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THE OPTIMAL PRODUCTION-RUN TIME FOR A STOCK-DEPENDENT IMPERFECT PRODUCTION PROCESS

Divya JAIN, Kishor KUMAR AGGARWAL

Department of Operational Research, Faculty of Mathematical Sciences, University of Delhi, Delhi 110007, India

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Abstract: This paper develops an inventory model for a hypothesized volume flexible manufacturing system in which the production rate is stock-dependent and the system produces both perfect and imperfect quality items. The demand rate of perfect quality items is known and constant, whereas the demand rate of imperfect (non-conforming to specifications) quality items is a function of discount offered in the selling price. In this paper, we determine an optimal production-run time and the optimal discount that should be offered in the selling price to influence the sale of imperfect quality items produced by the manufacturing system. The considered model aims to maximize the net profit obtained through the sales of both perfect and imperfect quality items subject to certain constraints of the system. The solution procedure suggests the use of 'Interior Penalty Function Method' to solve the associated constrained maximization problem. Finally, a numerical example demonstrating the applicability of proposed model has been included.

Keywords: Inventory, Imperfect Quality, Stock Dependent, Production-run, Volume-flexible, Constrained Optimization.

MSC: 90B05, 90B30.

1. INTRODUCTION

In the classical Economic Production Quantity (EPQ) model, the production rate of a manufacturing system is regarded to be pre-determined and inflexible. However, with changing trends, it has been noticed that the ability of manufacturing system to respond to variations has opened up a new level of competition. Today, flexibility has become an important tool to gain a competitive edge in a business environment.

Schweitzer and Seidmann[17] were the first who introduced the concept of flexibility in machine production rate in discussing the optimum processing rates for a Flexible Manufacturing System (FMS). Khouja and Mehrez[4] extended the Economic Production lot size model to an imperfect production process with a flexible production rate. Silver[18] discussed the reduced production rates in the context of manufacturing equipment dedicated to the production of a family of items, assuming a common cycle for all items. Controllable production rates in a family production context were also considered by Moon, Gallego and Simchi-Levi[7]. Gallego[2] extended the model of Silver[18] by removing the stipulation of a common cycle for all the items.

But, the above studies did not consider the production rate to depend on existing stock-level. In most real life situations, it is found that the production lot size also depends on the on-hand inventory level. The higher the existing stock level, the lower is the production. This type of volume flexibility can be easily seen as an important factor of a Flexible Manufacturing System (FMS) in which CNC machines are used to control the speed of production. Thus, in this paper we consider a volume flexible manufacturing system in which the production system is capable of adjusting the production rate depending on the existing stock level.

Moreover, in a manufacturing setup, it can be examined that incorporating flexibility into the system tends to decrease system efficiency due to which a certain percentage of total items produced could be found to be of imperfect quality (non-conforming to specifications). Several studies have been conducted by various researchers on an imperfect production/ordering system. Most researchers have considered that while producing a lot, a production process may go from the "in-control" state to the "out-of-control" state. As a result, the produced lot would contain both perfect and imperfect quality products. Rosenblatt and Lee[10] studied the effects of an imperfect production process on the optimal production-run time by assuming an elapsed time until shift is exponentially distributed, and Porteus[9] assumed that the probability of a shift from the "in-control" state to the "out-of-control" state has a given value for each item produced. Khouja and Mehrez[4] considered that in an "out-of-control" state, a certain percentage of total products is defective, which is reworked at the cost. Sana[16] extended their work by considering the shift to happen at any random time. Salameh and Jaber[11] developed an EOQ model considering the lot to contain a random fraction of imperfect quality items with a known probability distribution. They assumed that each shipment undergoes a 100% screening process, and that the items found to be of imperfect quality are sold at discounted price as a single batch at the end of the screening process, and thus are instantly removed from the system. Several researchers have extended the work of Salameh and Jaber[11]. Cárdenas-Barrón[1] observed a minor correction for the expression of optimal lot size. Goyal and Cárdenas-Barrón[3] presented a simple approach which approximately determines the order quantity when a random proportion of units are defective. Papachristos and Konstantaras[8] examined the model of Salameh and Jaber's model closely and discussed many of its assumptions, and in particular, those aimed at avoiding shortages. Maddah and Jaber[6] applied the concept of Renewal Theory to Salameh and Jaber's model to obtain a simple expression for the optimal order quantity and expected profit. Recently, Sana, Goyal and Chaudhuri[14] have extended an EPLS (Economic Production Lot Size) model which accounts for production system producing items of perfect as well as imperfect quality. The probability of imperfect quality items increases with the increase in production-run-time because of machinery problems, impatience of labor staff, and improper distribution of raw materials. They have assumed that the demand rate of perfect quality items is constant, whereas the demand rate of defective items which are not repaired is a function of reduction rate. In another model, Sana, Goyal and Chaudhuri[15] have developed a volume flexible inventory model with an imperfect production system where demand rate of conforming quality items is a random variable, and the demand rate of defective items is a function of a random variable and reduction rate.

