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OPTIMAL INVENTORY POLICIES FOR IMPERFECT INVENTORY WITH PRICE DEPENDENT STOCHASTIC DEMAND AND PARTIALLY BACKLOGGED SHORTAGES

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Abstract: The paper investigates a single period imperfect inventory model with price dependent stochastic demand and partial backlogging. The backorder rate is a nonlinear non-increasing function of the magnitude of shortage. Two special cases are considered assuming that the percentage of defective items follows a truncated exponential distribution and a normal distribution respectively. The optimal order quantity and the optimal mark up value are determined such that the expected total profit of the system is maximized. Numerical example is given to illustrate the proposed model which is compared with the traditional model of perfect stock. Sensitivity analysis is performed to explain the behavior of the proposed model with respect to the key parameters.

Keywords: Imperfect inventory, price-dependent stochastic demand, random defective units, partial backlogging.

MSC: 90B05.

1. INTRODUCTION

The traditional inventory models generally assume that an ordered lot contains all perfect items and hence, the possibility of shortage due to imperfect items in the accepted lot is ignored. But in reality, this concept is not fully acceptable in view of the extensive use of acceptance sampling in the quality control process in today's business and industry scenario. Ignoring this possibility may cause shortage during the selling season, and consequently increase the operating costs of the inventory system apart from loss of sales and customer goodwill in the highly competitive market. The objective of the paper is to analyze the optimal ordering and pricing policies for the retailer such that the expected total profit of the system is maximized.

Shih [33] analyzed two inventory models, a deterministic EOQ model, and a single period stochastic inventory model assuming that the ordered lot contains a random proportion of defective items. He developed optimal solutions to the modified systems and compared them numerically with the traditional models. Moinzadeh and Lee [21] investigated the effect of defective items on the order quantity and reorder point of a continuous-review inventory model with Poisson demand and constant lead time. Paknejad et al. [25] considered a random number of defective units in the ordered lot in a continuous review system (s, Q) with stochastic demand and constant lead time. They developed explicit results for the cases of exponential and uniform demand during lead time assuming that the number of defective items in a lot follows a binomial process. Affisco et al. [3] also investigated the effect of lower set up cost on the operating characteristics of the model. Porteus [29] analyzed the process of quality improvement and set up cost reduction, and determined the optimal lot size for an inventory model in his paper. Rosenblatt and Lee [31] developed a production inventory model with imperfect production process. Lin [20] presented a stochastic periodic review integrated inventory model involving defective items, backorder price discount, and variable lead time. Panda et al. [28] developed a single period inventory model with imperfect production and random demand under chance and imprecise constraints. Wee et al. [37] analyzed an optimal inventory model for defective items and shortage backordering. Chang et al. [6] studied Wee's [37] model to include the well known renewal-reward theorem and derived closed form solutions for the optimal lot size, backorder quantity and the maximum expected net profit. Hu et al. [14] investigated a two-echelon supply chain system with one retailer and one manufacturer for perishable products. They proposed two fuzzy random models for the newsboy problem with imperfect items in the centralized and decentralized systems. They used expectation theory and signed distance to transform the two fuzzy random models to the crisp ones. They showed that manufacturer's repurchase strategy can increase the whole supply chain profit. Nasri et al. [22] considered a basic EOQ model that allows stock out and backordering assuming random number of defective items. They gave closed form expressions for the cases when the proportion of defectives follows uniform and exponential distributions. Paknejad et al. [26] adjusted the EOQ model with planned shortages and quality factor. Nasri et al. [23] developed an EMQ model with planned shortage and random defective units. Cheng [10] discussed an EPQ model with process capability and quality assurance considerations. Goyal et al. [13] surveyed integrated production and quality control policies for EPQ inventory models. They provided closed form expressions when the proportion of defective units in a lot follows a one-sided truncated exponential distribution. In two recent papers Nasri et al. [22, 27] studied the relationship between order quantity and quality for processes that have not yet achieved the state of statistical control.

Khouza [17] gave a note on the single period newsboy problem with an emergency supply option. He did an extensive literature survey on the single period news-vendor problem in his paper [18], and suggested directions for future research.

Geunes et al. [12] considered an infinite horizon inventory system in the newsvendor model. In the present competitive market, the selling price of a product is one of the most important decisive factors to the buyers. Generally, higher selling price of a product negates the demand, and reasonable and low selling price increases the demand of the product. Whitin [39] first developed an inventory model with price-dependent demand. Chao et al. [9] discussed joint replenishment and pricing decisions in inventory systems with stochastically dependent supply capacity. They analyzed a single period periodic review system with price dependent stochastic demand. Recently Qin et al. [30] reviewed the newsvendor problem and provided directions for future research.

In many of the articles discussed in literature either shortages are not allowed, or if occur, they are considered to be completely backlogged. However, in today's highly competitive market providing varieties of products to the consumers due to globalization, partial backorder is a more realistic one. For fashionable items and high-tech products with short product life cycle, the willingness of a customer to wait for backlogging during the shortage period decreases with the waiting time. During the stock-out period, the backorder rate is generally considered as a non-increasing linear function of backorder replenishment lead time through the amount of shortages. The larger the expected shortage quantity is, the smaller the backorder rate would be. The remaining fraction of shortage is lost. This type of backlogging is called time-dependent partial backlogging. Abad discussed many pioneering and inspiring backlogging rates as functions of waiting time. Abad [1] developed an optimal pricing and lot-sizing inventory model for a reseller considering selling price dependent demand. Abad [2] formulated optimal lot sizing policies for perishable goods in a finite production inventory model with partial backlogging and lost sales. Liao et al. [19] investigated a distribution-free newsvendor model with balking and lost sales penalty. Zhou et al. [41] analyzed manufacturer-buyer co-ordination in an inventory system for newsvendor type products with two ordering opportunities and partial backorders. They developed a newsvendor type co-ordination model for a single-manufacturer single buyer channel with two ordering opportunities. The excessive demand after the first order is partially backlogged and both parties share the manufacturing setup cost of the second order (if occurs). It was showed that the decentralized system would perform best if the manufacturer covers utterly the second production setup cost, opposite to what was shown by Weng et al. [38]. They extended the model of Weng et al. [38] in the sense that the second order decision is made by the buyer based on the channel's benefit rather than only on the buyer's benefit. Chang et al. [7] investigated a partial backlogging inventory model for non-instantaneous deteriorating items. They assumed that the demand of the items are stock dependent, and proposed a mathematical model and a theorem to find minimum total relevant cost and optimal order quantity of the model under inflation. Chang et al. [8] deal with the optimal pricing and ordering policies for a deteriorating inventory model with limited shelf space. They considered that the demand of an item is dependent on the on-display stock level and the selling price per unit. They extended the traditional EOQ inventory models to two types of models for maximizing profits and derive the algorithms to find the optimal solution. Oberlaender et al. [24] analyzed dual sourcing strategies using an extended single-product newsvendor model with two order points. They used an exponential utility function to model different risk preferences. They showed that dual sourcing strategies are always preferable to an exclusive offshore approach, as long as the onshore ordering costs are smaller than the selling price of the product. Also, the more risk-averse the decision maker, the smaller the offshore order quantity is. Newsvendor models are widely used in literature assuming risk neutrality. Wang et al. [36] discussed a loss-averse newsvendor model and showed that when the shortage cost is not negligible, the optimal order quantity may increase the wholesale price and decrease the retailer's price, which can never occur in the risk neutral newsvendor model. Yang et al. [40] studied a newsvendor, who decides an order quantity and selling price to maximize the probability of achieving both profit and revenue targets simultaneously. They found that the probability depends critically on the relative magnitudes of the profit margin and the ratio between the profit target and the revenue target. They showed that if the product has greater price elasticity, the best strategy is always to price lower and order more. Tang et al. [34] investigated dynamic pricing in the newsvendor problem with yield risks. Arcelus et al. [4] evaluates the pricing and ordering policies of a retailer, facing a price-dependent stochastic demand for newsvendor type products under different degrees of risk tolerance and under a variety of optimizing objectives. Karakul [15] formulated joint pricing and procurement policies for fashion goods in the existence of clearance markets with random demand that follows a general distribution. The regular seasonal demand is assumed to be a linear decreasing function of the price of the product and excessive inventory at the end of the season is sold in the clearance market at a discounted price. He showed that the expected profit function is unimodal irrespective of the existence of clearance market. Donohue [11] analyzed efficient supply contracts in an inventory model for fashion goods with forecast updating and two production modes. Cachon [5] investigated allocation of inventory risks in a supply chain with push, pull, and advance-purchase discount contracts. Sahin et al. [32] proposed a single period newsvendor model where the inventory data capture process using the barcode system is prone to errors that lead to inaccurate data. They derived analytically the optimal policy in presence of errors when both demand and errors are uniformly distributed. In the second part, they examined the qualitative impact of record inaccuracies of an inventory system with additional coverage and shortage cost. Keren [16] developed a special form of the single period newsvendor problem with the known demand and random supply. He formulated general analytic solution for two types of yield risks, additive and multiplicative. Numerical examples are presented for the special case of uniformly distributed yield risk. Analysis of a two-tier supply chain of customer and producer revealed that when the customer orders more, it increases the producer's optimal production quantity. Wagner [35] discussed different inventory models and analyzed their applications in his book "Principles of Operations Research, with Applications to Managerial Decisions".

