

FUZZY ECONOMIC ORDER QUANTITY MODEL WITH RANKING FUZZY NUMBER COST PARAMETERS

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Abstract: In this paper, a multi-objective economic order quantity model with shortages and demand dependent unit cost under storage space constraint is formulated. In real life situation, the objective and constraint goals and cost parameters are not precisely defined. These are defined in fuzzy environment. The cost parameters are represented here as triangular shaped fuzzy numbers with different types of left and right branch membership functions. The fuzzy numbers are then expressed as ranking fuzzy numbers with best approximation interval. Geometric programming approach is applied to derive the optimal decisions in closed form. The inventory problem without shortages is discussed as a special case of the original problem. A numerical illustration is given to support the problem.

Keywords: Inventory, ranking fuzzy number, geometric programming.

MSC: 90B05.

1. INTRODUCTION

Classical inventory problems are generally formulated by considering that the demand rate of an item is constant and deterministic. The unit price of an item is usually considered to be constant and independent in nature (Hadley and Whitin (1958), Silver and Peterson (1985)). But in practical situation, unit price and demand rate of an item may be related to each other. When the demand of an item is high, items are produced in large numbers and fixed costs of production are spread over a large number of items. Hence the unit cost of the item decreases, i.e. the unit price of an item inversely relates to the demand of that item. So, demand rate of an item may be considered as a decision

variable. Cheng (1989), Jung and Klein (2001) formulated the economic order quantity (EOQ) problem with this idea, and solved it by using geometric programming (GP) method.

In real life, it is not always possible to obtain the precise information about inventory parameters. This type of imprecise data is not always well represented by random variables selected from probability distribution. So, decision making methods under uncertainty are needed. To deal with this uncertainty and imprecise data, the concept of fuzziness can be applied. The inventory cost parameters such as holding cost, set up cost, shortage cost are assumed to be flexible, i.e. fuzzy in nature. These parameters can be represented by fuzzy numbers. Efficient methods for ranking fuzzy numbers are very important to handle the fuzzy numbers in a fuzzy decision-making problem. Again, in real life situation, it is almost impossible to predict the total inventory cost and total available floor space precisely. These are also imprecise in nature. Decision maker may change these quantities within some limits as per the demand of the situation. Hence, these quantities may be assumed uncertain in non-stochastic sense but fuzzy in nature. In this situation, the inventory problem along with constraints can be developed with the fuzzy set theory.

In 1965, Zadeh first introduced the concept of fuzzy set theory. Later on, Bellman and Zadeh (1970) used the fuzzy set theory for the decision-making problem. Tanaka et. al. (1974) introduced the objectives as fuzzy goals over α -cut of a fuzzy constraint set, and Zimmermann (1976) gave the concept of solving multi-objective linear-programming problem by using fuzzy programming technique. Fuzzy set theory now has made an entry into the inventory control systems. Sommer (1981) applied the fuzzy concept to an inventory and production-scheduling problem. Park (1987) examined the EOQ formula in the fuzzy set theoretic perspective associating the fuzziness with the cost data. Roy et. al. (1997) solved a single objective fuzzy EOQ model using GP technique. De et. al. (2001) derived a replenishment policy for items with finite production rate and fuzzy deterioration rate represented by a triangular fuzzy number using extension principle. Jain (1976) first proposed the method of ranking fuzzy numbers. Yager (1981) proposed a procedure for ordering fuzzy subsets of the unit interval. A subjective approach for ranking fuzzy numbers was presented by Campos et. al. (1989). In 1999, Dubois et. al. proposed a unified view of ranking technique of fuzzy numbers. Wen et. al. (2005) used best approximation interval to rank fuzzy numbers.

GP method, as introduced by Duffin et. al. (1967), is an effective method to solve a non-linear programming problem. It has certain advantages over the other optimization methods. The advantage is that this method converts a problem with highly non-linear and inequality constraints (primal problem) to an equivalent problem with linear and equality constraints (dual problem). It is easier to deal with the dual problem consisting of linear and equality constraints than the primal problem with non-linear and inequality constraints. Kotchenberger (1971) first used GP method to solve the basic inventory problem. Warral and Hall (1982) utilized this technique to solve a multi-item inventory problem with several constraints. This method is now widely used to solve the optimization problem in inventories. But to solve a non-linear programming problem by GP method degree of difficulty (DD) plays a significant role. DD is defined as a total number of terms in objective function and constraints – (total number of decision variables + 1). It will be difficult to solve the problem for higher values of DD. If DD = 0, the dual variables can uniquely be determined from the normality and orthogonality

conditions. If $DD > 0$, computational complexity may rise. To avoid such complexity, one always tries to reduce the DD. Ata et al. ((1997), (2003)) and Chen (2000) developed some inventory problems and solved them by GP method. Hariri et. al. (1997) gave a new idea on GP to solve multi-item inventory problems (here, after, this new GP has been called modified geometric programming (MGP)). Mandal et. al. (2005) used MGP technique to solve multi-item inventory problem. S.T.Liu (2009) presented a profit maximization problem with interval coefficients and quantity discounts and solved them by GP method. K-N Francis Leung (2007) proposed an inventory problem with flexible and imperfect production process and used GP technique to obtain closed form optimal solution. Sadjadi et. al. (2010) proposed a new pricing and marketing planning problem where demand was a function of price and marketing expenditure with fuzzy environments, and the resulted problem was solved by GP method.

In this paper, a multi-objective economic order quantity problem with demand-dependent unit cost and shortages but fully backlogged is formulated along with total available storage space restriction. Due to volatile nature of the market, the cost parameters are represented here by fuzzy numbers with different types of left and right branches of membership function. These parameters are then expressed by ranking fuzzy numbers with best approximation interval. The objective goal and constraint goal can't be predicted precisely in real life. The authority may allow the flexibility of these goals to some extent. In this context, the objective function and constraint are considered here in fuzzy environment by giving some tolerance value. The problem has been expressed in posynomial problem. MGP technique is used here to solve the problem. As a particular case, we also investigate the case when shortages are not allowed. The problems are illustrated by numerical examples.

