G., "On Sharma-Swarup algorithm for time minimizing transportation problems", *Proc. Indian Acad. Sci.*, 89 (2) (1980) 101–102.

[5] Sharma, J.K., and Swarup, K., "Transportation fractional programming with respect to time", *Ricerca Operativa*, 7 (1978) 49–58.

[6] Sonia, and Puri, M.C., "Two level hierarchical time minimizing transportation problem", *TOP*, 12(2) (2004) 301–330.