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AN INTEGRATED SUPPLY CHAIN INVENTORY MODEL WITH IMPERFECT-QUALITY ITEMS, CONTROLLABLE LEAD TIME AND DISTRIBUTION-FREE DEMAND

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Abstract: In this paper, we consider an integrated vendor-buyer inventory policy for a continuous review model with a random number of defective items and screening process gradually at a fixed screening rate in buyer's arriving order lot. We assume that shortages are allowed and partially backlogged on the buyer's side, and that the lead time demand distribution is unknown, except its first two moments. The objective is to apply the minmax distribution free approach to determine the optimal order quantity, reorder point, lead time and the number of lots delivered in one production run simultaneously so that the expected total system cost is minimized. Numerical experiments along with sensitivity analysis were performed to illustrate the effects of parameters on the decision and the total system cost.

Keywords: Integrated model, minmax distribution free approach, defective items, controllable lead time.

MSC: 90B05.

1. INTRODUCTION

In recent years, most inventory problems have their focus on the integration between the vendor and the buyer. For supply chain management, establishing long-term strategic partnerships between the buyer and the vendor is advantageous for the two parties regarding costs, and therefore profits since both parties, to achieve improved benefits, cooperate and share information with each other. Several researchers have shown that the buyer and the vendor can achieve their own minimal total cost, or increase their mutual benefit through strategic cooperation with each other. Goyal [7] first developed an integrated inventory model for a single supplier-single buyer problem. In his study, the joint approach to the inventory problem faced by a single supplier-single buyer of a product has been formulated with the help of an integrated inventory model. Later, Banerjee [2] generalized Goyal's model [7] and developed a joint economic-lot-size model for the case in which a vendor produces on an order of a buyer on a lot-for-lot basis under deterministic conditions. Then Goyal [8] extended Banerjee's model [2] and suggested that the vendor's economic production quantity per cycle should be a positive integer multiple of the buyer's purchase quantity. A review of related literature on buyer-vendor coordination models prior to 1989 is given in [9].

Lu [20] relaxed the assumption [8] about completing a batch before starting shipments, and investigated a model that allowed shipments to take place during production, and the delivery quantity to the buyer is identical. Ha and Kim [11] further modified Goyal's model [8] and proposed an integrated JIT lot-splitting model to facilitate multiple shipments in small lots. In the same year, Hill [13] proposed a more general shipping policy for an integrated production inventory model by considering successive shipment sizes increased by a general fixed factor. And then, Hill [14] derived a globally-optimal batching and shipping policy for the single-vendor single-buyer integrated production-inventory problem. Goyal and Nebebe [10] further proposed an integrated inventory model in which the first shipment is smaller and is followed by shipments of the equal size. Pan and Yang [27] improved Goyal's model [8] by considering lead time as a decision variable, and obtained a lower joint total expected cost and shorter lead time. Recently, Ouyang et al. [23] extended Pan and Yang's model [27] by simultaneously optimizing ordering quantity, reorder point, lead time and the number of lots delivered in one production cycle. The aforementioned research on the integration vendor-buyer inventory problem focused on the production shipment schedule, in terms of the number and the size of batches transferred between both parties, neglecting the relationship between order lot and quality. A common unrealistic assumption of the above joint inventory models is that all the produced items are of good quality. However, as a result of imperfect production processes of the vendor, damage in transit, or other unforeseeable circumstances, an order lot arriving at the buyer often contains defective items. These defective items will affect the on hand inventory level, customer service level and the frequency of orders in the inventory system. So, production/shipment policy determined by conventional integrated inventory models may be inappropriate for the situation where an arriving lot contains some defective items. Therefore, it is worthwhile studying the effect of defective items on inventory problem. Since the pioneering work by Porteus [30] and Rosenblatt and Lee [31], in order to surmount the common unrealistic assumption of good quality, many researchers have attempted to develop various imperfect-quality inventory models on this important issue. Paknejad et al. [25] derived a modified EOQ model with constant lead time and stochastic demand (exponentially and uniformly distributed demand during lead time), and considered the number of non-defective items in a lot as a random variable. In their paper, the shortages are allowed and fully backordered, and the defective items in each lot are discovered and returned to the vendor at the time of delivery of the next lot. Wu and Ouyang [45] incorporated the assumption of a mixture of backorders and lost sales and variable lead time into Paknejad et al.'s model [25] and assumed that all goods are quickly inspected. There are more papers related to this issue such as

[1,5,17,29,35,40,44,46], and others. Though, the aforementioned inventory models tackled defective items focused on determining an optimal policy just from either the buyer's or the vendor's point of view. They considered just one-sided optimal inventory policies that neglected the complicated interaction and cooperation opportunity between the vendor and the buyer. This one-sided optimal strategy can be improved through forming an effective alliance with other parties. Huang [15,16] considered an integrated vendor-buyer cooperative inventory model for items with imperfect quality under equal-shipment policy, and assumed that the number of defective items follows a given probability density function, and that the vendor treats defective items as a single batch at the end of the buyer's 100% screening process. However, both shortages and lead time reduction were not considered. Recently, Ouyang et al. [24] developed an integrated inventory systems with fixed defective rate, and assumed that the buyer performs a 100% screening process immediately on receiving a lot, i.e., the length of inspection period is neglected here, and the vendor treats defective items as a single batch at the end of the buyer's 100% screening process. We notice that the reorder point and shortages were not considered. In many practical situations, lead time can be reduced, by an additional crashing cost, customer service level improved, inventory in safety stocks reduced, and the competitive edge in business increased; in other words, it is controllable. In addition, the information about the probability distribution of the lead time demand is often quite limited. There are many related studies such as [4,26,28,32,33,34,36,37,38,39,41,42], etc.

Based on the survey above, in this paper, we extend Wu and Ouyang's model [45] (the inspection process is considered to be a rapid action) and Ouyang et al.'s model [23] by considering an integrated supply chain inventory model with gradual screening process at a fixed screening rate for a random number of defective items in buyer's arriving order lot, in which the buyer's order quantity, reorder point, lead time and the number of lots delivered in one production cycle are decision variables. We assume that an arriving order lot may contain some defective items, and that the number of defective items is a binomial random number. Upon the arrival of an order, the buyer performs a non-destructive and error-free screening process gradually at a fixed screening rate on receiving a lot before selling, rather than inspecting through a rapid action; and all defective items in each lot are assumed to be discovered and returned to the vendor at the time of delivery of the next lot. So, the buyer will have two kinds of holding cost: non-defective items holding cost and defective items holding cost. Besides, the basic setting is a single-product continuous-review inventory system with a distribution free lead time demand; replenishments are made whenever the inventory level reaches the reorder point and also, as in [19], we assume that lead time is controllable and shortages, during the lead time, allowed. The purpose of this paper is to simultaneously optimize ordering quantity, reorder point, lead time and the number of lots delivered in one production cycle by using the minmax distribution free approach, originally addressed by Scarf [43] (popularized by Gallego and Moon [6]). In addition, we develop an algorithmic procedure to determine the optimal inventory policy. Finally, numerical experiments along with sensitivity analysis were performed to illustrate the effects of parameters on the decision and the total system cost.