2. MATHEMATICAL FORMULATION

A multi-item inventory model is developed under the following notations and assumptions

2.1. Notations

W	goal associated to storage space
p_w	tolerance of W
n	number of items
Parameters for i -th ($i=1,2,\dots,n$) item are	
D_i	demand per unit item (decision variable) ($D \equiv (D_1, D_2, \dots, D_n)^T$)
Q_i	lot size per unit item (decision variable) ($Q \equiv (Q_1, Q_2, \dots, Q_n)^T$)
S_i	shortage level per unit item (decision variable) ($S \equiv (S_1, S_2, \dots, S_n)^T$)
C_{1i}	holding cost per unit item
C_{2i}	shortage cost per unit item
C_{3i}	set up cost

- C_{4i} production cost per unit item
- W_i storage space
- $TC_i(D_i, Q_i, S_i)$ total average cost function
- $SS(Q_i)$ function of total available storage area
- TC_{0i} goal of the objective function
- p_{0i} tolerance value for the goal TC_{0i}

2.2. Assumptions

1. production rate is instantaneous
2. unit production cost is taken here as inversely related to the demand of the item. For the i -th item, unit price $C_{4i} = \psi_i D_i^{-\beta_i}$ where scaling constant of C_{4i} be $\psi_i (>0)$ and degree of economies of scale be $\beta_i (>1)$.

Let for the i -th($i=1,2,\dots,n$) item the amount of stock is R_i at time $t = 0$. In the interval $(0, T_i (= t_{1i} + t_{2i}))$, the inventory level gradually decreases to meet demands. By this process the inventory level reaches zero level at time t_{1i} and then shortages are allowed to occur in the interval (t_{1i}, T_i) . The cycle then repeats itself.

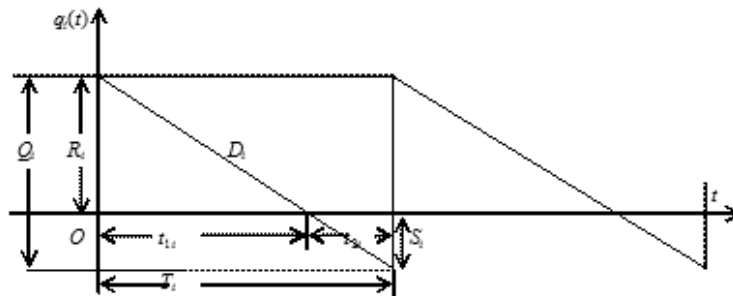


Figure-1: Inventory level of i -th item

The differential equation for the instantaneous inventory $q_i(t)$ at time t in $(0, T_i)$ is given by

$$\begin{aligned} \frac{dq_i(t)}{dt} &= -D_i && \text{for } 0 \leq t \leq t_{1i} \\ &= -D_i && \text{for } t_{1i} \leq t \leq T_i \end{aligned} \tag{1}$$

with the initial conditions $q_i(0) = R_i (= Q_i - S_i)$, $q_i(T_i) = -S_i$, $q_i(t_{1i}) = 0$.

For each period a fixed amount of shortage is allowed and there is a penalty cost C_{2i} per items of unsatisfied demand per unit time.

From (1), $q_i(t) = R_i - D_i t$ for $0 \leq t \leq t_{1i}$

$$= D_i(t_{1i} - t) \quad \text{for } t_{1i} \leq t \leq t_{2i}$$

$$\text{So, } D_i t_{1i} = R_i, \quad S_i = D_i t_{2i}, \quad Q_i = D_i T_i$$

$$\text{Holding cost} = C_{1i} \int_0^{t_{1i}} q_i(t) dt = \frac{C_{1i}(Q_i - S_i)^2}{2Q_i} T_i$$

$$\text{Shortage cost} = C_{2i} \int_{t_{1i}}^{T_i} (-q_i(t)) dt = \frac{C_{2i} S_i^2}{2Q_i} T_i$$

Therefore the total inventory cost = purchase cost + set up cost + holding cost + shortage cost

$$= C_{4i} Q_i + C_{3i} + C_{1i} \frac{(Q_i - S_i)^2}{2Q_i} T_i + \frac{C_{2i} S_i^2}{2Q_i} T_i$$

Total average inventory cost of the i -th item is

$$TC_i(D_i, Q_i, S_i) = \psi_i D_i^{1-\beta_i} + \frac{C_{3i} D_i}{Q_i} + \frac{(Q_i - S_i)^2}{2Q_i} C_{1i} + \frac{S_i^2}{2Q_i} C_{2i} \quad (2)$$

There is a limitation on the available warehouse floor space where the items are to be stored, i.e.