The author's survey of the relevant literature reveals that there is no published work that investigates an imperfect production process for volume flexibility defined by stock-dependent production rate. Therefore, this paper extends the work of Sana, Goyal and Chaudhuri[14]by considering the stock-dependent production rate for an imperfect production process in which a fixed percentage of the produced items is found to be of imperfect quality on inspection. The demand rate of perfect quality items is known and constant, whereas the demand rate of defective items which are not repaired is a function of discount offered in the selling price. In this paper, we determine an optimal production-run time and the optimal discount that should be offered in the selling price to influence the sale of imperfect quality items produced by the manufacturing system in which the production rate is stock-dependent. This is achieved by maximizing the net profit obtained through the sales of both perfect and imperfect quality items subject to certain constraints of the system.

This paper is organized as follows: 1. Introduction; 2. Notations and Assumptions; 3. Mathematical Model; 4. Numerical Example and Concavity of function; 5. Conclusion; 6. Appendix; 7. References.

2. NOTATIONS AND ASSUMPTIONS

The notations adopted in this paper are as follows:

d_1	demand rate of perfect quality items
d_2	deterministic factor of the demand rate of imperfect quality items
$Q_1(t)$	inventory of perfect quality items at any time $'t'(\geq 0)$
$Q_2(t)$	inventory of imperfect quality items at any time $'t'(\geq 0)$
C_h	holding cost per unit per unit time
I_c	unit inspection cost
η	unit production cost
S	unit selling price of a perfect quality item
r	discount (in %) given in selling price for imperfect quality items
n	fixed positive integer
T	length of an inventory cycle of perfect quality items
T'	length of an inventory cycle of imperfect quality items
t_1	production run time
$P(Q_1(t), Q_2(t))$	production rate is dependent on the on-hand inventory level of both perfect and imperfect quality items, i.e.

$P(Q_1(t), Q_2(t)) = \alpha - \beta_1 Q_1(t) - \beta_2 Q_2(t)$ where α, β_1 and β_2 are constants, $0 \le \beta_1 \le 1, 0 \le \beta_2 \le 1$.

λ percentage of perfect quality items in the produced lot, $0 \le \lambda \le 1$.

The following assumptions have been made:

1. The demand rate of perfect quality items is constant and deterministic.
2. The demand rate of imperfect quality items is a function of the discount given in selling price.
3. The produced items undergo a 100% inspection process that separates perfect and imperfect quality items.
4. Shortages are not allowed.
5. Lead time is negligible.
6. The discount to be offered in the selling price is considered as a decision variable.
7. The production run time is considered to be a decision variable.
8. Planning horizon is infinite.

3. MATHEMATICAL FORMULATION OF THE MODEL

Consider a volume flexible manufacturing system in which the production rate is dependent on the on-hand inventory level. The higher the existing stock, the lower the production rate is. During a production run, the system manufactures both perfect and imperfect quality items. The produced items simultaneously undergo a 100% inspection process at a cost 'I_c' per unit that separates perfect and imperfect quality items. The perfect quality items can be sold at a selling price of 'S' per unit and have a constant demand rate d_1. The produced lot contains a fixed percentage of imperfect quality items which can be sold at a discounted price $S(1-r)$ per unit, where $0 < r < 1$ is the discount given to influence the demand of imperfect quality items. Moreover, the demand rate of imperfect quality items is $\left(\frac{r^n}{1-r}\right)d_2$; $(0 < r < 1, d_2 \ge 0)$, which is an increasing function of r as n is a fixed positive integer (Refer to Sana, Goyal and Chaudhuri[14]).