The remainder of this paper is organized as follows. Section 2 details the notation and assumptions. In Section 3, we formulate the integrated inventory model involving imperfect-quality items and controllable lead time, and then develop an algorithmic procedure to find the optimal solution. Section 4 provides a numerical example and discussion of the results. In Section 5, we draw some conclusions and give suggestions for some future research.

2. NOTATION AND ASSUMPTIONS

The following notation and assumptions are used, throughout the paper, to develop the proposed models.

Notation:

S	Vendor's set-up cost per set-up
D	Expected demand per unit time on the buyer (for non-defective items)
P	Production rate on the vendor
A	Buyer's ordering cost per order
F	Transportation cost per delivery
ω	Vendor's unit treatment cost of defective items
π	Buyer's shortage cost per unit short
π_0	Buyer's marginal profit (i.e., cost of lost demand) per unit
β	Fraction of the demand during the stock-out period that will be backordered, $\beta \in [0,1]$
h_v	Vendor's holding cost per item per unit time
h_{b1}	Buyer's holding cost per non-defective item per unit time
h_{b2}	Buyer's holding cost per defective item per unit time, $h_{b2} \le h_{b1}$
s	Buyer's unit screening cost
x	Buyer's screening rate
Q	Order quantity of the buyer for non-defective items (decision variable)
q	Order quantity of the buyer per order including defective items, i.e., shipping quantity from the vendor to the buyer per shipment (decision variable)
r	Reorder point of the buyer for non-defective items (decision variable)
L	Length of lead time for the buyer (decision variable)
n	The number of lots in which the product is delivered from the vendor to the buyer in one production run, a positive integer (decision variable)
X	The lead time demand which has a p.d.f. f_X with finite mean DL and standard deviation $\sigma\sqrt{L}$, where σ denotes the standard deviation of the demand per unit time
Y	The number of defective items in a lot size q is a random variable
$E[\cdot]$	Mathematical expectation
x^+	Maximum value of x and 0, i.e., $x^+ = \max\{x, 0\}$
$*$	The superscript representing optimal value

Assumptions

1. There is single-vendor and single-buyer for a single-product in this model.

2. Inventory is continuously reviewed. The buyer places an order or requests for successive shipments when on hand inventory level (based on the number of non-defective items) falls to the reorder point r.
3. The reorder point r = expected demand during lead time + safety stock (SS), and $SS = k \times$ (standard deviation of lead time demand), that is, $r = DL + k\sigma\sqrt{L}$, where k is known as the safety factor.
4. The lead time L consists of m mutually independent components. The i th component has the normal duration b_i, the minimum duration a_i and the crashing cost per unit time c_i. Furthermore, these c_i are assumed to be arranged such that $c_1 \le c_2 \le \cdots \le c_m$.
5. The components of lead time are crashed one at a time starting with the component of least c_i, and so on.
6. If we let L_i be the length of lead time with components 1, 2, ..., i crashed to their minimum duration, then $L_{\min} = \sum_{i=1}^{m} a_i \le L \le \sum_{i=1}^{m} b_i = L_{\max}$, $L_i = L_{\max} - \sum_{j=1}^{i} (b_j - a_j)$, and the lead time crashing cost per cycle $C(L)$ for a given $L \in (L_i, L_{i-1}]$ is given by $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.
7. The extra costs incurred by the vendor will be fully transferred to the buyer if shortened lead time is requested.
8. The buyer orders a lot of size Q (for non-defective items) and will receive the batch quantity in n equally-sized shipments of size q, where n is a positive integer.
9. An arriving lot may contain some defective items. We assume that the number of defective items, Y, in an arriving order of size q is a random variable which has a binomial distribution with parameters q and γ where γ ($0 \le \gamma < 1$) represents the defective rate in the order lot. Upon the arrival of the order, all the items in the lot are inspected with the screening rate x by the buyer, and defective items in each lot are discovered and returned to the vendor at the time of delivery of the next lot.
10. Vendor's production rate for the non-defective items is greater than buyer's demand rate, i.e., $(1-\gamma)P > D$.
11. The screening process and the demand proceed simultaneously, but the screening rate is greater than the demand rate.

3. THE BASIC MODEL

The information about the form of d.f. of the lead time demand is often limited in practice. Thus, the conventional assumption of a full knowledge about the form of d.f.s for lead time demand may not provide the best protection against the occurrence of other distributions. Thus, in this section, we establish an integrated vendor-buyer inventory model involving a mixture of backorders and lost sales, inspections of defective items and controllable lead time when only the first two moments of the demand distribution are known. During the production run, as soon as the first q units have been produced, the vendor will deliver them to the buyer. After that, the vendor will make a delivery, on average, every $E[(q-Y)/D]$ units of time until the inventory level falls to zero, where we have assumed that each lot contains a random number of defectives, Y. Upon order arrival, the buyer inspects all the items at the fixed screening rate, x, and all defective items in each lot are discovered and returned to the vendor at the time of delivery of the next lot. In order to reduce the production cost, the vendor manufactures nq at one set-up with a finite production rate P when the buyer orders quantity q, and each batch is dispatched to the buyer in n equally-sized shipments, where n is a positive integer. Therefore, the expected length of each ordering cycle for the buyer is $E[T]=E[(q-Y)/D]$, and the expected length of each production cycle for the vendor is $E[nT]=nE[(q-Y)/D]$.