$$SS(Q) = \sum_{i=1}^n W_i Q_i \leq W.$$

The problem is to find demand levels, lot sizes, shortage amounts so as to minimize the total average inventory cost subject to the storage space restriction. It may be written as

$$\text{Min } TC_i(D_i, Q_i, S_i) \quad i = 1, 2, \dots, n \quad (3)$$

$$\text{subject to } SS(Q) \leq W, \quad D, Q, S > 0.$$

Fuzzy Model: In a multi-item inventory system, a manufacturing company may initially have a storage capacity $W \text{ m}^2$ to store the items. But in course of business, to take the advantage of special discount or minimum transportation cost, etc. he/she may have to augment the storage area, if the situation demands, i.e. in that case, the warehouse capacity becomes uncertain in non-stochastic sense and the storage area can be expressed by a fuzzy set. Depending upon different aspects, inventory cost parameters fluctuate. So, holding costs, set-up costs and shortage costs are assumed as fuzzy numbers. When the cost parameters are represented by fuzzy numbers and total average inventory cost and total storage space parameter are characterized by fuzzy sets. The crisp model (3) can be formulated as

$$\text{Min } T\tilde{C}(D_i, Q_i, S_i) = \psi_i D_i^{1-\beta_i} + \frac{\tilde{C}_{3i} D_i}{Q_i} + \frac{(Q_i - S_i)^2}{2Q_i} \tilde{C}_{1i} + \frac{S_i^2}{2Q_i} \tilde{C}_{2i} \quad (4)$$

subject to

$$SS(Q) \leq W, \quad D, Q, S > 0$$

where \sim represents the fuzzification of the parameters and goals. The fuzzy cost coefficients \tilde{C}_{ji} , $j = 1, 2, 3$, $i = 1, 2, \dots, n$ are represented by triangular shaped fuzzy numbers.

3. RANKING FUZZY NUMBER OF COST PARAMETERS WITH BEST APPROXIMATION INTERVAL

Fuzzy number: A real number \tilde{A} described as a fuzzy subset on the real line \mathfrak{R} whose membership function $\mu_{\tilde{A}}(x)$ has the following characteristics with $-\infty < a_1 \leq a_2 \leq a_3 < \infty$

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu_{\tilde{A}}^L(x) & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } x = a_2 \\ \mu_{\tilde{A}}^R(x) & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

where left branch membership function $\mu_{\tilde{A}}^L(x): [a_1, a_2] \rightarrow [0, 1]$ is continuous and strictly increasing; right branch membership function $\mu_{\tilde{A}}^R(x): [a_2, a_3] \rightarrow [0, 1]$ is continuous and strictly decreasing.

α -level set: The α -level set of a fuzzy number \tilde{A} is defined as a crisp set $A(\alpha)$ which is a non-empty bounded closed interval contained in X and can be denoted by $A(\alpha) = [A_L(\alpha), A_R(\alpha)] = [\inf\{x \in \mathfrak{R} : \mu_{\tilde{A}}(x) \geq \alpha\}, \sup\{x \in \mathfrak{R} : \mu_{\tilde{A}}(x) \geq \alpha\}]$ where $A_L(\alpha)$ and $A_R(\alpha)$ are the lower and upper bounds of the closed interval, respectively $\forall \alpha \in [0, 1]$.

Interval number: An interval number A is defined by an ordered pair of real numbers as follows $A = [a_L, a_R] = \{x : a_L \leq x \leq a_R, x \in \mathfrak{R}\}$ where a_L and a_R are the left and right bounds of interval A , respectively.

Here we want to approximate a fuzzy number by a crisp model. Suppose \tilde{A} and \tilde{B} are two fuzzy numbers with α -cuts that are $A(\alpha) = [A_L(\alpha), A_R(\alpha)]$ and $B(\alpha) = [B_L(\alpha), B_R(\alpha)]$ respectively. The distance $d(A(\alpha), B(\alpha))$ between $A(\alpha)$ and $B(\alpha)$ is given according to Wen and Quen (2005)

$$\begin{aligned}
 d^2(A(\alpha), B(\alpha)) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ \left[\frac{A_L(\alpha) + A_R(\alpha)}{2} + x(A_R(\alpha) - A_L(\alpha)) \right] \right. \\
 &\quad \left. - \left[\frac{B_L(\alpha) + B_R(\alpha)}{2} + x(B_R(\alpha) - B_L(\alpha)) \right] \right\}^2 dx \\
 &= \left(\frac{A_L(\alpha) + A_R(\alpha)}{2} - \frac{B_L(\alpha) + B_R(\alpha)}{2} \right)^2 + \frac{1}{12} [(A_R(\alpha) - A_L(\alpha)) - (B_R(\alpha) - B_L(\alpha))]^2
 \end{aligned}$$

Let the distance between fuzzy numbers \tilde{A} and \tilde{B} be defined by $D(\tilde{A}, \tilde{B})$

$$\text{where } D^2(\tilde{A}, \tilde{B}) = \frac{\int_0^1 d^2(A(\alpha), B(\alpha)) f(\alpha) d\alpha}{\int_0^1 f(\alpha) d\alpha}.$$

The weight function $f(\alpha)$ (> 0) is a continuous function defined on $[0, 1]$ $\forall \alpha \in (0, 1]$.

Best approximation intervals of fuzzy cost parameters:

The cost parameters \tilde{C}_{ji} are represented by fuzzy numbers. The α -level interval of \tilde{C}_{ji} is $\tilde{C}_{ji}(\alpha) = [C_{jiL}(\alpha), C_{jiR}(\alpha)]$, $\forall \alpha \in (0, 1]$. Since each interval is also a fuzzy number with constant α -cuts, we can find a best approximation interval $C_D(\tilde{C}_{ji}) = [C_{jiL}, C_{jiR}]$ which is nearest to \tilde{C}_{ji} with respect to metric D . Now we have to minimize $g(C_{jiL}, C_{jiR}) = D^2(\tilde{C}_{ji}, C_D(\tilde{C}_{ji}))$ i.e.

$$\begin{aligned}
 g(C_{jiL}, C_{jiR}) &= \int_0^1 \left\{ \left[\frac{1}{2}(C_{jiL}(\alpha) + C_{jiR}(\alpha)) - \frac{1}{2}(C_{jiL} + C_{jiR}) \right]^2 + \frac{1}{12} [(C_{jiR}(\alpha) - C_{jiL}(\alpha)) \right. \right. \\
 &\quad \left. \left. - (C_{jiR} - C_{jiL}) \right]^2 \right\} f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha
 \end{aligned}$$

with respect to C_{jiL} and C_{jiR} .