The production cycle starts at time $t = 0$ with zero inventory level and is continued till time t_1. During the time interval $(0, t_1)$, system manufactures both perfect and imperfect quality items at a stock-dependent rate. The system simultaneously inspects all the manufactured items and filters them into perfect and imperfect quality items. Till time t_1, manufactured items are utilized to serve the demand of perfect and imperfect quality items as well as pile up inventories. After time t_1, the piled inventories are utilized to meet the demands. The inventory of perfect and imperfect quality items fall to the zero level at time $t = T$ and $t = T'$, respectively. Figure 1 shows the behavior of the inventory system during a production cycle.

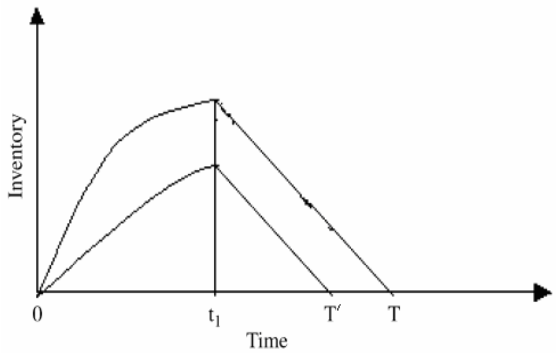

Fig. 1. Production Cycle

Therefore, the differential equations are:

$$\frac{\mathrm{d}Q_1(t)}{\mathrm{d}t} = \lambda.[\alpha - \beta_1 Q_1(t) - \beta_2 Q_2(t)] - d_1 \quad ; \quad 0 \le t \le t_1,\ 0 \le \beta_1 \le 1, 0 \le \beta_2 \le 1, \tag{1}$$

with $Q_1(0) = 0$,

$$\frac{\mathrm{d}Q_1(t)}{\mathrm{d}t} = -d_1 \quad ; \quad t_1 \le t \le T, \tag{2}$$

with $Q_1(T) = 0$

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = (1-\lambda).[\alpha - \beta_1 Q_1(t) - \beta_2 Q_2(t)] - \frac{r^n}{1-r} d_2 \quad ; \quad 0 \le t \le t_1,\ 0 \le \beta_1 \le 1, 0 \le \beta_2 \le 1, \tag{3}$$

with $Q_2(0) = 0$ and

$$\frac{\mathrm{d}Q_2(t)}{\mathrm{d}t} = -\frac{r^n}{1-r} d_2 \quad ; \quad t_1 \le t \le T', \tag{4}$$

with $Q_2(T') = 0$.

Let, $\frac{\mathrm{d}}{\mathrm{d}t} \equiv D$ then for $0 \le t \le t_1$, equation (1) and (3) can be written as

$$[D + \lambda\beta_1]Q_1(t) + \lambda\beta_2 Q_2(t) = \lambda\alpha - d_1 \tag{5}$$

$$(1-\lambda)\beta_1 Q_1(t) + [D + (1-\lambda)\beta_2]Q_2(t) = (1-\lambda)\alpha - \frac{r^n}{1-r} d_2 \tag{6}$$

Multiply (5) by $[D + (1-\lambda)\beta_2]$ & (6) by $\lambda\beta_2$ and then subtract, so we get

$$\left[D^2 + (\lambda\beta_1 + (1-\lambda)\beta_2)D\right]Q_1(t) = \lambda\beta_2 d_2 \frac{r^n}{1-r} - (1-\lambda)\beta_2 d_1$$

i.e. $$\frac{d^2Q_1(t)}{dt^2}+(\lambda\beta_1+(1-\lambda)\beta_2)\frac{dQ_1(t)}{dt}=\lambda\beta_2 d_2\frac{r^n}{1-r}-(1-\lambda)\beta_2 d_1 \tag{7}$$

Let, $z=e^{\frac{-1}{2}(\lambda\beta_1+(1-\lambda)\beta_2)t}$ and $Q_1=v.z$ where v and z are functions of t. Then equation (7) becomes

$$\frac{d^2v}{dt^2}-\frac{1}{4}(\lambda\beta_1+(1-\lambda)\beta_2)^2 v=\left[\lambda\beta_2 d_2\frac{r^n}{1-r}-(1-\lambda)\beta_2 d_1\right].e^{\frac{1}{2}(\lambda\beta_1+(1-\lambda)\beta_2)t} \tag{8}$$