3.1 Buyer's expected average total cost per unit time

In this paper, the basic setting is a continuous-review inventory system, and we have assumed that shortages are allowed. An order of size q for successive shipment is placed as soon as the buyer's inventory position (based on the number of non-defective items) reaches the reorder point, r. From assumption 3, we can also consider the safety factor k as a decision variable instead of r. Therefore, the expected shortage quantity at the end of the cycle is given by $E[(X-r)^+]$. Thus, the expected number of backorders per ordering cycle is $\beta E[(X-r)^+]$ and the expected loss in sales per ordering cycle is $(1-\beta)E[(X-r)^+]$. Thus, the stock-out cost per ordering cycle is $[\pi+\pi_0(1-\beta)]E[(X-r)^+]$. Upon order arrival, the buyer inspects all the items at the fixed screening rate, x, and all defective items in each lot are discovered and returned to the vendor at the time of delivery of the next lot. Therefore, the buyer has two kinds of holding costs: non-defective items holding cost and defective items holding cost. The average inventory level of non-defective items (involving those defective items which are not detected in a flaw yet before the end of the screening time, q/x) of q units order per cycle, given that there are y defective items in an arriving order of size q, can be approximated by

$$\begin{aligned}&\frac{qy}{2x(q-y)/D}+\left\{\frac{q-y}{2}+r-DL+(1-\beta)E[(X-r)^+]\right\}\\&=\frac{Dqy}{2x(q-y)}+\frac{q-y}{2}+k\sigma\sqrt{L}+(1-\beta)E[(X-r)^+].\end{aligned} \tag{1}$$

Hence, the non-defective holding cost per cycle is

$$h_{b1}\left(\frac{q-y}{D}\right)\left\{\frac{Dqy}{2x(q-y)}+\frac{q-y}{2}+k\sigma\sqrt{L}+(1-\beta)E[(X-r)^{+}]\right\}. \quad (2)$$

The buyer’s defective inventory pattern is shown in Figure 1.

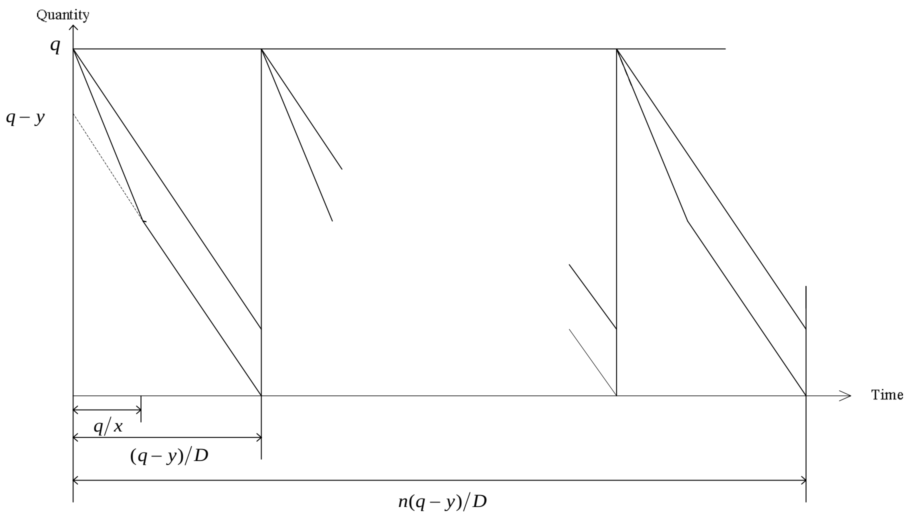

Figure 1: Defective inventory pattern of the buyer

Similarly, given that there are y defective items in an arriving order of size q, the buyer’s average inventory of defective items per cycle can be obtained as follows. The number of non-defective items in each shipment is $q-y$, and inspection period time is q/x. Note that all the received items are accounted non-defective until they are gradually detected in a flaw. Hence, the buyer’s average inventory of defective items per cycle is

$$\frac{1}{(q-y)/D}\left[\frac{(q-y)y}{D}-\frac{qy}{2x}\right] \quad (3)$$

and the defective holding cost per cycle is

$$h_{b2}\left[\frac{(q-y)y}{D}-\frac{qy}{2x}\right]. \quad (4)$$

Therefore, the buyer's total cost per cycle, given that there are y defective items in an arriving shipment of size q, is the sum of the ordering cost, transportation cost, total holding cost, stock-out cost, screening cost and lead time crashing cost. Symbolically, the buyer's total cost per cycle can be expressed as:

$$\begin{aligned} C_b(q,r,L;y) &= A+F+h_{b1}\left(\frac{q-y}{D}\right) \\ &\times\left\{\frac{Dqy}{2x(q-y)}+\frac{q-y}{2}+k\sigma\sqrt{L}+(1-\beta)E[(X-r)^+]\right\} \\ &+h_{b2}\left[\frac{(q-y)y}{D}-\frac{qy}{2x}\right]+\left[\pi+\pi_0(1-\beta)\right]E[(X-r)^+]+sq+C(L). \end{aligned} \tag{5}$$

If we assume that all items are quickly inspected, i.e., $x\to\infty$, then the length of inspection period $q/x=0$ and Equation (5) reduces to

$$\begin{aligned} C_b(q,r,L;y) &= A+F+h_{b1}\left(\frac{q-y}{D}\right) \\ &\times\left\{\frac{q-y}{2}+k\sigma\sqrt{L}+(1-\beta)E[(X-r)^+]\right\} \\ &+\frac{h_{b2}(q-y)y}{D}+[\pi+\pi_0(1-\beta)]E[(X-r)^+]+sq+C(L), \end{aligned} \tag{6}$$

which is the cost per cycle, given that there are y defective items in an arriving order of size q, in Wu and Ouyang' model [45]. Furthermore, when the vendor promises that the arriving order contains no defective items, i.e., $y=0$, and hence $s=0$, $h_{b2}=0$ then, Equation (6) can be reduced to Moon and Choi's model [22], and further, if $\sigma>0$ and k is sufficiently large, we get $E[(X-r)^+]\to 0$ (see Proposition below); then Equation (6) can be reduced to Ben-Daya and Raouf's model [3].

Let the number of defective items in a lot q be a binomial random variable with parameters q and γ, where γ ($0\le\gamma<1$) represents the defective rate in an order lot. That is,

$$P_r(Y)=C_Y^q\gamma^Y(1-\gamma)^{q-Y}\text{, for } Y=0,1,2,\ldots,q\,. \tag{7}$$

In this case

$$E[Y]=q\gamma \text{ and } E[Y^2]=q^2\gamma^2+q\gamma(1-\gamma)\,. \tag{8}$$

The expected length of the cycle time and the expected cycle cost under the lot of size q are

$$E[T]=E\left[\frac{q-Y}{D}\right]=\frac{q(1-\gamma)}{D} \tag{9}$$

and

$$
\begin{aligned}
&EC_b(q,k,L) \equiv E[C_b(q,k,L;Y)] = A + F + \frac{h_{b1}q^2\gamma}{2x} + \frac{h_{b1}q(1-\gamma)(q-q\gamma+\gamma)}{2D} \\
&+\frac{h_{b1}q(1-\gamma)}{D}\Big[k\sigma\sqrt{L}+(1-\beta)E[(X-r)^+]\Big] \\
&+\frac{h_{b2}q\gamma(1-\gamma)(q-1)}{D}-\frac{h_{b2}q^2\gamma}{2x} \\
&+[\pi+\pi_0(1-\beta)]E[(X-r)^+]+sq+C(L).
\end{aligned} \tag{10}
$$

Therefore, the expected average total cost per unit time for the buyer is

$$
\begin{aligned}
&EC_b^U(q,k,L) = \frac{EC_b(q,k,L)}{E(T)} = \frac{EC_b(q,k,L)D}{q(1-\gamma)} \\
&=\frac{D}{q(1-\gamma)}\left\{A+F+\bar{\pi}E[(X-r)^+]+C(L)\right\}+\frac{h_{b1}q\gamma D}{2x(1-\gamma)}+\frac{h_{b1}\left(q-q\gamma+\gamma\right)}{2} \\
&+h_{b1}\left\{k\sigma\sqrt{L}+\left(1-\beta\right)E[(X-r)^+]\right\}+h_{b2}\gamma\left(q-1\right) \\
&-\frac{h_{b2}q\gamma D}{2x\left(1-\gamma\right)}+\frac{sD}{1-\gamma},
\end{aligned} \tag{11}
$$

where $\bar{\pi} = \pi + \pi_0(1-\beta)$.