The present paper develops a single period inventory model assuming that the percentage of defective items in the order quantity is a random variable. Two special cases are considered assuming that the percentage of defectives follows truncated exponential distribution and normal distribution, respectively. The demand of the product is dependent on the selling price and has a random component, which follows a general probability distribution. Shortage may occur, either due to the presence of defective items in the ordered lot, or due to the uncertainty of demand. Shortage, if occurs, is partially backlogged and the remaining fraction is lost. The backorder rate is a negative exponential function of the magnitude of shortage. The optimal order quantity and the optimal selling price are determined.

The rest of this paper is organized as follows. In the next section, the assumptions and notations used in the paper are stated. In Section 3, the proposed inventory model is developed, and two special cases are considered in section 4. Numerical examples and sensitivity analysis carried out to examine the sensitivity of the optimal solution in the neighborhood of the key parameters of the model are given in section 5. Section 6 suggests directions for future research in the related area.

2. NOTATIONS AND MODELING ASSUMPTIONS

The mathematical models for the proposed stochastic inventory models are based on the following notations and assumptions:

2.1. Assumptions

i. This is a single period inventory model for seasonal items.
ii. Demand per season, Y is a continuous random variable dependent on retailer's selling price p.
iii. The ordered lot contains a random number of defective items, which follows a general probability distribution.
iv. Shortage may occur in the proposed inventory model either due to the unexpected presence of defective units in the accepted lot or due to the uncertainty of demand.
v. Shortages, if occur are partially backlogged. The fraction of shortage backordered is a negative exponential function of the magnitude of shortage.
Units unsold at the end of the season, if any, are removed from the retail shop to the outlet discount store and are sold at a lower price than the cost price of the item viz. the salvage value.

2.2. Notations

Q	the order quantity (a decision variable)
Z	the percentage of defective units in the ordered lot which is a random variable
z	the value of Z
Q_1	the percentage of non-defective / perfect items in the ordered lot i.e., $Q_1 = Q(1-z)$
c	the unit cost price for the retailer
m	the mark up value (a decision variable)
p	the unit selling price for the retailer where p = m c
X	a continuous random variable
x	the value of X
$f(x)$	the probability density function of X
Y	the demand per season, given by $Y = a - b \quad p + X$ where a, b are real numbers such that $a >> b > 0$.

y the value of Y i.e., $y = a - b \ p + x = a - b \ c\,m + x \geq 0$ i.e. $m < \frac{a}{(bc)}$

$g(z)$ the probability density function of Z

$\beta(y - Q_1)$ the fraction of shortages which is backordered i.e., $\beta(y - Q_1) = e^{-\varepsilon(y-Q_1)}$ where ε is a positive real number. When $\beta(y - Q_1) = 1(or\ 0)$ then shortages are completely backlogged (or completely lost).

(Q^*, m^*) the optimal order quantity Q^* and optimal mark up value m^* which maximize the expected total profit ETP(Q, m).

C_b the unit backorder cost in case of shortage

C_1 the unit cost of lost sales in case of shortage, $C_1 = p - c + \eta$, where η is a nonnegative real number

λ $1/\lambda$ is the average value of the r.v. X when X follows exponential distribution

θ $1/\theta$ is the parameter of the p.d.f. of the r.v. Z when the percentage of defectives Z in the ordered lot follows a truncated exponential distribution

μ mean of Z when Z follows normal distribution

σ standard deviation of Z when Z follows normal distribution

3. MODEL FORMULATION

In traditional models, it is generally assumed that the order quantity contains all perfect and usable units. But, in reality, there exist a random percentage of defective units in the delivered lot. If the probability of imperfect units in stock is not considered while formulating inventory policies, then it might increase the operating costs of the inventory system apart from stock outs and loss of customer goodwill. In this paper, a stochastic inventory model is developed assuming random percentage of defective units in the accepted lot. Two special cases are considered presuming that the percentage of defective units in the order quantity follows truncated exponential distribution and truncated normal distribution, respectively. The results are compared with the traditional model of all perfect items.

In the classic single period problem (SPP, newsboy problem or newsvendor problem), the retailer makes orders for seasonal items per unit cost c, and prepares well before the beginning of the selling season since the items generally have a very long replenishment lead time. The items are sold during the season at the unit selling price $p = mc$. The order quantity Q and the mark up value m are considered as the decision variables in the problem. Demand is probabilistic in nature and also depends on the selling price p.

3.1. Model I: Inventory with imperfect items

Let z represent the random percentage of defective items in the ordered lot Q. Then $Q_1 = Q(1 - z)$ is the available perfect or usable unit in the stock. The defective items are discovered at the time of sale and are returned to the vendor for refund at his cost. Now, there may be two kinds of shortages. Shortage may occur when the expected demand $y = (a - b\ p + x)$ is less than or equal to the order quantity Q, but greater than Q_1, the available perfect units. Again, shortage may occur when the expected demand y exceeds the order quantity Q. The retailer has to sell unsold units, if there be any, at the end of the season at a price lower than the cost price of the item viz. the salvage value and incur loss. If $y > Q_1$, the retailer incurs a shortage cost for each unit shortage during the season. Here shortage is assumed to be partially backlogged. The parameter β represents the fraction of shortage, which is backordered. The remaining fraction is lost. The partial backlogging rate is given by

$$\beta(y - Q_1) = e^{-\varepsilon(y - Q_1)}, \varepsilon > 0$$

The magnitude of shortage is equivalent to the backorder replenishment lead time. As backorder replenishment lead time increases, the expected shortage amount increases, and people tend to order less. The expected shortage amount is given by $(y - Q_1) = (a - b\ p + x - Q_1) = (x - q)$, say, where $q = (Q_1 - a + b\ p)$.