Solving the second order differential equation in (8), we get

$$v=c_1e^{\frac{1}{2}(\lambda\beta_1+(1-\lambda)\beta_2)t}+c_2e^{\frac{-1}{2}(\lambda\beta_1+(1-\lambda)\beta_2)t}+\left[\frac{\lambda\beta_2 d_2\frac{r^n}{1-r}-(1-\lambda)\beta_2 d_1}{\lambda\beta_1+(1-\lambda)\beta_2}\right].te^{\frac{1}{2}(\lambda\beta_1+(1-\lambda)\beta_2)t};$$

Since $Q_1=v.z$

$$\therefore\ Q_1(t)=c_1+c_2e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t}+\left[\frac{\lambda\beta_2 d_2\frac{r^n}{1-r}-(1-\lambda)\beta_2 d_1}{\lambda\beta_1+(1-\lambda)\beta_2}\right].t \quad ;0\le t\le t_1 \tag{9}$$

$$\Rightarrow \frac{dQ_1(t)}{dt}=-c_2(\lambda\beta_1+(1-\lambda)\beta_2)e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t}+\left[\frac{\lambda\beta_2 d_2\frac{r^n}{1-r}-(1-\lambda)\beta_2 d_1}{\lambda\beta_1+(1-\lambda)\beta_2}\right] \quad ;0\le t\le t_1 \tag{10}$$

From equations (1), (9) and (10) we get

$$Q_2(t)=\frac{1}{\lambda\beta_2}\left\{\begin{array}{l}(1-\lambda)\beta_2c_2e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t}-\\ \left[\frac{\lambda\beta_2 d_2\frac{r^n}{1-r}-(1-\lambda)\beta_2 d_1}{\lambda\beta_1+(1-\lambda)\beta_2}\right].(1+\lambda\beta_1 t)+\lambda(\alpha-\beta_1c_1)-d_1\end{array}\right\} \quad ;0\le t\le t_1 \tag{11}$$

Using initial condition $Q_1(0)=0$ and $Q_2(0)=0$, we get

$$Q_1(t)=\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t}-1\right) \tag{12}$$
$$+\frac{\beta_2}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1\right\}.t \quad ;0\le t\le t_1$$

and

$$Q_2(t)=\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t}-1\right)$$
$$-\frac{\beta_1}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1\right\}.t \qquad ;0\le t\le t_1 \tag{13}$$

Now, for $t_1 \le t \le T$ using equations (2), (12) and initial condition $Q_1(T)=0$, we get

$$Q_1(t)=\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)$$
$$+\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1-d_1\mathrm{t} \qquad ;t_1\le t\le \mathrm{T} \tag{14}$$

Similarly for $t_1 \le t \le T'$ using equations (4), (13) and initial condition $Q_2(T')=0$, we get

$$Q_2(t)=\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)$$
$$+\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1-\frac{r^n}{1-r}d_2\mathrm{t} \qquad ;t_1\le t\le \mathrm{T}' \tag{15}$$

Equation (14) and $Q_1(T)=0$ implies

$$T=\frac{1}{d_1}\left\{\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.+\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1\right\} \tag{16}$$

Similarly, equation (15) and $Q_2(T')=0$ implies

$$T'=\left(\frac{1-r}{r^n}\right).\frac{1}{d_2}\left\{\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.+\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1\right\} \tag{17}$$

For feasibility of t_1 in the practical situation, we must have

(i) $0<t_1<T$ i.e. $0<t_1$ and $T-t_1>0$.

(ii) $t_1\le T'$ i.e. $T'-t_1\ge 0$

Also $T'\le T$ otherwise, the inventory of imperfect quality items will get carried to the next cycle.