To solve the problem, we find partial derivatives for $g(C_{jiL}, C_{jiR})$ with respect to C_{jiL} and C_{jiR} .

$$\frac{\partial g(C_{jiL}, C_{jiR})}{\partial C_{jiL}} = -\frac{1}{3} \int_0^1 [2C_{jiL}(\alpha) + C_{jiR}(\alpha)] f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha + \frac{1}{3} (2C_{jiL} + C_{jiR}),$$

$$\frac{\partial g(C_{jiL}, C_{jiR})}{\partial C_{jiR}} = -\frac{1}{3} \int_0^1 [C_{jiL}(\alpha) + 2C_{jiR}(\alpha)] f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha + \frac{1}{3} (C_{jiL} + 2C_{jiR})$$

Solving

$$\frac{\partial g(C_{jiL}, C_{jiR})}{\partial C_{jiL}} = 0, \text{ and } \frac{\partial g(C_{jiL}, C_{jiR})}{\partial C_{jiR}} = 0.$$

$$\text{We have } C_{jiL} = \int_0^1 C_{jiL}(\alpha) f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha \text{ and } C_{jiR} = \int_0^1 C_{jiR}(\alpha) f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha.$$

$$\text{Since } \det(\nabla^2 g(C_{jiL}, C_{jiR})) = \frac{1}{3} > 0 \text{ and } \frac{\partial^2 g(C_{jiL}, C_{jiR})}{\partial C_{jiL}^2} = \frac{2}{3} > 0,$$

which ensures that $g(C_{jiL}, C_{jiR})$ is a strictly convex function. Therefore, the best approximation interval fuzzy number \tilde{C}_{ji} with respect to distance D is

$$C_D(\tilde{C}_{ji}) = \left[\int_0^1 C_{jiL}(\alpha) f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha, \int_0^1 C_{jiR}(\alpha) f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha \right].$$

Note: If $f(\alpha) = 1, \forall \alpha \in (0,1]$ the best approximation interval

$$C_D(\tilde{C}_{ji}) = \left[\int_0^1 C_{jiL}(\alpha) d\alpha, \int_0^1 C_{jiR}(\alpha) d\alpha \right] \text{ which was defined by Campos et. al. (1989).}$$

Ranking fuzzy numbers of cost parameters with best approximation interval:

The best approximation interval of \tilde{C}_{ji} is $[C_{jiL}, C_{jiR}]$. The ranking fuzzy number of the best approximation interval $[C_{jiL}, C_{jiR}]$ is defined as a convex combination of lower and upper boundary of the best approximation interval. Let $\lambda \in [0,1]$ is a pre-assigned parameter, called degree of optimism. Therefore, the ranking fuzzy number of \tilde{C}_{ji} is defined by $R_{\lambda, f}(\tilde{C}_{ji}) = \lambda C_{jiR} + (1 - \lambda) C_{jiL}$. A large value of $\lambda \in [0,1]$ specifies the higher degree of optimism. When $\lambda = 0$, $R_{0, f}(\tilde{C}_{ji}) = C_{jiL}$ expresses that the decision maker's viewpoint is completely pessimistic. When $\lambda = 1$, $R_{1, f}(\tilde{C}_{ji}) = C_{jiR}$ expresses that the decision maker's attitude is completely optimistic. When $\lambda = \frac{1}{2}$, $R_{\frac{1}{2}, f}(\tilde{C}_{ji}) = \frac{1}{2} [C_{jiR} + C_{jiL}]$ reflects moderately optimistic or neutral attitude of the decision maker. To find the ranking fuzzy numbers of $\tilde{C}_{ji}, i=1,2,\dots,n, j=1,2,3$ first, transform these fuzzy numbers into best approximation interval numbers

$C_D(\tilde{C}_{ji}) = [C_{jiL}, C_{jiR}]$ by means of the best approximation operator C_D . Then by using the convex combination of the boundaries of $C_D(\tilde{C}_{ji}) = [C_{jiL}, C_{jiR}]$, we change these interval numbers into real values. Ranking fuzzy numbers of \tilde{C}_{ji} is as follows:

$$R_{\lambda, f}(\tilde{C}_{ji}) = \int_0^1 [\lambda C_{jiR}(\alpha) + (1-\lambda)C_{jiL}(\alpha)] f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha$$

Taking $f(\alpha) = \alpha$, ($\forall \alpha \in (0,1)$) then

$$R_{\lambda, f}(\tilde{C}_{ji}) = 2 \int_0^1 \alpha [\lambda C_{jiR}(\alpha) + (1-\lambda)C_{jiL}(\alpha)] d\alpha$$

This is a ranking function introduced by Campos and Munoz (1989).

If $\tilde{C}_{ji} = (C_{ji1}, C_{ji2}, C_{ji3})$ is a linear fuzzy number (LFN) then $C_{jiL}(\alpha) = C_{ji1} + \alpha(C_{ji2} - C_{ji1})$ and $C_{jiR}(\alpha) = C_{ji3} - \alpha(C_{ji3} - C_{ji2})$.