If we assume that lead time is prescribed and if $\sigma > 0$ and k is sufficiently large, then Equation (11) can be reduced to Paknejad et al.'s model [25].

3.2. Vendor's expected average total cost per unit time

During the production period, when the first q units have been produced, the vendor will deliver them to the buyer, after that the vendor will make the delivery on average every $E[T]$ units of time until the vendor's inventory level reaches zero. Because the production rate of vendor's non-defective items is greater than the buyer's demand rate, vendor's inventory level will increase gradually. When the total required amount nq is fulfilled, the vendor stops producing immediately. Therefore, the vendor's inventory per production cycle can be obtained by subtracting the accumulated buyer inventory level from the accumulated vendor inventory level as follows.

$$
\begin{aligned}
&\frac{nq}{2}\left\{\left[\left(n-1\right)T+\frac{q}{P}\right]+\left[\left(n-1\right)T+\frac{q}{P}-\frac{nq}{P}\right]\right\}-\left[1+2+\cdots+(n-1)\right]qT \\
&=\frac{nq^2}{P}+\frac{n(n-1)qT}{2}-\frac{n^2q^2}{2P}
\end{aligned} \tag{12}
$$

(refer to [18]).

The vendor's total cost per production cycle is the sum of the set-up cost, defective item treatment cost and holding cost. Symbolically, the vendor's total cost per production cycle can be expressed as:

$$C_v(q,n) = S + nY\omega + h_v\left[\frac{nq^2}{P} + \frac{n(n-1)qT}{2} - \frac{(nq)^2}{2P}\right]. \tag{13}$$

Therefore, the expected average total cost per unit time for the vendor can be obtained as

$$\begin{aligned} EC_v^U(q,n) &= \frac{E[C_v(q,n)]}{E[nT]} = \left\{S + nq\gamma\omega + h_v\left[\frac{nq^2}{P} + \frac{n(n-1)q^2(1-\gamma)}{2D} - \frac{(nq)^2}{2P}\right]\right\} \\ &\times \frac{D}{nq(1-\gamma)} = \frac{SD}{nq(1-\gamma)} + \frac{D\gamma\omega}{1-\gamma} + \frac{h_v Dq}{1-\gamma}\left[\frac{1}{P} + \frac{(n-1)(1-\gamma)}{2D} - \frac{n}{2P}\right]. \end{aligned} \tag{14}$$

3.3. The joint total expected average cost per unit time

Just-in-time (JIT) systems focus primarily on purchasing and manufacturing required items for immediate consumption. JIT requires a spirit of co-operation between the buyer and the vendor. Once the buyer and the vendor have built up a long-term strategic partnership, they can coordinate their production and inventory strategies and share information with each other to determine the best policy for both parties. The integrated inventory model is useful particularly for JIT inventory systems where the buyer and the vendor form a strategic alliance for profit sharing.

This concept of joint optimization for the buyer and the vendor was initiated by Goyal [7] and reinforced by Banerjee [2] and Monahan [21]. Following their approach, we get the joint total expected average cost per unit time as follows:

$$\begin{aligned} JEC^U(q,k,L,n) &= EC_b^U(q,k,L) + EC_v^U(q,n) \\ &= \frac{D}{q(1-\gamma)}\left[\frac{S}{n} + A + F + \bar{\pi}E[(X-r)^+] + C(L)\right] \\ &+ \frac{D}{1-\gamma}\left[s + \gamma\omega - \frac{h_{b2}q\gamma}{2x} + h_v q\left(\frac{1}{P} + \frac{(n-1)(1-\gamma)}{2D} - \frac{n}{2P}\right)\right] \\ &+ h_{b1}\left[k\sigma\sqrt{L} + (1-\beta)E[(X-r)^+]\right] + h_{b2}\gamma(q-1) \\ &+ \frac{h_{b1}q\gamma D}{2x(1-\gamma)} + \frac{h_{b1}(q-q\gamma+\gamma)}{2}. \end{aligned} \tag{15}$$

Then, in this one case, we should mainly discuss Equation (15) and find the optimal values of q (Q), k (r), L, and n such that $JEC^U(q,k,L,n)$ in Equation (15) is minimum. Note that when the vendor promises that the arriving order contains no defective items and the shortages are fully backordered, Equation (15) can be reduced to Ouyang et al.'s model [23].

As mentioned earlier, we make no assumption on the distribution other than saying that it has given finite first and second moments; i.e., the c.d.f. F of X belongs to the class Ω of c.d.f.'s with finite mean DL and standard deviation $\sigma\sqrt{L}$, so we cannot find the exact value of the expected demand shortage quantity at the end of each cycle, $E[(X-r)^{+}]$. Therefore, the minmax distribution free procedure is used to find the least favorable c.d.f. F in Ω for each (q,r,L,n) and then minimize the joint total expected average cost per unit time over (q,k,L,n). More precisely, our problem is to solve:

$$\min_{(q,k,L,n)} \max_{F\in\Omega} JEC^{U}(q,k,L,n)\,. \tag{16}$$

To this end, we need to use the following proposition as in [6]:

Proposition. *For any* $F\in\Omega$

$$E[(X-r)^{+}] \le \frac{1}{2}\left[\sqrt{\sigma^{2}L+(r-DL)^{2}}-(r-DL)\right]. \tag{17}$$

Moreover, the upper bound, equation (17), is tight.