The parameter β satisfies the following properties:

(i) $\lim_{(y-Q_1)\to 0} \beta(y - Q_1) = 1$

i.e. the complete backorder case.

(ii) $\lim_{(y-Q_1)\to \infty} \beta(y - Q_1) = 0$

i.e. the complete lost case.

The backorders are replenished through emergency orders during the period incurring additional cost per unit backorder to avoid lost sales penalty and loss of customer goodwill. The backordered units are assumed to contain all perfect units. The different costs associated with the inventory model are ordering cost, expected backorder cost and expected cost of lost sales.

The expected overstock i.e. the expected number of unsold units at the end of the season is

$$H = \int \int_0^q (q - x) f(x) dx g(z) dz \tag{3.1}$$

where $0 < z \le 1$

The expected shortage is

$$S = \int \int_q^\infty (x - q) f(x) dx g(z) dz \tag{3.2}$$

The expected backorder in case of shortage is

$$B = \int \int_q^\infty (x - q) e^{-\varepsilon(x-q)} f(x) dx g(z) dz \tag{3.3}$$

where $\varepsilon (> 0)$ is a real number.

Hence, expected lost sales

$$L = (S - B) \tag{3.4}$$

The expected number of perfect units in inventory

$$G = \int_0^1 Q_1 g(z)dz$$

Now, expected revenue earned in No Shortage case

R_1 = Expected revenue earned from sold units + Expected salvage value of unsold perfect units

$$= p(G - H) + v\,H \tag{3.5}$$

where p = m c

Expected revenue earned in case of shortage

$$\begin{aligned} R_2 &= p\int\int_q^\infty \left\{Q_1 + (x-q)e^{-\varepsilon(x-q)}\right\} f(x)dxg(z)dz \\ &= p\left\{\int\int_q^\infty Q_1 f(x)dxg(z)dz + B\right\} \end{aligned} \tag{3.6}$$

Therefore, expected total revenue earned during the season

$$ETR = R_1 + R_2 \tag{3.7}$$

Expected total cost of the system

ETC = Ordering cost of perfect units + Expected backorder cost + Expected cost of lost sales

$$\begin{aligned} &= c\,G + C_b B + C_1 L \\ &= c\,G + (C_b - C_1)B + C_l S \end{aligned} \tag{3.8}$$

Therefore, the expected total profit of the system

$$ETP = ETR - ETC$$

$$= R_1 + R_2 - c\,G - (C_b - C_l)B - C_l S \tag{3.9}$$

Maximizing ETP with respect to the decision variables Q and m, we get the optimal values of the decision variables denoted by Q* and m* satisfying the necessary conditions

$$\frac{\partial ETP}{\partial Q} = 0, \frac{\partial ETP}{\partial m} = 0 \tag{3.10}$$

Let $\Delta = \begin{pmatrix} ETP_{QQ} & ETP_{Qm} \\ ETP_{mQ} & ETP_{mm} \end{pmatrix}$ be the Hessian matrix, where $ETP_{QQ} \equiv \dfrac{\partial^2 ETP}{\partial Q^2}$

The expected total profit ETP will be maximum at the point (Q^*, m^*) if and only if Δ is negative definite at the point (Q^*, m^*) so that the principal minors of Δ are alternatively negative and positive, i.e.,

$$\begin{aligned}&\Delta_1 = ETP_{QQ|(Q^*,m^*)} < 0, \Delta_2 = ETP_{mm|(Q^*,m^*)} < 0\\ &and \qquad ,\\ &\Delta_3 = |\Delta| = (ETP_{QQ}ETP_{mm} - ETP_{Qm}{}^2)\Big|_{(Q^*,m^*)} > 0\end{aligned} \tag{3.11}$$

3.2. Model II: Inventory with all perfect units

Assuming $z = 0$ in the above model, we get the traditional model of all perfect units.

4. SPECIAL CASES

The total demand per season is given by $Y = a - b\,p + X$. Let the random variable X follow exponential distribution and the probability density function of X is given by

$$f(x) = \lambda e^{-\lambda x} \; when \; 0 \le x < \infty \tag{4.1}$$

$= 0$, otherwise where $\lambda > 0$ and $E(x) = \dfrac{1}{\lambda}$.

4.1. Percentage of defectives in the ordered lot follows truncated exponential distribution

In this case, the probability density function of Z is

$$g(z) = \frac{\theta}{(1-e^{-\theta})} e^{-\theta z}, 0 < z \le 1, \theta > 0 \tag{4.2}$$

Then the expected total profit of the system from equation (3.9) is given by

ETP=

$$\begin{aligned}&(p-c)\alpha Q + (v-p)\beta\left[\frac{e^{\lambda d}(e^{-\theta} - e^{-\lambda Q})}{\lambda^2(\lambda Q - \theta)} - \frac{d}{\beta} + Q(\frac{1}{\gamma} - \frac{1}{\theta\beta}) - \frac{1}{\beta\lambda}\right] +\\ &p\frac{Q\beta}{\lambda(Q\lambda - \theta)^2} e^{\lambda d}\left\{e^{-\theta} - (\lambda Q + 1 - \theta)e^{-\lambda Q}\right\} + \left\{\frac{p - C_b + C_l}{\eta^2} - \frac{C_l}{\lambda^2}\right\}\\ &\frac{\beta e^{\lambda d}}{(\lambda Q - \theta)}(e^{-\theta} - e^{-\lambda Q})\end{aligned} \tag{4.3}$$

where $\eta = \lambda + \varepsilon, \gamma = \theta\,\lambda, \beta = \gamma/(1-e^{-\theta}), \alpha = \left\{\frac{\beta}{\gamma} - \frac{1}{\theta}\right\}, d = a - b\,p, p = c\,m$ (4.4)

Here we have

$$\frac{\partial^2 ETP}{\partial m^2} = -\frac{c^2\beta}{(\theta-\lambda Q)} e^{(a-b\,c\,m-Q)\lambda} \begin{bmatrix} \frac{(b\lambda)^2}{\theta}(1-e^{-\theta+\lambda Q})\left\{\rho(2mc+u-c)+(\frac{c_b}{\eta^2}-\frac{v}{\lambda^2})\right\} \\ +\frac{bQ}{\theta}(1-e^{-\theta+\lambda Q})(2-bcm\lambda)\left\{\frac{e^{\theta}}{(e^{\theta}-e^{\lambda Q})}-\frac{1}{(\theta-\lambda Q)}\right\} \\ +\frac{2b}{\theta\lambda}\left\{e^{(bcm+Q-a)\lambda}(1-e^{-\theta})(\theta-\lambda Q)-2\rho\lambda^2(1-e^{-\theta+\lambda Q})\right\} \end{bmatrix} \tag{4.5}$$

Now $\frac{\partial^2 ETP}{\partial m^2} < 0$ when $Q < \frac{(\theta-2)}{\lambda}, \theta > 2$, and $\frac{(a-Q)}{(bc)} < m < \frac{a}{(bc)}$.