The lower limit of the interval is

$$C_{jiL} = \int_0^1 C_{jiL}(\alpha) f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha = \frac{1}{3}(C_{ji1} + 2C_{ji2})$$

and the upper limit of the interval is

$$C_{jiR} = \int_0^1 C_{jiR}(\alpha) f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha = \frac{1}{3}(2C_{ji2} + C_{ji3}).$$

Therefore the interval number considering \tilde{C}_{ji} as a LFN is $\left[\frac{1}{3}(C_{ji1} + 2C_{ji2}), \frac{1}{3}(2C_{ji2} + C_{ji3}) \right]$ and the corresponding ranking fuzzy number is

$$R_{\lambda, f}(\tilde{C}_{ji}) = \frac{1}{3}[(1-\lambda)a_1 + 2a_2 + \lambda a_3], \quad \forall \lambda \in [0,1].$$

If \tilde{C}_{ji} is a parabolic fuzzy number (PFN), then $C_{jiL}(\alpha) = C_{ji2} - (C_{ji2} - C_{ji1})\sqrt{1-\alpha}$ and $C_{jiR}(\alpha) = C_{ji2} + (C_{ji3} - C_{ji2})\sqrt{1-\alpha}$ and the approximated interval is $\left[\frac{1}{15}(8C_{ji1} + 7C_{ji2}), \frac{1}{15}(7C_{ji2} + 8C_{ji3}) \right]$.

The ranking fuzzy number is $R_{\lambda, f}(\tilde{C}_{ji}) = \frac{1}{15}[8(1-\lambda)C_{ji1} + 7C_{ji2} + 8\lambda C_{ji3}]$, $\forall \lambda \in [0,1]$.

When \tilde{C}_{ji} is an exponential fuzzy number (EFN), then $C_{jiL}(\alpha) = C_{ji1} - \frac{(C_{ji2} - C_{ji1})}{\delta_1} \log\left(1 - \frac{\alpha}{\nu_1}\right)$ and $C_{jiR}(\alpha) = C_{ji3} + \frac{(C_{ji3} - C_{ji2})}{\delta_2} \log\left(1 - \frac{\alpha}{\nu_2}\right)$

where $\delta_1, \delta_2 > 0$, $\nu_1, \nu_2 > 1$

and the approximated interval is

$$\left[C_{ji1} + \frac{(C_{ji2} - C_{ji1})}{\delta_1} \left\{ (\nu_1^2 - 1) \log\left(1 - \frac{1}{\nu_1}\right) + \nu_1 + \frac{1}{2} \right\}, C_{ji3} - \frac{(C_{ji3} - C_{ji2})}{\delta_2} \left\{ (\nu_2^2 - 1) \log\left(1 - \frac{1}{\nu_2}\right) + \nu_2 + \frac{1}{2} \right\} \right]$$

The ranking fuzzy number is $R_{\lambda, f}(\tilde{C}_{ji}) = \lambda C_{jiR} + (1 - \lambda) C_{jiL}$, $\forall \lambda \in [0, 1]$.

4. GEOMETRIC PROGRAMMING TECHNIQUE TO SOLVE FUZZY INVENTORY PROBLEM

The triangular shaped fuzzy numbers \tilde{C}_{ji} are represented by $\tilde{C}_{ji} = (C_{ji1}, C_{ji2}, C_{ji3})$ for $j = 1, 2, 3$ then the objective functions are represented by $\tilde{TC}_i = (TC_{i1}, TC_{i2}, TC_{i3})$, $i = 1, 2, \dots, n$ where

$$TC_{ik} = \psi_i D_i^{1-\beta_i} + \frac{C_{3ik} D_i}{Q_i} + \frac{(Q_i - S_i)^2}{2Q_i} C_{1ik} + \frac{S_i^2}{2Q_i} C_{2ik}, \quad k = 1, 2, 3.$$

Now using ranking fuzzy numbers cost parameters, objective function becomes

$$R_{\lambda, f}(\tilde{TC}_i) = \psi_i D_i^{1-\beta_i} + \frac{R_{\lambda, f}(\tilde{C}_{3i}) D_i}{Q_i} + \frac{R_{\lambda, f}(\tilde{C}_{1i}) (Q_i - S_i)^2}{2Q_i} + \frac{R_{\lambda, f}(\tilde{C}_{2i}) S_i^2}{2Q_i}.$$

According to Werners (1987) the objective functions should be fuzzy in nature because of fuzzy inequality constraints. So, for given $\lambda \in [0, 1]$, (4) is equivalent to the following fuzzy goal programming problem

Find D, Q, S

$$\text{subject to } R_{\lambda, f}(\tilde{TC}_i) \leq TC_{0i}, \quad \text{for } i = 1, 2, \dots, n \quad (5)$$

$$SS(Q) \leq W, \quad D, Q, S > 0.$$

In this formulation, it is assumed that the manufacturer has a target of expenditure TC_{01} for the first item. As before, it may happen that in course of business, he / she may be compelled to augment some more capital to spend more; say, p_{01} for the first item to take some business advantages, if such a situation occurs. Similar cases may also happen for other items. Here, we assume that the objective goals are imprecise

having a minimum targets TC_{01}, \dots, TC_{0n} with positive tolerances p_{01}, \dots, p_{0n} for $\lambda \in [0,1]$. The above stated multi-item, multi-objective fuzzy EOQ model is solved by GP method.

In fuzzy set theory, the imprecise objectives and constraints are defined by their membership functions, which also may be linear and / or non-linear. For simplicity, we assume here $\mu_i(R_{\lambda,f}(T\tilde{C}_i))$ and $\mu_{SS}(Q)$ are linear membership functions for the n -objectives and constraint respectively. These are

$$\mu_i(R_{\lambda,f}(T\tilde{C}_i)) = \begin{cases} 0 & \text{for } R_{\lambda,f}(T\tilde{C}_i) > TC_{0i} + p_{0i} \\ 1 - \frac{R_{\lambda,f}(T\tilde{C}_i) - TC_{0i}}{p_{0i}} & \text{,, } TC_{0i} \leq R_{\lambda,f}(T\tilde{C}_i) \leq TC_{0i} + p_{0i} \\ 1 & \text{,, } R_{\lambda,f}(T\tilde{C}_i) < TC_{0i} \end{cases}$$