Since $r = DL + k\sigma\sqrt{L}$, and using Proposition, and considering the safety factor k as a decision variable instead of the reorder point r, our problem is reduced to minimizing the cost function for the worst distribution

$$\begin{aligned}
&JEC_{W}^{U}(q,k,L,n)\\
&=\frac{D}{q(1-\gamma)}\left[\frac{S}{n}+A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k^{2}}-k)+C(L)\right]\\
&+\frac{D}{1-\gamma}\left[s+\gamma\omega-\frac{h_{b2}q\gamma}{2x}+h_{v}q\left(\frac{1}{P}+\frac{(n-1)(1-\gamma)}{2D}-\frac{n}{2P}\right)\right]\\
&+h_{b1}\sigma\sqrt{L}\left[k+(1-\beta)\frac{\sqrt{1+k^{2}}-k}{2}\right]+h_{b2}\gamma(q-1)\\
&+\frac{h_{b1}q\gamma D}{2x(1-\gamma)}+\frac{h_{b1}(q-q\gamma+\gamma)}{2}.
\end{aligned} \tag{18}$$

First, for fixed (q,k,L), the effect of n on the joint total expected average cost per unit time $JEC_{W}^{U}(q,k,L,n)$ will be examined. Let us take the second-order partial derivative of $JEC_{W}^{U}(q,k,L,n)$ with respect to n. Then we have

$$\frac{\partial^{2} JEC_{W}^{U}(q,k,L,n)}{\partial n^{2}} = \frac{2Ds}{q(1-\gamma)n^{3}} > 0. \tag{19}$$

Therefore, $JEC_{W}^{U}(q,k,L,n)$ is convex in n, for fixed (q,k,L), and consequently the search for the optimal shipment number, n^{*}, is reduced to finding a local minimum.

Furthermore, for fixed integer n, taking the partial derivatives of $JEC_{W}^{U}(q,k,L,n)$ with respect to q, k and $L\in(L_{i},L_{i-1})$, we get

$$\begin{aligned}\frac{\partial JEC_W^U(q,k,L,n)}{\partial q} &= -\frac{D}{q^2(1-\gamma)}\left[\frac{S}{n}+A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k^2}-k)+C(L)\right] \\ &+\frac{D}{1-\gamma}\left[-\frac{h_{b2}\gamma}{2x}+h_v\left(\frac{1}{P}+\frac{(n-1)(1-\gamma)}{2D}-\frac{n}{2P}\right)\right] \\ &+h_{b2}\gamma+\frac{h_{b1}\gamma D}{2x(1-\gamma)}+\frac{h_{b1}(1-\gamma)}{2},\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial JEC_W^U(q,k,L,n)}{\partial k} &= \frac{D\bar{\pi}\sigma\sqrt{L}}{2q(1-\gamma)}\left(\frac{k}{\sqrt{1+k^2}}-1\right) \\ &+\frac{h_{b1}\sigma\sqrt{L}}{2}\left[1+\beta+(1-\beta)\left(\frac{k}{\sqrt{1+k^2}}\right)\right]\end{aligned} \tag{21}$$

and

$$\begin{aligned}\frac{\partial JEC_W^U(q,k,L,n)}{\partial L} &= \frac{D}{q(1-\gamma)}\left[\frac{\bar{\pi}\sigma(\sqrt{1+k^2}-k)}{4\sqrt{L}}-c_i\right] \\ &+\frac{h_{b1}\sigma\left[k+\frac{1}{2}(1-\beta)(\sqrt{1+k^2}-k)\right]}{2\sqrt{L}}.\end{aligned} \tag{22}$$

However, for fixed (q,k,n), $JEC_W^U(q,k,L,n)$ is a concave function in $L\in[L_i,L_{i-1}]$, because

$$\begin{aligned}\frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial L^2} &= -\frac{D\bar{\pi}\sigma}{8q(1-\gamma)}(\sqrt{1+k^2}-k)L^{-\frac{3}{2}} \\ &-\frac{1}{4}h_{b1}\sigma\left[k+\frac{1}{2}(1-\beta)(\sqrt{1+k^2}-k)\right]L^{-\frac{3}{2}}<0.\end{aligned} \tag{23}$$

Therefore, for fixed (q,k,n), the minimum joint total expected average cost per unit time will occur at the end points of the interval. On the other hand, it can be shown that for fixed n and $L\in[L_i,L_{i-1}]$, $JEC_W^U(q,k,L,n)$ is convex in both q and k (see Appendix for the proof). Therefore, for fixed n and $L\in[L_i,L_{i-1}]$, the minimum value of $JEC_W^U(q,k,L,n)$ will occur at the point (q,k) which satisfies

$\partial JEC_W^U(q,k,L,n)/\partial q=0$ and $\partial JEC_W^U(q,k,L,n)/\partial k=0$ simultaneously. Solving these two equations, we obtain

$$q=\left\{\frac{D\left(\frac{S}{n}+A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k^2}-k)+C(L)\right)}{Dh_v\left(\frac{1}{P}+\frac{(n-1)(1-\gamma)}{2D}-\frac{n}{2P}\right)+\frac{D\gamma(h_{b1}-h_{b2})}{2x}+\frac{h_{b1}(1-\gamma)^2}{2}+h_{b2}\gamma(1-\gamma)}\right\}^{1/2} \tag{24}$$

and

$$\frac{k}{\sqrt{1+k^2}}=1-\frac{2h_{b1}q(1-\gamma)}{D\bar{\pi}+h_{b1}q(1-\gamma)(1-\beta)}. \tag{25}$$

We note that explicit general solutions for (q,k) are not possible because the evaluation for Equations (24) and (25) requires a knowledge of the value of the other. The optimal value of (q,k) can be obtained by adopting a similar graphical technique used in [12].

Therefore, we establish the following iterative algorithm to find the optimal solution of (q,k,L,n).

Algorithm.

Step 1. Set $n=1$.

Step 2. For each L_i, $i=0,\ 1,\ 2,\dots,\ m$, *perform (i) to (iv).*

Start with $k_{i1}=0$

(i) Substituting k_{i1} into Equation (24) to evaluate q_{i1}.

(ii) Utilizing q_{i1} to determine k_{i2} from Equation (25).

(iii) Set $k_{i1}=k_{i2}$ and repeat (i) and (ii) until no change occurs in the values of q_i and k_i.

(iv) Compute the corresponding $JEC_W^U(q_i,k_i,L_i,n)$, $i=0,\ 1,\ 2,\dots,\ m$.

Step3. Find $\min_{i=0,1,\dots,m} JEC_W^U(q_i,k_i,L_i,n)$.

If $JEC_W^U(q_n^*,k_n^*,L_n^*,n)=\min_{i=0,1,\dots,m} JEC_W^U(q_i,k_i,L_i,n)$, *then* (q_n^*,k_n^*,L_n^*) *is the optimal solution for fixed* n.

Step 4. Set $n=n+1$, *and repeat Step 2 to Step 3 to get* $JEC_W^U(q_n^*,k_n^*,L_n^*,n)$.