Moreover, the total production of each cycle must be greater than or equal to the demand of perfect and imperfect quality items occurring in each cycle, i.e. it must be greater than or equal to $d_1+\frac{r^n}{1-r}d_2$.

$$\text{Total Inventory} = \int_0^{t_1} Q_1(t)\mathrm{d}t + \int_{t_1}^{T} Q_1(t)\mathrm{d}t + \int_0^{t_1} Q_2(t)\mathrm{d}t + \int_{t_1}^{T'} Q_2(t)\mathrm{d}t \cdot$$

where,

$$\int_0^{t_1} Q_1(t)\mathrm{d}t = \int_0^{t_1} \left\{ \begin{array}{l} \frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t}-1\right) \\ +\frac{\beta_2}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2 r^n}{1-r}-(1-\lambda)d_1\right\}.t \end{array} \right\}\mathrm{d}t$$

$$\begin{aligned} &= \frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^3}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(1-e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}\right) \\ &\quad -\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.t_1 \\ &\quad +\frac{\beta_2}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2 r^n}{1-r}-(1-\lambda)d_1\right\}.\frac{t_1^2}{2} \end{aligned} \tag{18}$$

$$\int_{t_1}^{T} Q_1(t)\mathrm{d}t = \int_{t_1}^{T} \left\{ \begin{array}{l} \frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right) \\ +\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2 d_2 r^n}{1-r}+\beta_1 d_1\right\}.t_1 - d_1\mathrm{t} \end{array} \right\}\mathrm{d}t = \frac{d_1}{2}(T-t_1)^2 \tag{19}$$

$$\int_0^{t_1} Q_2(t)\mathrm{d}t = \int_0^{t_1} \left\{ \begin{array}{l} \frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\} \\ .\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t}-1\right)-\frac{\beta_1}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2 r^n}{1-r}-(1-\lambda)d_1\right\}.t \end{array} \right\}\mathrm{d}t$$

$$\begin{aligned} &= \frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^3}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(1-e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}\right) \\ &\quad -\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.t_1 \\ &\quad -\frac{\beta_1}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2 r^n}{1-r}-(1-\lambda)d_1\right\}.\frac{t_1^2}{2} \end{aligned} \tag{20}$$

$$\int_{t_1}^{T'} Q_2(t)\mathrm{d}t = \int_{t_1}^{T'} \left\{ \begin{array}{l} \frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right) \\ +\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2 d_2 r^n}{1-r}+\beta_1 d_1\right\}.t_1 - \frac{r^n}{1-r}d_2\mathrm{t} \end{array} \right\}\mathrm{d}t \tag{21}$$

$$= \frac{1}{2}\cdot\frac{r^n}{1-r}d_2(T'-t_1)^2$$

Thus,

$$\text{Total Inventory} = \frac{1}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)^3}\left\{\beta_2\left[\frac{d_2 r^n}{1-r} - (1-\lambda)\alpha\right] + \beta_1\left[d_1 - \lambda\alpha\right]\right\}.\left(1 - e^{-\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)t_1}\right)$$
$$- \frac{1}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r} - (1-\lambda)\alpha\right] + \beta_1\left[d_1 - \lambda\alpha\right]\right\}.t_1 \tag{22}$$
$$+ \frac{\left(\beta_2 - \beta_1\right)}{2\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)}\left\{\frac{\lambda d_2 r^n}{1-r} - (1-\lambda)d_1\right\}.t_1^2$$
$$+ \frac{1}{2}\left[d_1\left(T - t_1\right)^2 + \frac{r^n}{1-r}d_2\left(T' - t_1\right)^2\right]$$

where,

$$T - t_1 = \frac{1}{d_1}\left\{\frac{\lambda}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r} - (1-\lambda)\alpha\right] + \beta_1\left[d_1 - \lambda\alpha\right]\right\}.\left(e^{-\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)t_1} - 1\right)\right.$$
$$\left. + \frac{\beta_2}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)}\left\{\frac{\lambda d_2 r^n}{1-r} - (1-\lambda)d_1\right\}.t_1\right\} \tag{23}$$

and

$$T' - t_1 = \left(\frac{1-r}{r^n}\right).\frac{1}{d_2}\left\{\frac{(1-\lambda)}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r} - (1-\lambda)\alpha\right] + \beta_1\left[d_1 - \lambda\alpha\right]\right\}.\left(e^{-\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)t_1} - 1\right)\right. \tag{24}$$
$$\left. - \frac{\beta_1}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)}\left\{\frac{\lambda d_2 r^n}{1-r} - (1-\lambda)d_1\right\}.t_1\right\}$$

$$\text{Total Production cost is} = \eta\int_0^{t_1}\left[\alpha - \beta_1 Q_1(t) - \beta_2 Q_2(t)\right]dt$$
$$= \eta\left[\frac{-1}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r} - (1-\lambda)\alpha\right] + \beta_1\left[d_1 - \lambda\alpha\right]\right\}.\left(1 - e^{-\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)t_1}\right)\right. \tag{25}$$
$$\left. + \frac{1}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)}\left\{\frac{\beta_2 d_2 r^n}{1-r} + \beta_1 d_1\right\}.t_1\right]$$