for $i = 1, 2, \dots, n$
and it is graphically represented as

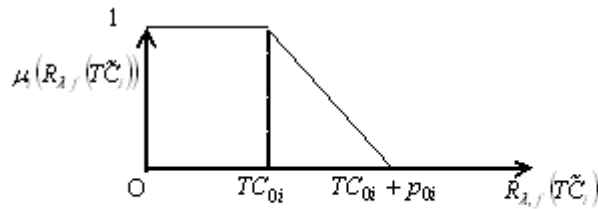


Figure-2: Membership function of $R_{\lambda,f}(T\tilde{C}_i)$

$$\text{and } \mu_{SS}(Q) = \begin{cases} 0 & \text{for } \sum_{i=1}^n W_i Q_i > W + p_W \\ 1 - \frac{\sum_{i=1}^n W_i Q_i - W}{p_W} & \text{,, } W \leq \sum_{i=1}^n W_i Q_i \leq W + p_W \\ 1 & \text{,, } \sum_{i=1}^n W_i Q_i < W, \end{cases}$$

where p_W is the admissible tolerance of total space.

Following Bellman and Zadeh's (1970), max-min operator or convex combination operator the fuzzy goal programming problem (5) may be reduced to a crisp Primal Geometric Programming (PGP) problem. To reduce the DD, here convex combination operator is used. So, the problem (5) can be formulated as

$$\text{Max } V(D, Q, S) = \sum_{i=1}^n \sigma_i \mu_i(R_{\lambda, f}(T\tilde{C}_i)) + \sigma_S \mu_{SS}(Q)$$

subject to

$$\mu_i(R_{\lambda, f}(T\tilde{C}_i)) = 1 - \frac{R_{\lambda, f}(T\tilde{C}_i) - TC_{0i}}{p_{0i}},$$

$$\mu_{SS}(Q) = 1 - \frac{\sum_{i=1}^n w_i Q_i - W}{p_W}, \quad (6)$$

$$\mu_i(R_{\lambda, f}(T\tilde{C}_i)), \mu_{SS}(Q) \in [0, 1], \quad i = 1, 2, \dots, n.$$

Here σ_i, σ_S may be taken as positive normalized preference values (i.e. weights) of objective functions, storage space restrictions respectively, i.e.

$$\sum_{i=1}^n \sigma_i + \sigma_S = 1.$$

Problem (6) is equivalent to

$$\text{Min } U(D, Q, S) = \sum_{i=1}^n U_i(D_i, Q_i, S_i) \quad (7)$$

subject to $D_i, Q_i, S_i > 0$

$$\text{where } V(D, Q, S) = \sum_{i=1}^n \left(\sigma_i \left(1 + \frac{TC_{0i}}{p_{0i}} \right) + \sigma_S \left(1 + \frac{W}{p_W} \right) \right) - U(D, Q, S)$$

$$\text{and } U_i(D_i, Q_i, S_i) = \theta_{1i} D_i^{1-\beta_i} + \frac{\theta_{2i} D_i}{Q_i} + \theta_{3i} Q_i + \frac{\theta_{4i} S_i^2}{Q_i} - \theta_{5i} S_i,$$

$$\theta_{1i} = \frac{\sigma_i \psi_i}{p_{0i}}, \quad \theta_{2i} = \frac{\sigma_i R_{\lambda, f}(\tilde{C}_{3i})}{p_{0i}}, \quad \theta_{3i} = \frac{\sigma_i R_{\lambda, f}(\tilde{C}_{1i})}{2p_{0i}} + \frac{\sigma_S \omega_i}{p_W},$$

$$\theta_{4i} = \frac{\sigma_i}{2p_{0i}} (R_{\lambda, f}(\tilde{C}_{1i}) + R_{\lambda, f}(\tilde{C}_{2i})), \quad \theta_{5i} = \frac{\sigma_i}{p_{0i}} R_{\lambda, f}(\tilde{C}_{1i}), \quad i = 1, 2, \dots, n.$$

Problem (7) is an unconstrained signomial PGP problem with $DD = (2n - 1)$, which is difficult to solve by formulating its Dual Problem (DP) for higher values of n . MGP method can be used to reduce the DD. Consider only the terms of the i -th function ($i = 1, 2, \dots, n$) instead of all the terms of the objective function. Here number of terms in the i -th function is 5 and the number of decision variables (namely D_i, Q_i, S_i) of that function is 3. So, DD is now reduced to 1. Under this consideration, the corresponding DP is

$$\text{Max } dw(w_1, \dots, w_5) = \prod_{i=1}^n \left[\left(\frac{\theta_{1i}}{w_{1i}} \right)^{w_{1i}} \left(\frac{\theta_{2i}}{w_{2i}} \right)^{w_{2i}} \left(\frac{\theta_{3i}}{w_{3i}} \right)^{w_{3i}} \left(\frac{\theta_{4i}}{w_{4i}} \right)^{w_{4i}} \left(\frac{\theta_{5i}}{w_{5i}} \right)^{-w_{5i}} \right] \quad (8)$$

subject to the normality and orthogonality conditions

$$w_{1i} + w_{2i} + w_{3i} + w_{4i} - w_{5i} = 1,$$

$$(1 - \beta_i)w_{1i} + w_{2i} = 0,$$

$$-w_{2i} + w_{3i} - w_{4i} = 0,$$

$$2w_{4i} - w_{5i} = 0$$

where $w_{ki} > 0$ for $i = 1, 2, \dots, n$ and $k = 1, 2, \dots, 5$,

Solving the above linear equations in terms of w_{3i} ,

$$w_{1i} = \frac{1}{2\beta_i - 1}, \quad w_{2i} = \frac{\beta_i - 1}{2\beta_i - 1}, \quad w_{4i} = w_{3i} - \frac{\beta_i - 1}{2\beta_i - 1} \quad \text{and} \quad w_{5i} = 2w_{3i} - \frac{2(\beta_i - 1)}{2\beta_i - 1}$$

for $i = 1, 2, \dots, n$.