Step 5. If $JEC_W^U(q_n^*,k_n^*,L_n^*,n)\le JEC_W^U(q_{n-1}^*,k_{n-1}^*,L_{n-1}^*,n-1)$, *then go to Step 4, otherwise go to Step 6.*

Step 6. Set $JEC_W^U(q_n^*,k_n^*,L_n^*,n)=JEC_W^U(q_{n-1}^*,k_{n-1}^*,L_{n-1}^*,n-1)$. *Then* (q^*,k^*,L^*,n^*) *is the optimal solution and the optimal reorder point is* $r^*=DL^*+k^*\sigma\sqrt{L^*}$ *and the optimal* *effective order quantity (i.e., the optimal quantity of non-defective or salable items)* $Q^* = n^* E[q^* - Y] = n^* q^* (1-\gamma)$ *follows.*

4. NUMERICAL EXAMPLE

In order to illustrate the above solution procedure, let us consider an integrated inventory system with the following data: $D = 600$ units/year, $A = \$200$ /order, $S = \$1500$ /setup, $P = 2000$ units/year, $h_v = \$2$ /unit/year, $h_{b1} = \$4$ /unit/year, $h_{b2} = \$3$ /unit /year, $F = \$25$ /shipment, $w = \$4$ /unit, $s = \$0.5$ /unit, $x = 175200$ unit/year, $\pi = \$30$ /unit, $\pi_0 = \$50$ /unit, $\sigma = 7$ units/week, the lead time has three components with data shown in Table 1, and for understanding the effects of various values of the defective rate, γ , and the backorder rate, β , on the entire integrated inventory system, we consider that seven different values of γ (ranging from 0.005 to 0.200) and four different values of β (0.0, 0.5, 0.8 and 1.0). Applying the proposed Algorithm procedure yields the results shown in Table 2. From the results in Table 2, it is interesting to observe that as we fix the backorder rate, β , an increase in the value of the percentage of defective items, γ , results in an increase in all the buyer's expected average annual total cost, the vendor's expected average annual total cost and the joint expected average annual total cost. Therefore, the vendor should make every effort to reduce the rate of defective items so as to decrease his/her own cost and the cost of the entire supply chain system. In addition, increasing the value γ will result in an increase in the reorder point and in the number of shipments per production run from the vendor to the buyer. On the other hand, we fix the value of the percentage of defective items, γ , as the backorder rate, β , increases, both the buyer's expected average annual total cost and the joint expected average annual total cost decrease, and the vendor's expected average annual total cost increases. It is also interesting to observe that decreasing the value β will result in an increase in the reorder point and the shipping quantity from the vendor to the buyer per shipment.

Table 1: Lead time data

Lead time component, i	Normal duration, b_i (days)	Minimum duration, a_i (days)	Unit crashing cost, c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 2: Summary of the optimal solution (L^* in weeks)

β	q^*	r^*	L^*	n^*	$EC^U_{b,w}(q^*,r^*,L^*)$	$EC^U_v(q^*,n^*)$	$JEC^U_W(q^*,r^*,L^*,n^*)$
$\gamma = 0.005$							
0.0	371	85	4	3	1766.85	1454.81	3221.66
0.5	367	78	4	3	1707.32	1456.69	3164.01
0.8	365	72	4	3	1664.09	1458.07	3122.16
1.0	363	68	4	3	1629.89	1459.19	3089.08
$\gamma = 0.015$							
0.0	373	85	4	3	1780.04	1485.67	3265.71
0.5	369	78	4	3	1720.31	1487.58	3207.89
0.8	366	72	4	3	1676.95	1488.98	3165.93
1.0	364	68	4	3	1642.65	1490.12	3132.77
$\gamma = 0.025$							
0.0	374	85	4	3	1793.43	1517.11	3310.53
0.5	370	78	4	3	1733.50	1519.06	3252.56
0.8	368	72	4	3	1690.00	1520.48	3210.48
1.0	366	68	4	3	1655.60	1521.63	3177.23
$\gamma = 0.035$							
0.0	376	85	4	3	1807.00	1549.14	3356.14
0.5	372	78	4	3	1746.89	1551.13	3298.02
0.8	369	72	4	3	1703.25	1552.57	3255.82
1.0	367	68	4	3	1668.74	1553.75	3222.49
$\gamma = 0.045$							
0.0	377	85	4	3	1820.79	1581.81	3402.60
0.5	373	78	4	3	1760.47	1583.82	3344.29
0.8	371	72	4	3	1716.69	1585.30	3301.99
1.0	369	68	4	3	1682.08	1586.48	3268.56
$\gamma = 0.100$							
0.0	386	86	4	3	1900.51	1773.37	3673.88
0.5	323	82	4	4	1817.93	1794.30	3612.23
0.8	321	76	4	4	1771.44	1794.49	3565.93
1.0	319	71	4	4	1734.76	1794.68	3529.44
$\gamma = 0.200$							
0.0	345	91	4	4	2041.06	2191.43	4232.49
0.5	342	83	4	4	1974.81	2191.90	4166.71
0.8	339	77	4	4	1926.74	2192.34	4119.08
1.0	337	72	4	4	1888.86	2192.73	4081.59

On the other hand, for comparison purposes, we present the following analysis. If the buyer and the vendor do not choose to cooperate with each other, they will determine their own optimal policy separately. First, the buyer makes his/her own decision without any intention to cooperate with the vendor; he/she discusses the total cost per unit time of the Equation (11). Our problem is to minimize the cost function for the worst distribution

$$EC_{b,W}^{U}(q,k,L)=\frac{D}{q(1-\gamma)}\left[A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k^2}-k)+C(L)\right]+\frac{h_{b1}q\gamma D}{2x(1-\gamma)}$$
$$+\frac{h_{b1}\left(q-q\gamma+\gamma\right)}{2}+h_{b1}\sigma\sqrt{L}\left[k+\frac{1}{2}(1-\beta)(\sqrt{1+k^2}-k)\right] \tag{26}$$
$$+h_{b2}\gamma\left(q-1\right)-\frac{h_{b2}q\gamma D}{2x\left(1-\gamma\right)}+\frac{sD}{1-\gamma}.$$

By analogous arguments as in Appendix, it can be readily shown that for fixed q and k, $EC_{b,W}^{U}(q,k,L)$ is a concave function in $L\in\left[L_i,L_{i-1}\right]$. Thus, the minimum value of $EC_{b,W}^{U}(q,k,L)$ will occur at the end points of the interval $\left[L_i,L_{i-1}\right]$. Furthermore, we can show that $EC_{b,W}^{U}(q,k,L)$ is convex in both q and k. Thus for fixed $L\in\left[L_i,L_{i-1}\right]$, the optimal solutions of q and k (denoted by (q_b^*,k_b^*)) which minimize the expected average total cost per unit time for the buyer will satisfy