Since

$$(\frac{c_b}{\eta^2} - \frac{v}{\lambda^2}) > 0 \text{ and } \rho\lambda^2 = \left\{1-(\frac{\lambda}{\lambda+\varepsilon})^2\right\} < 1 \tag{4.6}$$

$$\frac{\partial^2 ETP}{\partial Q^2} = -\frac{\beta}{(\theta-\lambda Q)} e^{(a-bcm-Q)\lambda} \begin{bmatrix} cm\left\{\frac{2\theta(1-e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^3}-\frac{2\lambda Q e^{2\lambda Q}}{(e^{\theta}-e^{\lambda Q})^2}-\frac{(2+\lambda Q)^{e\lambda Q}}{(e^{\theta}-e^{\lambda Q})}\right\} \\ +\frac{\lambda^2}{(\theta-\lambda Q)^2}\left\{(\theta-\lambda Q)(\theta-\lambda Q-2)+2(1-e^{-\theta+\lambda Q})\right\} \\ \left\{\rho(2mc+u-c)+(\frac{c_b}{\eta^2}-\frac{v}{\lambda^2})\right\} \\ +\frac{2cm(\theta-\lambda Q-1+e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^2}\left\{\begin{matrix}\frac{\lambda Q(\theta-\lambda Q)e^{\theta+\lambda Q}}{(e^{\theta}-e^{\lambda Q})^2}+ \\ \frac{\theta e^{\theta}}{(e^{\theta}-e^{\lambda Q})}-\frac{(\theta+\lambda Q)}{(\theta-\lambda Q)}\end{matrix}\right\} \\ +cm\lambda Q\left\{\frac{1}{(\theta-\lambda Q)}-\frac{e^{\theta}}{(e^{\theta}-e^{\lambda Q})}\right\} \end{bmatrix} \tag{4.7}$$

$\frac{\partial^2 ETP}{\partial Q^2} < 0$ when $\theta > 4$ and $Q < \frac{2}{\lambda}$ since $(\frac{c_b}{\eta^2} - \frac{v}{\lambda^2}) > 0$, $(2mc + u - c) = c(2m-1) + u > 0$ (see Appendix I) (4.8)

Also

$$\frac{\partial^2 ETP}{\partial Q \partial m} =$$

$$\frac{c\beta}{(\theta - \lambda Q)} e^{(a-bcm-Q)\lambda} \left[\begin{array}{l} -(bcm\lambda - 1)(1 - e^{-\theta + \lambda Q}) \left\{ \frac{Qe^{\theta + \lambda Q}}{(e^{\theta} - e^{\lambda Q})^2} + \frac{e^{\theta}}{\lambda(e^{\theta} - e^{\lambda Q})} - \frac{\theta}{\lambda(\theta - \lambda Q)^2} \right\} \\ + \frac{1}{(\theta - \lambda Q)} (\theta - \lambda Q - 1 + e^{-\theta + \lambda Q}) \\ \left\{ \begin{array}{l} (bcm\lambda - 1)(Q(\frac{e^{\theta}}{(e^{\theta} - e^{\lambda Q})} - \frac{1}{(\theta - \lambda Q)}) - 2\rho\lambda) \\ -b\lambda^2 (\rho(u - c) + (\frac{c_b}{\eta^2} - \frac{v}{\lambda^2})) \end{array} \right\} \end{array} \right]$$

$(bcm\lambda - 1) > 0$ when $m > \frac{1}{bc\lambda}$

It can be shown that $(\frac{\partial^2 ETP}{\partial m^2})(\frac{\partial^2 ETP}{\partial Q^2}) > (\frac{\partial^2 ETP}{\partial Q \partial m})^2$ (see Appendix II) (4.9)

Therefore, maximizing ETP with respect to the decision variables Q and m, we get the optimal values of the decision variables Q* and m*, respectively, satisfying the required necessary and sufficient conditions given by (3.10) and (3.11).

4.2. Percentage of defectives in the ordered lot follows truncated normal distribution with mean μ and standard deviation σ

The probability density function of Z is as follows:

$$g(z) = \frac{K}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}, \quad 0 < z \leq 1$$
$$= 0, \quad otherwise$$

where

$$K = \frac{2}{\left\{Erf\left[\frac{K_2}{\sqrt{2}}\right] - Erf\left[\frac{K_1}{\sqrt{2}}\right]\right\}}, K_1 = -\frac{\mu}{\sigma}, K_2 = \frac{(1-\mu)}{\sigma},$$

$$u = \frac{(z-\mu)}{\sigma}$$

and $Erf\left[\frac{K_2}{\sqrt{2}}\right] = \frac{2}{\sqrt{\pi}} \int_0^{\frac{K_2}{\sqrt{2}}} e^{-t^2} dt$

Making a transformation of variable from Z to the standard normal variable U, the p.d.f. of U is obtained as $\varphi(u) = \frac{K}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$, where $u = \frac{(z-\mu)}{\sigma}$

Proceeding similarly as in Section 4.1, we get the optimal inventory policy of the proposed model in this case also.

5. NUMERICAL EXAMPLES AND SENSITIVITY ANALYSIS

5.1. Numerical Example

Example1: The following values of the given set of parameters are taken c = 100, C_b = 130, C_l = p – c + 50, a = 1000, b = 3, ε = 0.001, v = 50, λ = 1/400, θ =1/ 0.2, μ = 0.2, σ = 0.05 in appropriate units. The outputs (optimal solutions) generated by computer are presented in Table 1 and Table 2.

Table 1: Exponential distribution results

Optimal order quantity Q*	723.11
Optimal mark up value m*	2.612
Expected total profit ETP*	117504.0
Expected perfect units G*	583.393
Expected overstock H*	137.506
Expected Shortage S*	170.338
Expected backorder B*	86.907
Expected Loss of sales L*	83.431

Table 2: Normal distribution results

Optimal order quantity Q*	735.428
Optimal mark up value m*	2.577
Expected total profit ETP*	122362.0
Expected perfect units G*	588.337
Expected overstock H*	124.237
Expected Shortage S*	162.683
Expected backorder B*	83.001
Expected Loss of sales L*	79.681

Tables 1 and 2 show that the expected total profit ETP* is greater ($122362.0 – $117504.0 = $4858.0) in case of normally distributed percentage of defectives than the exponential case, though the optimal mark up value m* is higher in the second case.

Table 3: Traditional case (perfect stock)

Optimal order quantity Q*	587.887
Optimal mark up value m*	2.5757
Expected total profit ETP*	122637
Expected overstock H*	122.986
Expected Shortage S*	162.383
Expected backorder B*	82.848
Expected Loss of sales L*	79.534

Comparing the results in Table 3 with the results in Tables 1 and 2, we see that the optimal order quantity Q* increases significantly in imperfect cases (23% (approx.), in case of exponential distribution, and 25% (approx.) in case of normal distribution) compared to the perfect case. The expected total profit ETP* is, however, greater in the traditional case than in the defective cases, which is expected. If the traditional optimal policy (Q*, m*) = (587.887, 2.5757) is substituted in the profit functions of cases with exponentially and normally distributed defective items, then ETP = $115167.0 and ETP = $119210.0, respectively are obtained. Therefore, if a retailer adopts the traditional optimal policy when the order quantity contains imperfect items then he/she will incur a potential loss of profit (2% (approx.) in case of exponential distribution, and 2.6% (approx.) in case of normal distribution), which depends on the probability distribution of the percentage of defective items in the order quantity. The numerical results are further explained with the help of the following 3D graphs drawn below by using MATHEMATICA.