The dual variables $w_{ki} > 0$ ensures that $\beta_i > 1$.

Putting the values of dual variables w_{ki} , $i = 1, 2, \dots, n$, $k = 1, 2, \dots, 5$ into the dual function of (8),

$$\text{Max } dw(w_3) = \prod_{i=1}^n \left((2\beta_i - 1)\theta_{1i} \right)^{\frac{1}{2\beta_i - 1}} \left(\frac{(2\beta_i - 1)\theta_{2i}}{\beta_i - 1} \right)^{\frac{\beta_i - 1}{2\beta_i - 1}} \left(\frac{\theta_{3i}}{w_{3i}} \right)^{w_{3i}} \left(\frac{\theta_{4i}}{w_{3i} - \frac{\beta_i - 1}{2\beta_i - 1}} \right)^{w_{3i} - \frac{\beta_i - 1}{2\beta_i - 1}} \times \left(\frac{2w_{3i} - \frac{2(\beta_i - 1)}{2\beta_i - 1}}{\theta_{5i}} \right)^{2w_{3i} - \frac{2(\beta_i - 1)}{2\beta_i - 1}} \quad (9)$$

subject to $w_{3i} > 0$, $i = 1, 2, \dots, n$.

To find the optimal value of (9), differentiate the dual function $dw(w_3)$ with respect to w_{3i} and then setting to zero i.e., $\frac{\partial \log dw(w_3)}{\partial w_{3i}} = 0$ yields

$$w_{3i}^* = \frac{4\theta_{3i}\theta_{4i}(\beta_i - 1)}{(4\theta_{3i}\theta_{4i} - \theta_{5i}^2)(2\beta_i - 1)}. \quad \text{As } w_{3i}^* > 0, \quad \text{the optimality criteria is } 4\theta_{3i}\theta_{4i} > \theta_{5i}^2 \quad \text{and}$$

$$\beta_i > 1. \quad \text{Using the values of } w_{3i}^*, \quad \text{other optimal dual variables are } w_{1i}^* = \frac{1}{2\beta_i - 1},$$

$$w_{2i}^* = \frac{\beta_i - 1}{2\beta_i - 1}, \quad w_{4i}^* = \frac{\theta_{5i}^2(\beta_i - 1)}{(4\theta_{3i}\theta_{4i} - \theta_{5i}^2)(2\beta_i - 1)} \quad \text{and} \quad w_{5i}^* = \frac{2\theta_{5i}^2(\beta_i - 1)}{(4\theta_{3i}\theta_{4i} - \theta_{5i}^2)(2\beta_i - 1)}$$

for $i = 1, 2, \dots, n$.

The optimal values of the primal function and decision variables are obtained from the relations

$$U^* = n(dw^*)^{\frac{1}{n}},$$

$$\begin{aligned} \frac{\theta_{1i} D_i^{*1-\beta_i}}{w_{1i}^*} &= \frac{U^*}{n}, \\ \frac{\theta_{3i} Q_i^*}{w_{3i}^*} &= \frac{U^*}{n}, \quad \text{for } i = 1, 2, \dots, n \\ \frac{\theta_{5i} S_i^*}{w_{5i}^*} &= \frac{U^*}{n}. \end{aligned}$$

which give optimal values of the decision variables as

$$\begin{aligned} D_i^* &= \left(\frac{w_{1i}^* U^*}{n \theta_{1i}} \right)^{\frac{1}{1-\beta_i}}, \\ Q_i^* &= \frac{w_{3i}^* U^*}{n \theta_{3i}}, \quad \text{for } i = 1, 2, \dots, n \\ S_i^* &= \frac{w_{5i}^* U^*}{n \theta_{5i}} \end{aligned}$$

With the help of the above optimum decision variables D^* , Q^* , S^* we can obtain the optimal values of the cost functions $TC_i^*(D_i^*, Q_i^*, S_i^*)$, $i = 1, 2, \dots, n$ for given $\lambda (\in [0, 1])$.

Special Case (fuzzy inventory model without shortages):

The fuzzy inventory model without shortages (i.e. $S_i = 0$) may be considered as a special case of (4) by allowing $\tilde{C}_{2i} \rightarrow \infty$. In that case the problem (4) is reduced to

$$\text{Min } T\tilde{C}_i(D_i, Q_i, S_i) = \psi_i D_i^{1-\beta_i} + \frac{\tilde{C}_{3i} D_i}{Q_i} + \frac{Q_i}{2} \tilde{C}_{1i}, \quad i = 1, 2, \dots, n$$

subject to $SS(Q) \leq W$, $D, Q, S > 0$.

The dual variables of the corresponding dual problem are reduced to $w_{1i}^* = \frac{1}{2\beta_i - 1}$, $w_{2i}^* = \frac{\beta_i - 1}{2\beta_i - 1} = w_{3i}^*$, $w_{4i}^* = w_{5i}^* = 0$ for $i = 1, 2, \dots, n$.

Optimal value of dual function is

$$dw_0^* = \prod_{i=1}^n \left[\theta_{1i} (\beta_i - 1) \left\{ \frac{\theta_{2i} \theta_{3i} (2\beta_i - 1)^2}{(\beta_i - 1)^2} \right\}^{\beta_i - 1} \right]^{\frac{1}{2\beta_i - 1}}$$

Optimal values of the primal function and decision variables are obtained from the following relations

$$U_0^* = n(dw_0^*)$$

$$D_i^* = \left(\frac{w_{1i}^* U_0^*}{n \theta_{1i}} \right)^{\frac{1}{1 - \beta_i}}, \text{ for } i = 1, 2, \dots, n$$

$$Q_i^* = \frac{w_{3i}^* U_0^*}{n \theta_{3i}},$$

With the help of the above optimum decision variables D_i^* , Q_i^* we can obtain the optimal values of the cost functions $TC_i^*(D_i^*, Q_i^*)$, $i = 1, 2, \dots, n$ for given $\lambda \in [0, 1]$.