$\partial EC_{b,W}^{U}(q,k,L)/\partial q\,|_{(q,k)=(q_b^*,k_b^*)}=0$ and $\partial EC_{b,W}^{U}(q,k,L)/\partial k\,|_{(q,k)=(q_b^*,k_b^*)}=0$

simultaneously. Solving these equations, we obtain

$$q_b^*=\left\{\frac{D\left(A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k_b^{*2}}-k_b^*)+C(L)\right)}{\frac{D\gamma(h_{b1}-h_{b2})}{2x}+\frac{h_{b1}(1-\gamma)^2}{2}+h_{b2}\gamma(1-\gamma)}\right\}^{1/2} \tag{27}$$

and

$$\frac{k_b^*}{\sqrt{1+k_b^{*2}}}=1-\frac{2h_{b1}q_b^*(1-\gamma)}{D\bar{\pi}+h_{b1}q_b^*(1-\gamma)(1-\beta)}. \tag{28}$$

Following the analogous method of the Step 2 of Algorithm, we can find (q_b^*,k_b^*) for each L_i, $i=0,\ 1,\ 2,\ldots,\ m$.

Then set $EC_{b,W}^{U}(q_b^*,k_b^*,L_b^*)=\min_{i=0,1,\ldots,m}EC_{b,W}^{U}(q_i,k_i,L_i)$. Thus (q_b^*,k_b^*,L_b^*) is the optimal solution for the buyer. On the other hand, the expected average total cost per unit time for the vendor is Equation (14). Since the best production quantity is a positive integer multiple of the buyer's ordering quantity, we compute Equation (14) by using buyer's optimal ordering quantity and set $n=1,\ 2,\ 3,\ldots.$ We can find the best choice of

n for the vendor and $EC_v^U(q_b^*, n_v^*)$. Using the data, as stated above, the optimal number of vendor product in one production run is $n_v^* = 4$, and other results are presented in Table 3. From the results shown in Tables 2 and 3, it indicates that the total cost and the vendor's cost will be lower and the buyer's will be higher for integrated models than for those without integration. This leads to an important issue of the vendor's compensation for the loss of the buyer in integrated models. Therefore, they should cooperate and jointly determine the best solution, and the total savings can and should be shared in some equitable manner. Goyal [7] suggested a judicious method for allocating costs that the total annual cost $JEC_W^U(q^*, r^*, L^*, n^*)$ should be allocated to the vendor and the buyer as follows.

Cost to the buyer $= \xi \cdot JEC_W^U(q^*, r^*, L^*, n^*)$,

Cost to the vendor $= (1-\xi) \cdot JEC_W^U(q^*, r^*, L^*, n^*)$,

where $\xi = \dfrac{EC_{b,W}^U(q_b^*, r_b^*, L_b^*)}{EC_{b,W}^U(q_b^*, r_b^*, L_b^*) + EC_v^U(q_b^*, n_v^*)}$; q_b^*, r_b^* and L_b^* respectively denote the buyer's optimal ordering quantity, optimal reorder point and optimal lead time, and n_v^* denotes the vendor's optimal number of lots delivered to the buyer. The results of allocated total cost are also shown in Table 3.

Table 3: Summary of the comparison between the policies

β	Model			Type			R[a]
	Independent			Integrated			
	Buyer's cost	Vendor's cost	Total cost	Allocated buyer's cost	Allocated vendor's cost	Total cost	
$\gamma = 0.005$							
0.0	1753.10	1485.65	3238.75	1743.85	1477.81	3221.66	100.531
0.5	1691.18	1484.67	3175.85	1684.87	1479.13	3164.00	100.374
0.8	1646.07	1484.52	3130.59	1641.64	1480.52	3122.16	100.270
1.0	1610.36	1484.71	3095.07	1607.24	1481.84	3089.08	100.194
$\gamma = 0.015$							
0.0	1765.84	1515.47	3281.31	1757.45	1508.26	3265.71	100.478
0.5	1703.69	1514.65	3218.34	1698.16	1509.73	3207.89	100.326
0.8	1658.42	1514.60	3173.02	1654.71	1511.22	3165.93	100.224
1.0	1622.58	1514.88	3137.46	1620.16	1512.61	3132.77	100.150
$\gamma = 0.025$							
0.0	1778.76	1545.88	3324.64	1771.21	1539.32	3310.53	100.426
0.5	1716.38	1545.20	3261.58	1711.63	1540.93	3252.56	100.277
0.8	1670.94	1545.27	3216.21	1667.97	1542.51	3210.48	100.178
1.0	1634.98	1545.63	3180.61	1633.24	1543.99	3177.23	100.106
$\gamma = 0.035$							
0.0	1791.86	1576.88	3368.74	1785.16	1570.98	3356.14	100.376
0.5	1729.25	1576.36	3305.61	1725.28	1572.74	3298.02	100.230
0.8	1683.65	1576.53	3260.18	1681.40	1574.42	3255.82	100.134
1.0	1647.56	1576.98	3224.54	1646.52	1575.97	3222.49	100.064
$\gamma = 0.045$							
0.0	1805.16	1608.49	3413.65	1799.32	1603.28	3402.60	100.325
0.5	1742.31	1608.13	3350.44	1739.11	1605.18	3344.29	100.184
0.8	1696.54	1608.41	3304.95	1695.02	1606.97	3301.99	100.090
1.0	1660.33	1608.94	3269.27	1659.97	1608.59	3268.56	100.022
$\gamma = 0.100$							
0.0	1881.96	1794.20	3676.16	1880.79	1793.09	3673.88	100.062
0.5	1817.80	1794.73	3612.53	1817.64	1794.59	3612.23	100.008
0.8	1771.08	1795.65	3566.73	1770.68	1795.25	3565.93	100.023
1.0	1734.15	1796.64	3530.79	1733.49	1795.95	3529.44	100.038
$\gamma = 0.200$							
0.0	2040.54	2235.69	4276.23	2019.67	2212.82	4232.49	101.034
0.5	1973.74	2195.23	4168.97	1972.67	2194.04	4166.71	100.054
0.8	1925.15	2197.38	4122.53	1923.54	2195.54	4119.08	100.084
1.0	1886.79	2199.28	4086.07	1884.72	2196.87	4081.59	100.110

[a] R denotes the ratio of the total cost of the best independent policy to the total cost of the best integrated policy expressed as a percentage.

5. CONCLUDING REMARKS

The purpose of this paper is to investigate an integrated vendor-buyer inventory policy for a continuous review model including a random number of defective items in buyer's arrival order lot with a mixture of backorders and lost sales when only the mean and the variance of the distribution of the lead time demand are known. Analyzing the joint total expected cost function, we develop an algorithmic procedure to determine the optimal order quantity, reorder point, lead time and the number of lots delivered in one production run. The effects of parameters are also studied for the decision-making references. Moreover, the results of the numerical example indicate that both parties can benefit on condition that the buyer and the vendor make their decisions cooperatively. In addition, the total cost of the entire supply chain system decreases when the backorder rate increases and the defective rate decreases. Therefore, the vendor should endeavor to enhance the production quality to reduce defective rate so as to decrease the total cost of the entire supply chain system.