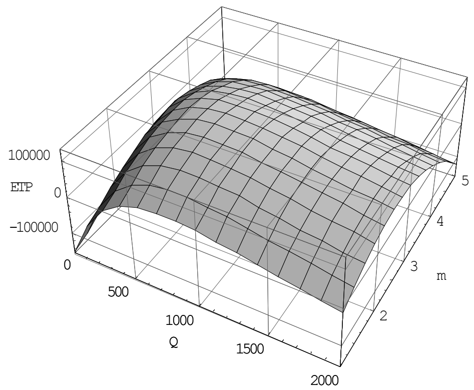

Figure 1: Expected profit function ETP vs. order quantity Q and mark up value m for normally distributed defective items.

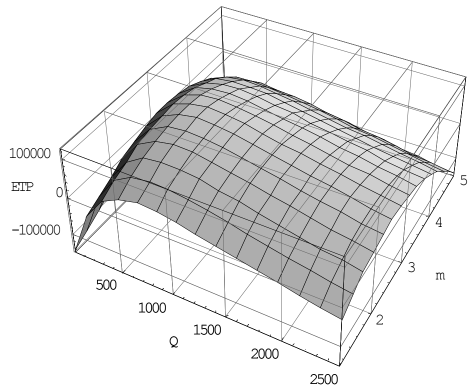

Figure 2: Expected profit function ETP vs. order quantity Q and mark up value m for exponentially distributed defective items.

Figures 1- 2 indicate that the expected profit functions are strictly concave with respect to the decision variables Q and m for cases when the percentage of defectives in the order quantity follows truncated normal distribution and one-sided truncated exponential distribution. The expected profit of the proposed model is maximum at the point (Q*, m*) i.e., (735.428, 2.577) and (723.110, 2.612) for normally distributed and exponentially distributed percentage of defectives respectively.

5.2. Sensitivity analysis

The proposed inventory model is further analyzed through sensitivity analysis in case of exponentially distributed percentage of defectives (θ = 1/0.05) in the order quantity with respect to the key parameters of the model using the following Tables 4-10 and the corresponding Figures 3-9:

Table 4: Effect of unit cost of lost sales $C_l = p - c + \eta$

η	Q*	m*	ETP*	H*	G*	S*	B*	L*
30	608.702	2.5639	124151	115.766	578.267	168.329	85.882	82.447
35	611.314	2.5673	123713	117.800	580.748	166.871	85.138	81.732
40	613.913	2.5706	123279	119.830	583.217	165.436	84.406	81.030
45	616.499	2.5739	122849	121.855	585.674	164.026	83.687	80.339
50	619.074	2.5771	122424	123.878	588.120	162.637	82.978	79.659
55	621.637	2.5802	122002	125.896	590.555	161.272	82.281	78.990
60	624.188	2.5834	121584	127.911	592.979	159.927	81.595	78.332
65	626.728	2.5864	121170	129.921	595.392	158.604	80.9203	77.684

Table 5: Effect of unit backorder cost Cb

C_b	Q*	m*	ETP*	H*	G*	S*	B*	L*
100	602.780	2.5561	125153	111.178	572.641	171.704	87.604	84.099
110	608.266	2.5633	124225	115.427	577.853	168.575	86.008	82.567
120	613.697	2.5703	123315	119.660	583.012	165.555	84.467	81.088
130	619.074	2.5771	122424	123.878	588.120	162.637	82.978	79.659
140	624.400	2.5836	121549	128.079	593.180	159.816	81.539	78.277
150	629.678	2.5899	120692	132.262	598.194	157.086	80.146	76.940
160	634.907	2.5961	119850	136.429	603.162	154.443	78.798	75.646
170	640.091	2.6020	119024	140.576	608.086	151.884	77.492	74.392

Table 6: Effect of unit salvage value v

v	Q*	m*	ETP*	H*	G*	S*	B*	L*
0	561.898	2.4975	118133	80.685	533.803	197.638	100.836	96.802
10	570.872	2.5113	118840	87.213	542.328	191.488	97.698	93.790
20	580.855	2.5261	119612	94.599	551.812	184.960	94.367	90.592
30	592.038	2.5419	120459	103.011	562.436	178.011	90.822	87.189
40	604.667	2.5588	121391	112.673	574.434	170.591	87.036	83.555
50	619.074	2.5771	122424	123.878	588.120	162.637	82.978	79.659
60	635.709	2.5968	123573	137.027	603.924	154.070	78.607	75.463
70	655.215	2.6182	124863	152.694	622.454	144.783	73.869	70.914

Table 7: Effect of a

a	Q*	m*	ETP*	H*	G*	S*	B*	L*
1000	619.074	2.5771	122424	123.878	588.120	162.637	82.978	79.659
1100	645.777	2.7097	148910	103.893	613.488	177.486	90.554	86.932
1200	670.051	2.8345	178335	82.862	636.548	195.966	99.983	95.984
1300	692.056	2.9495	210977	61.438	657.453	219.131	111.802	107.33
1400	711.942	3.0523	247212	40.575	676.345	248.531	126.801	121.729
1500	729.849	3.1396	287561	21.757	693.357	286.517	146.182	140.335
1600	745.908	3.2068	332769	7.449	708.613	336.781	171.827	164.954
1700	760.230	3.2479	383943	1.965	722.219	405.368	206.820	198.547

Table 8: Effect of b

b	Q*	m*	ETP*	H*	G*	S*	B*	L*
0.5	786.796	14.0130	1070150	179.264	747.456	131.153	66.915	64.238
1.0	745.362	7.1485	499199	162.538	708.094	139.593	71.221	68.372
1.5	709.041	4.8612	309710	149.517	673.589	146.748	74.872	71.877
2.0	676.435	3.7183	215539	139.141	642.613	152.867	77.994	74.874
2.5	646.649	3.0332	159466	130.742	614.317	158.123	80.675	77.448
3.0	619.074	2.5771	122424	123.878	588.120	162.637	82.978	79.659
3.5	593.273	2.2518	96244	118.236	563.309	166.507	84.952	81.554
4.0	568.919	2.0083	76846	113.591	540.473	169.806	86.636	83.170

Table 9: Effect of θ

1/θ	Q*	m*	ETP*	H*	G*	S*	B*	L*
0.03	606.168	2.5762	122567	123.294	587.983	162.463	82.890	79.574
0.04	612.550	2.5765	122506	123.545	588.048	162.535	82.926	79.609
0.05	619.074	2.5771	122424	123.878	588.120	162.637	82.978	79.659
0.06	625.735	2.5777	122316	124.283	588.191	162.782	83.052	79.730
0.07	632.524	2.5787	122180	124.802	588.248	162.953	83.139	79.814
0.08	639.427	2.5798	122012	125.396	588.275	163.183	83.257	79.926
0.09	646.428	2.5812	121809	126.082	588.259	163.469	83.403	80.067
0.10	653.503	2.5828	121569	126.855	588.182	163.820	83.582	80.238