5. NUMERICAL EXAMPLE

A manufacturing company produces two types of machines A and B in lots. The company has a warehouse whose total floor area is $(W) = 300 \text{ m}^2$. The company has also an additional stock capacity $(p_W) = 100 \text{ m}^2$ to store any excess spare parts, if necessary. From the past records, it was found that the production cost of the two machines are $15000D_1^{-1.7}$ and $18000D_2^{-1.8}$ respectively, where D_1 and D_2 are the monthly demands of the corresponding items. The holding cost of the machine A is near about \$0.8 but never less than \$0.3 and never above than \$1.3 (i.e. $\tilde{c}_{11} \equiv \$(0.3, 0.8, 1.3)$). Similarly, holding cost of machine B is $\tilde{c}_{12} \equiv \$(0.2, 0.5, 1)$. The shortage and set up costs of two machines are $\tilde{c}_{21} \equiv \$(12, 20, 25)$, $\tilde{c}_{22} \equiv \$(15, 25, 30)$ and $\tilde{c}_{31} \equiv \$(50, 75, 100)$, $\tilde{c}_{32} \equiv \$(70, 100, 150)$ respectively. The space required for two types of machines are 1.6 m^2 and 1.2 m^2 respectively. The authority decides to spend \$470 to produce machine A and \$380 to produce machine B and allows a tolerance of \$200 for each machine.

Table 1: Left and right branches of fuzzy cost parameters

Branch	\tilde{C}_{11}	\tilde{C}_{12}	\tilde{C}_{21}	\tilde{C}_{22}	\tilde{C}_{31}	\tilde{C}_{32}
Left	L	E	L	P	E	P
Right	E	P	L	L	L	E

L, P, E stands for linear, parabolic and exponential membership functions respectively.

Table 2: Values of (ν_1, δ_1) and (ν_2, δ_2) for the membership functions of $\tilde{C}_{11}, \tilde{C}_{12}, \tilde{C}_{31}, \tilde{C}_{32}$

Branch	\tilde{C}_{11}	\tilde{C}_{12}	\tilde{C}_{31}	\tilde{C}_{32}
Left: (ν_1, δ_1)	-	(1.4,1.5)	(1.4,1.2)	-
Right: (ν_2, δ_2)	(1.2,1.6)	-	-	(1.6,1.4)

Table 3: Best approximation interval of fuzzy cost parameters

\tilde{C}_{11}	\tilde{C}_{12}	\tilde{C}_{21}	\tilde{C}_{22}	\tilde{C}_{31}	\tilde{C}_{32}
[0.633, 1.015]	[0.339, 0.767]	[17.333, 21.667]	[19.667, 26.667]	[64.528, 83.333]	[84, 129.646]

Table 4: Optimal Solution for the fuzzy model with shortages (taking $\lambda = 0.6$)

Preference values $(\sigma_1, \sigma_2, \sigma_S)$	i	D_i^*	Q_i^*	S_i^*	TC_i^* (\$)
Equal preferences on objectives and constraint					
(1/3,1/3,1/3)	1	216.4252	67.38566	2.794497	619.1748
	2	176.2651	85.42286	2.080490	541.9718
More preferences to the 1 st objective function than the other					
(0.5,0.3,0.2)	1	296.7979	115.2730	4.780396	521.6874
	2	201.8865	109.0606	2.656193	495.6344
More preferences to the 2 nd objective function than the other					
(0.3,0.5,0.2)	1	250.3887	86.33546	3.580349	569.6456
	2	236.6665	145.1848	3.536004	450.7372

Table 5: Optimal Solution for the fuzzy model without shortages (taking $\lambda = 0.6$)

Preference values $(\sigma_1, \sigma_2, \sigma_S)$	i	D_i^*	Q_i^*	TC_i^* (\$)
Equal preferences on objectives and constraint				
(1/3,1/3,1/3)	1	215.9805	67.15043	621.1304
	2	176.0826	85.26376	542.9728
More preferences to the 1 st objective function than the other				
(0.4,0.3,0.3)	1	239.5784	80.09495	585.2520
	2	176.0826	85.26376	542.9728
More preferences to the 2 nd objective function than the other				
(0.3,0.5,0.2)	1	312.3595	125.7353	511.6022
	2	296.1940	217.4263	406.1861

6. CONCLUSION

In this paper a multi-objective inventory problem with shortages along-with space constraint is formulated. In real life inventory control system, the inventory cost parameters such as holding cost are very often defined as ‘holding cost is about $\$C_{11}$ / near $\$C_{11}$ / more-or-less $\$C_{11}$ ’ etc; that is, the inventory cost parameters are fuzzy in nature. The cost components are considered here as triangular shaped fuzzy numbers with linear, parabolic and exponential type of left and right membership functions. These fuzzy numbers are then defined by ranking fuzzy numbers with respect to the best approximation interval number. The objective goals and constraint goal are not precise. The authority allows some flexibility to attain his target. The company can achieve its target varying the level of optimistic value (λ) from 0 and 1. The model is illustrated with a practical example (manufacturing company). MGP method is used here to solve the problem. We have also derived the model without shortages as a special case of the original problem. Moreover, it is to be noted that the application of classical GP method leads to the problem with $(2n-1)$ DD whereas MGP method reduces the DD to 1, which does not create any problem for solution. The model can be easily extended to generic inventory problems with other constraints. The method presented here is quite general and can be applied to the real inventory problems faced by the practitioners in industry or in other areas. This method may be applied to several type of fuzzy model in engineering optimization (like structural optimization).

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