Regarding some future research, we propose the adoption of the random sub-lot sampled inspection policy to inspecting the selected items. In order to show the uncertainties, we could extend the present model so to apply stochastic demand and production rate in each member of the supply chain.

Appendix: The proof of $JEC_W^U(q,k,L,n)$ is convex in (q,k) for fixed n and $L \in [L_i, L_{i-1}]$.

For fixed n and $L \in [L_i, L_{i-1}]$, we first obtain the Hessian matrix **H** as follows.

$$\mathrm{H} = \begin{bmatrix} \dfrac{\partial^2 JEC_W^U(q,k,L,n)}{\partial q^2} & \dfrac{\partial^2 JEC_W^U(q,k,L,n)}{\partial q \partial k} \\ \dfrac{\partial^2 JEC_W^U(q,k,L,n)}{\partial k \partial q} & \dfrac{\partial^2 JEC_W^U(q,k,L,n)}{\partial k^2} \end{bmatrix}.$$

Then we proceed by evaluating the principal minor of **H**.

The first principal minor of **H** is $|H_{11}| > 0$, since

$$\frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial q^2} = \frac{2D}{q^3(1-\gamma)} \times \left[\frac{S}{n} + A + F + \frac{\bar{\pi}\sigma\sqrt{L}}{2}\left(\sqrt{1+k^2} - k\right) + C(L)\right] > 0 \tag{A.1}$$

and

$$\frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial k^2} = \frac{\sigma\sqrt{L}}{2}\left[\frac{D\bar{\pi}}{q(1-\gamma)} + h_{b1}(1-\beta)\right]\left(1+k^2\right)^{-\frac{3}{2}} > 0. \tag{A.2}$$

Next, computing the second principal minor of **H,** we get

$$\frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial k\partial q}=\frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial q\partial k}$$
$$=-\frac{D\bar{\pi}\sigma\sqrt{L}}{2q^2(1-\gamma)}\left(\frac{k}{\sqrt{1+k^2}}-1\right) \tag{A.3}$$

and it follows from Equations (A.1)-(A.3) that

$$|H_{22}|=\begin{vmatrix}\frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial q^2} & \frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial q\partial k}\\ \frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial k\partial q} & \frac{\partial^2 JEC_W^U(q,k,L,n)}{\partial k^2}\end{vmatrix}$$
$$=\frac{2D}{q^3(1-\gamma)}\left[\frac{S}{n}+A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k^2}-k)+C(L)\right]$$
$$\times\left[\frac{D\bar{\pi}\sigma\sqrt{L}}{2q(1-\gamma)}\left(1+k^2\right)^{-\frac{3}{2}}+\frac{h_{b1}\sigma\sqrt{L}\left(1-\beta\right)}{2}(1+k^2)^{-\frac{3}{2}}\right]$$
$$-\frac{1}{4}\left(\frac{D\bar{\pi}\sigma\sqrt{L}}{q^2(1-\gamma)}\right)^2\left(\frac{2k^2-2k\sqrt{1+k^2}+1}{1+k^2}\right)$$
$$=\frac{D^2\bar{\pi}\sigma\sqrt{L}}{q^4(1-\gamma)^2}(1+k^2)^{-\frac{3}{2}}\left[\frac{S}{n}+A+F+C(L)\right]$$
$$+\frac{Dh_{b1}\sigma\sqrt{L}}{q^3(1-\gamma)}(1-\beta)(1+k^2)^{-\frac{3}{2}}\left[\frac{S}{n}+A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k^2}-k)+C(L)\right]$$
$$+\frac{1}{2}\left[\frac{D\bar{\pi}\sigma\sqrt{L}}{q^2(1-\gamma)}\right]^2\frac{\sqrt{1+k^2}-k}{(\sqrt{1+k^2})^3}-\frac{1}{2}\left[\frac{D\bar{\pi}\sigma\sqrt{L}}{q^2(1-\gamma)}\right]^2\frac{2k^2-2k\sqrt{1+k^2}+1}{2(1+k^2)}$$
$$=\frac{D^2\bar{\pi}\sigma\sqrt{L}}{q^4(1-\gamma)^2}\left(1+k^2\right)^{-\frac{3}{2}}\left[\frac{S}{n}+A+F+C(L)\right]$$
$$+\frac{Dh_{b1}\sigma\sqrt{L}}{q^3(1-\gamma)}(1-\beta)\left(1+k^2\right)^{-\frac{3}{2}}\left[\frac{S}{n}+A+F+\frac{\bar{\pi}\sigma\sqrt{L}}{2}(\sqrt{1+k^2}-k)+C(L)\right] \tag{A.4}$$
$$+\frac{1}{2}\left[\frac{D\bar{\pi}\sigma\sqrt{L}}{q^2(1-\gamma)}\right]^2\frac{(2k^3-2k^2\sqrt{1+k^2}+\sqrt{1+k^2})}{2(1+k^2)^{\frac{3}{2}}}>0.$$

Let $G(k)=2k^3-2k^2\sqrt{1+k^2}+\sqrt{1+k^2}$. Since $1+k^2<\left(k+1/2k\right)^2$,

$$\frac{dG(k)}{dk}=\frac{6k^2\left(\sqrt{1+k^2}-k-\frac{1}{2k}\right)}{\sqrt{1+k^2}}<0,$$

thus $G(k)$ is a decreasing function of k. Furthermore, $G(0)=1$ and it follows from L'Hospital's rule that

$$G(\infty) \equiv \lim_{k\to\infty} G(k) = \lim_{k\to\infty}(2k^3 - 2k^2\sqrt{1+k^2} + \sqrt{1+k^2}) = \lim_{k\to\infty}(2k^2-1)(k-\sqrt{1+k^2}) + k$$
$$= \lim_{k\to\infty} \frac{(2k^2-1)(k^2-1-k^2) + k(k+\sqrt{1+k^2})}{k+\sqrt{1+k^2}}$$
$$= \lim_{k\to\infty} \frac{k}{(k+\sqrt{1+k^2})^2} = \lim_{k\to\infty} \frac{1}{k\left(1+\sqrt{1+\frac{1}{k^2}}\right)^2} = 0.$$

Consequently, $G(k) > 0,\ \forall k \in [0,\infty)$ and so $|H_{22}| > 0$. Therefore, it is clearly that the Hessian matrix **H** is positive definite at point (q,k).

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