Table 10: Effect of (μ, σ) taking σ = 0.02

μ	Q*	m*	ETP*	H*	G*	S*	B*	L*
0.03	607.848	2.5759	122614	123.092	587.925	162.408	82.861	79.547
0.04	613.135	2.5759	122610	123.111	587.932	162.412	82.863	79.549
0.05	619.111	2.5759	122608	123.122	587.937	162.415	82.865	79.550
0.06	625.527	2.5759	122606	123.130	587.940	162.417	82.866	79.551
0.07	632.207	2.5759	122605	123.135	587.941	162.417	82.866	79.551
0.08	639.070	2.5759	122604	123.137	587.943	162.419	82.866	79.552
0.09	646.093	2.5759	122604	123.142	587.944	162.419	82.866	79.552
0.10	653.273	2.5759	122603	123.145	587.946	162.420	82.867	79.553

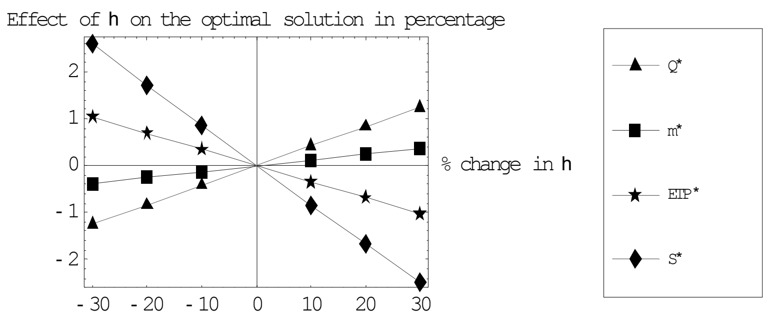

Figure 3: Percentage change in the optimal solution vs. percentage change in η

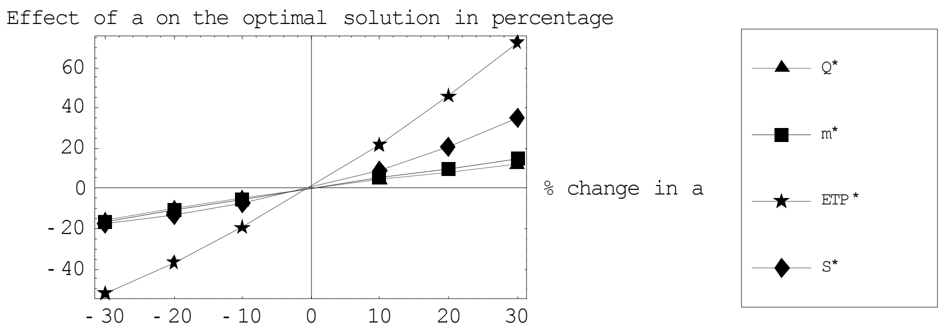

Figure 4: Percentage change in the optimal solution vs. percentage change in a

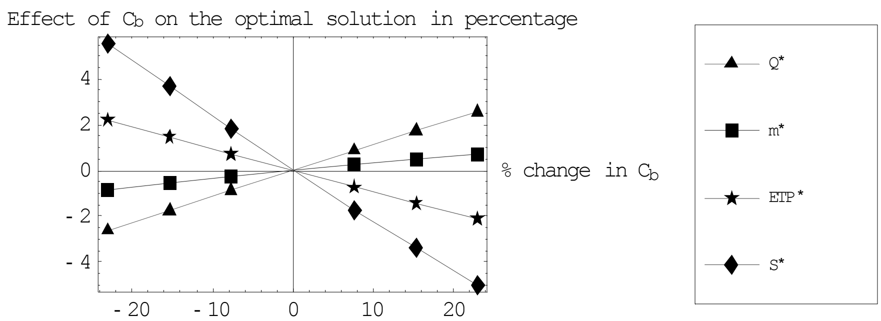

Figure 5: Percentage change in the optimal solution vs. percentage change in C_b

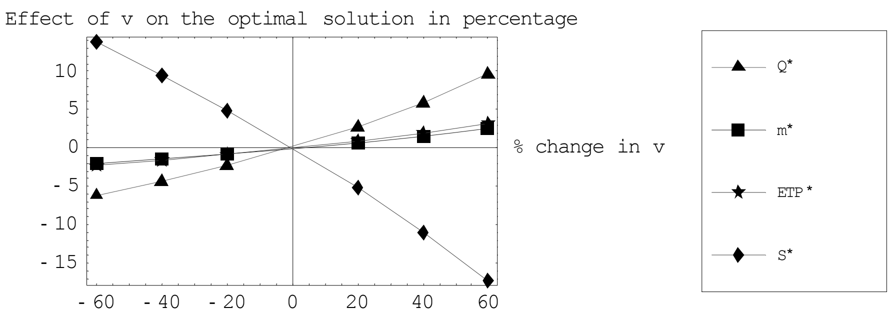

Figure 6: Percentage change in the optimal solution vs. percentage change in v

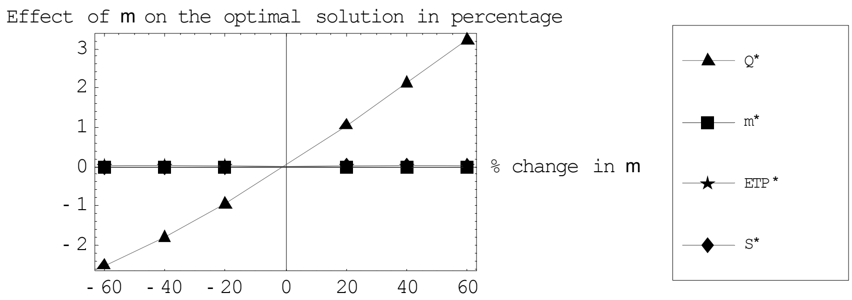

Figure 7: Percentage change in the optimal solution vs. percentage change in μ

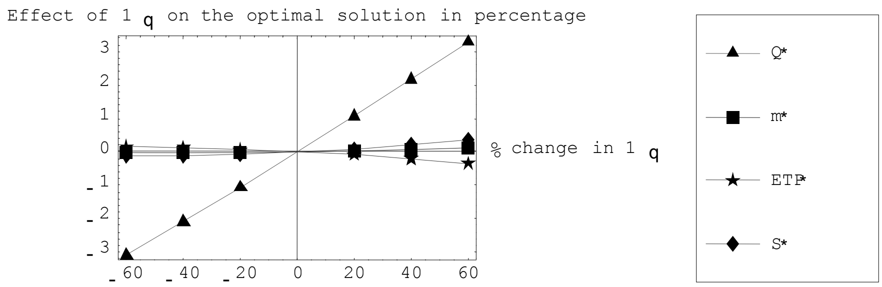

Figure 8: Percentage change in the optimal solution vs. percentage change in 1/ θ

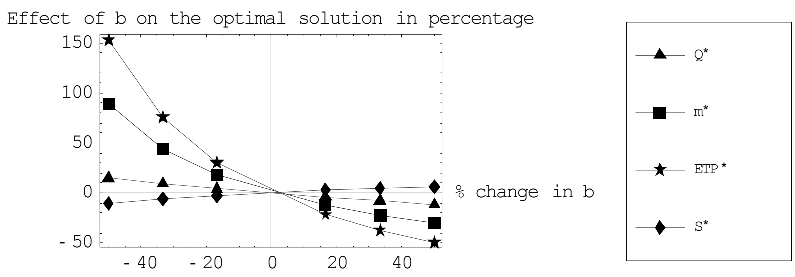

Figure 9: Percentage change in the optimal solution vs. percentage change in b

From the Figures 3 – 9, we observe that

i. From Fig. 3, we see that S* is slightly sensitive to η and decreases as η increases, which is expected since the cost of lost sales C_{ls} rises as η increases. The other variables Q*, m*, and ETP* are insensitive to η.

ii. From Fig. 4, it is found that, the optimal expected profit ETP* is highly sensitive to a and increases rapidly as a increases. Further, the rate of increase rises for higher values of a. The variables Q*, m*, and S* are moderately sensitive to a and increase as a increases.

iii. From Fig. 5, S* is moderately sensitive to C_b and decreases as C_b increases. Q* and ETP* are slightly sensitive to C_b. Q* increases as C_b increases, whereas ETP* decreases. However, m* is insensitive to C_b.

iv. From Fig. 6, S* is moderately sensitive to the salvage value v and decreases as v increases. This is justified, since higher value of v motivates overstock and hence lower expected shortage. Q* is also moderately sensitive to v. ETP* and m* are slightly sensitive to v and increase with v.

v. From Fig. 7, Q* is slightly sensitive to μ and increases as μ increases. However, the other variables are insensitive to μ.

vi. From Fig. 8, Q* is slightly sensitive to 1/θ and increases with 1/θ. Though, the other variables are insensitive to 1/ θ.

vii. From Fig. 9, ETP* is highly sensitive to b and decreases rapidly as b increases. The rate of decrease falls with b. The optimal mark-up m* is again highly sensitive to b, whereas, Q* and S* are moderately sensitive to b.

From the above sensitivity analysis we conclude that the parameters a, b, and v should be estimated carefully since the optimal solution is highly/moderately sensitive with respect to these parameters.

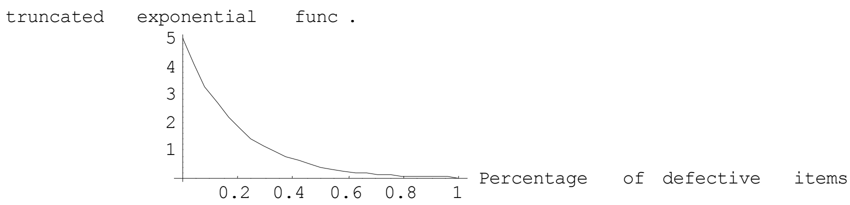

Figure 10: The truncated exponential p.d.f. (θ = 1/0.2) vs. the percentage of defectives

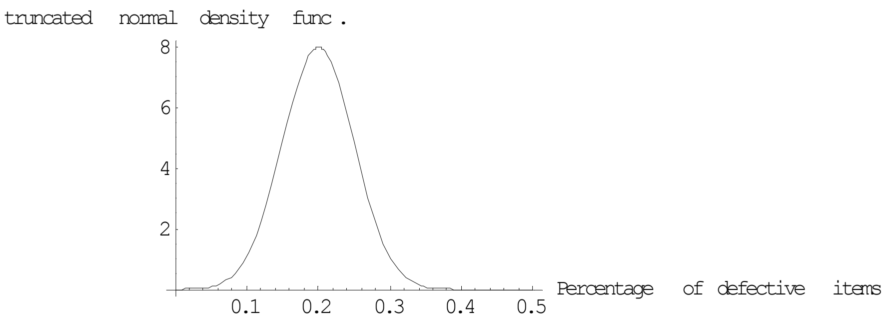

Figure 11: The truncated normal p.d.f. ($\mu = 0.2$, $\sigma = 0.05$) vs. the percentage of defectives

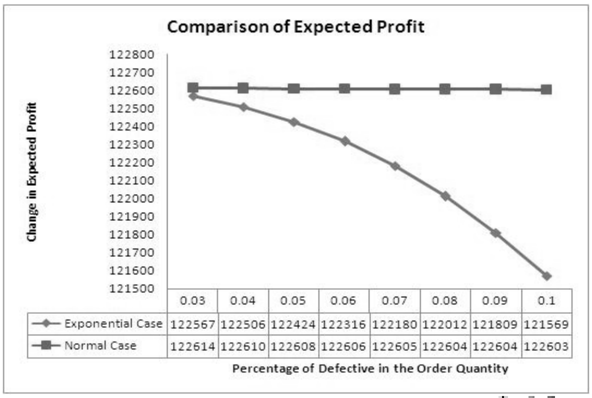

Figure 12: Comparison of expt. profit for different values of $(1/\theta)$ and μ in [0.03, 0.10] taking $\sigma = 0.02$

Figure 12 shows that in case of exponentially distributed percentage of defectives as $(1/\theta)$ increases from 3% to 10%, the expected profit decreases from \$122567.0 to \$121569. However, for normally distributed percentage of defectives, the decrease in expected profit is negligible. Therefore, while formulating optimal inventory policies extra care is needed in the exponential case for the given set of parameter values.

6. CONCLUDING REMARKS

In this paper, a single period stochastic inventory model is developed assuming random percentage of defective units in the order quantity. The demand of the product is random as well as sensitive to the selling price of the product. The optimal order quantity and selling price are obtained to maximize the expected total profit. In the present scenario of globalization and stiff competition, the business firms are giving more and more importance to reduction of on hand stock and acceptance of good quality and perfect items only in inventory to maximize the expected profit of the system. This is the reason behind the evolution of “Just In Time inventory” or “JIT”. Consequently, quality inspection and acceptance sampling has become very important in the industry nowadays. Rigorous inspection of each item in the ordered lot is impossible in most cases and sometimes it might be destructive. The quality inspection process is thus based on sample inspection and hence, it is not full proof. There is always the possibility of having defective items in inventory and additional measures must be taken to minimize the expected total cost. The paper can be extended by considering quality improvement policies, capacity planning of the system or production inventory models with variable set up costs.

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APPENDIX

Appendix I: $(\frac{\partial^2 ETP}{\partial Q^2}) < 0$

(i) $\left[\frac{2\theta(1-e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^3} - \frac{2\lambda Q e^{2\lambda Q}}{(e^\theta - e^{\lambda Q})^2} - \frac{(2+\lambda Q)e^{\lambda Q}}{(e^\theta - e^{\lambda Q})}\right] > 0$

since $2\theta(e^\theta - e^{\lambda Q})^3 > (\theta-\lambda Q)^3 e^{\theta+\lambda Q}\left\{(2+\lambda Q)e^\theta - (2-\lambda Q)e^{\lambda Q}\right\} > 0$

(ii)

$$\begin{bmatrix} \frac{2cm(\theta-\lambda Q-1+e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^2}\left\{\frac{\lambda Q(\theta-\lambda Q)e^{\theta+\lambda Q}}{(e^\theta-e^{\lambda Q})^2} + \frac{\theta e^\theta}{(e^\theta-e^{\lambda Q})} - \frac{(\theta+\lambda Q)}{(\theta-\lambda Q)}\right\} \\ cm\lambda Q\left\{\frac{1}{(\theta-\lambda Q)} - \frac{e^\theta}{(e^\theta-e^{\lambda Q})}\right\} \end{bmatrix}$$

$$= cm\begin{bmatrix} \frac{2\lambda Q e^{\theta+\lambda Q}(\theta-\lambda Q-1+e^{-\theta+\lambda Q})}{(\theta-\lambda Q)(e^\theta-e^{\lambda Q})^2} + \frac{2(\theta-\lambda Q-1+e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^2}\left\{\frac{\theta e^\theta}{(e^\theta-e^{\lambda Q})} - \frac{(\theta+\lambda Q)}{(\theta-\lambda Q)}\right\} \\ +\lambda Q\left\{\frac{1}{(\theta-\lambda Q)} - \frac{e^\theta}{(e^\theta-e^{\lambda Q})}\right\} \end{bmatrix}$$

$$= cm\begin{bmatrix} \frac{2\lambda Q e^{\theta+\lambda Q}(\theta-\lambda Q-1+e^{-\theta+\lambda Q})}{(\theta-\lambda Q)(e^\theta-e^{\lambda Q})^2} + \frac{(\theta-\lambda Q-1+e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^3(e^\theta-e^{\lambda Q})} \\ \left\{e^\theta(\theta-\lambda Q)(\theta(2-\lambda Q)+(\lambda Q)^2) - 2(\theta+\lambda Q)(e^\theta-e^{\lambda Q})\right\} \end{bmatrix} > 0$$

Because $\left[e^\theta(\theta-\lambda Q)(\theta(2-\lambda Q)+(\lambda Q)^2) - 2(\theta+\lambda Q)(e^\theta-e^{\lambda Q})\right] > 0$

That implies $e^\theta(\theta-\lambda Q)(\theta(2-\lambda Q)+(\lambda Q)^2) > 2(\theta+\lambda Q)(e^\theta-e^{\lambda Q})$

i.e., $(\theta-\lambda Q)(\theta(2-\lambda Q)+(\lambda Q)^2) > 2(\theta+\lambda Q)(1-e^{-\theta+\lambda Q})$

i.e., $\frac{(\theta-\lambda Q)}{2}.\frac{(\theta(2-\lambda Q)+(\lambda Q)^2)}{(\theta+\lambda Q)} > (1-e^{-\theta+\lambda Q})$

which is true when $Q < \frac{(\theta-2)}{\lambda}$ and $Q < \frac{2}{\lambda}$.

Assuming $(\theta-2) > 2$ i.e. $\theta > 4$ we get $(\frac{\partial^2 ETP}{\partial Q^2}) < 0$ if $\theta > 4$ and $Q < \frac{2}{\lambda}$.

Appendix II: We have

$$(\frac{\partial^2 ETP}{\partial Q\partial m})^2 = \frac{(c\beta)^2}{(\theta-\lambda Q)^2} e^{2(a-b\,c\,m-Q)\lambda} \left[\begin{array}{l} (bcm\lambda-1)^2(1-e^{-\theta+\lambda Q})^2 \left\{ \begin{array}{l} \frac{Qe^{\theta+\lambda Q}}{(e^{\theta}-e^{\lambda Q})^2} + \frac{e^{\theta}}{\lambda(e^{\theta}-e^{\lambda Q})} \\ -\frac{\theta}{\lambda(\theta-\lambda Q)^2} \end{array} \right\}^2 \\ +\frac{1}{(\theta-\lambda Q)^2}(\theta-\lambda Q-1+e^{-\theta+\lambda Q})^2 \\ \left\{ \begin{array}{l} -b\lambda^2(\rho(u-c)+(\frac{c_b}{\eta^2}-\frac{v}{\lambda^2})) \\ +(bcm\lambda-1)(Q(\frac{e^{\theta}}{(e^{\theta}-e^{\lambda Q})}-\frac{1}{(\theta-\lambda Q)})-2\rho\lambda) \end{array} \right\}^2 \\ -2\frac{(bcm\lambda-1)}{(\theta-\lambda Q)}(1-e^{-\theta+\lambda Q})(\theta-\lambda Q-1+e^{-\theta+\lambda Q}) \\ \times \left[\begin{array}{l} \left\{ \frac{Qe^{\theta+\lambda Q}}{(e^{\theta}-e^{\lambda Q})^2} + \frac{e^{\theta}}{\lambda(e^{\theta}-e^{\lambda Q})} - \frac{\theta}{\lambda(\theta-\lambda Q)^2} \right\} \\ \left\{ \begin{array}{l} (bcm\lambda-1)(Q(\frac{e^{\theta}}{(e^{\theta}-e^{\lambda Q})}-\frac{1}{(\theta-\lambda Q)})-2\rho\lambda) \\ -b\lambda^2(\rho(u-c)+(\frac{c_b}{\eta^2}-\frac{v}{\lambda^2})) \end{array} \right\} \end{array} \right] \end{array} \right] \quad (4.10)$$

Again

$$(\frac{\partial^2 ETP}{\partial Q^2})(\frac{\partial^2 ETP}{\partial m^2}) = \frac{(c\beta)^2}{(\theta-\lambda Q)^2} e^{2(a-bcm-Q)\lambda} \left[\begin{array}{l} cm\lambda \left\{ \begin{array}{l} \frac{2(\theta-\lambda Q-1+e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^2}(\frac{Q(\theta-\lambda Q)e^{\theta+\lambda Q})}{(e^{\theta}-e^{\lambda Q})^2} \\ +\frac{\theta e^{\theta}}{\lambda(e^{\theta}-e^{\lambda Q})} - \frac{(\theta+\lambda\theta)}{\lambda(\theta-\lambda\theta)}) \\ +Q(\frac{1}{(\theta-\lambda Q)}-\frac{e^{\theta}}{(e^{\theta}-e^{\lambda Q})}) \end{array} \right\} \\ +\frac{\lambda^2}{(\theta-\lambda Q)^2}\left\{(\theta-\lambda Q)(\theta-\lambda Q-2)+2(1-e^{-\theta+\lambda Q})\right\} \\ \left\{\rho(2mc+u-c)+(\frac{c_b}{\eta^2}-\frac{v}{\lambda^2})\right\} \\ +cm\left\{\frac{2\theta(1-e^{-\theta+\lambda Q})}{(\theta-\lambda Q)^3} - \frac{2\lambda Qe^{2\lambda Q}}{(e^{\theta}-e^{\lambda Q})^2} - \frac{(2+\lambda Q)e^{\lambda Q}}{(e^{\theta}-e^{\lambda Q})}\right\} \end{array} \right] \quad (4.11)$$

$$\times \left[\begin{array}{l} \frac{(b\lambda)^2}{\theta}(1-e^{-\theta+\lambda Q})\left\{\rho(2mc+u-c)+(\frac{C_b}{\eta^2}-\frac{v}{\lambda^2})\right\} \\ +\frac{bQ}{\theta}(1-e^{-\theta+\lambda Q})(2-bmc\lambda)\left\{\frac{e^{\theta}}{(e^{\theta}-e^{\lambda Q})}-\frac{1}{(\theta-\lambda Q)}\right\} \\ +\frac{2b}{\theta\lambda}\left\{e^{-(a-bcm-Q)\lambda}(1-e^{-\theta})(\theta-\lambda Q)-2\rho\lambda^2(1-e^{-\theta+\lambda Q})\right\} \end{array} \right]$$

Comparing the terms of (4.10) and (4.11) we conclude that $(\frac{\partial^2 ETP}{\partial m^2})(\frac{\partial^2 ETP}{\partial Q^2}) > (\frac{\partial^2 ETP}{\partial Q\partial m})^2$

When $\theta > 4$ i.e., $\frac{1}{\theta} < 0.25$, $0 < Q < \frac{2}{\lambda}$, and $\frac{(a-Q)}{bc} < m < \frac{a}{(bc)}$.