Similarly,

$$\text{Total Inspection cost is} = I_c\int_0^{t_1}\left[\alpha - \beta_1 Q_1(t) - \beta_2 Q_2(t)\right]dt$$
$$= I_c\left[\frac{-1}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2 r^n}{1-r} - (1-\lambda)\alpha\right] + \beta_1\left[d_1 - \lambda\alpha\right]\right\}.\left(1 - e^{-\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)t_1}\right)\right. \tag{26}$$
$$\left. + \frac{1}{\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)}\left\{\frac{\beta_2 d_2 r^n}{1-r} + \beta_1 d_1\right\}.t_1\right]$$

$$\text{Revenue from perfect quality items is} = Sd_1T$$

$$= S\left[\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\} \right.$$
$$\left. .\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right) + \frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1\right] \tag{27}$$

and

$$\text{Revenue from imperfect quality items is} = S(1-r)\left(\frac{r^n}{1-r}\right)d_2T'$$

$$= S(1-r)\left[\frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}\right.$$
$$\left. .\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right) + \frac{(1-\lambda)}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1\right] \tag{28}$$

The Net Profit from each production cycle is given by

$\pi(t_1,r) =$ Revenue from perfect quality items + Revenue from imperfect quality items

– Total production cost – Total Inspection cost – Total Inventory cost.

$$\therefore\ \pi(t_1,r) = \left\{S[1-r(1-\lambda)]-(\eta+I_c)+\frac{C_h}{(\lambda\beta_1+(1-\lambda)\beta_2)}\right\}\times$$
$$\left\{\frac{1}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.+\frac{1}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1\right\}$$
$$-C_h\left\{\frac{\alpha}{(\lambda\beta_1+(1-\lambda)\beta_2)}t_1+\frac{(\beta_2-\beta_1)}{2(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1\right\}.t_1^2\right.$$
$$\left.+\frac{1}{2}\left[d_1(T-t_1)^2+\frac{r^n}{1-r}d_2(T'-t_1)^2\right]\right\} \tag{29}$$

where,

$$T-t_1 = \frac{1}{d_1}\left\{\frac{\lambda}{(\lambda\beta_1+(1-\lambda)\beta_2)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1[d_1-\lambda\alpha]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.+\frac{\beta_2}{(\lambda\beta_1+(1-\lambda)\beta_2)}\left\{\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1\right\}.t_1\right\}$$

and

$$T'-t_1=\left(\frac{1-r}{r^n}\right).\frac{1}{d_2}\left\{\frac{(1-\lambda)}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1\left[d_1-\lambda\alpha\right]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.-\frac{\beta_1}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)}\left\{\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1\right\}.t_1\right\}$$

Hence, to obtain the optimal values of t_1 and r, we have to

Maximize $\pi(t_1,r)$

Subject to the Constraints

(i) Total production of a cycle $\geq d_1+\frac{r^n}{1-r}d_2$

i.e. $\int_0^{t_1}\left[\alpha-\beta_1Q_1(t)-\beta_2Q_2(t)\right]dt\geq d_1+\frac{r^n}{1-r}d_2$,

i.e. $$\frac{1}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1\left[d_1-\lambda\alpha\right]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)$$
$$+\frac{1}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1\ \geq\ d_1+\frac{r^n}{1-r}d_2$$

(ii) $0<t_1<T$ i.e. $0<t_1$ and $T-t_1>0$.

i.e.

$$\frac{1}{d_1}\left\{\frac{\lambda}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1\left[d_1-\lambda\alpha\right]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.+\frac{\beta_2}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)}\left\{\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1\right\}.t_1\right\}\ >\ 0.$$

(iii) $t_1\leq T'$ i.e. $T'-t_1\geq 0$

i.e.

$$\left(\frac{1-r}{r^n}\right).\frac{1}{d_2}\left\{\frac{(1-\lambda)}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1\left[d_1-\lambda\alpha\right]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.-\frac{\beta_1}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)}\left\{\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1\right\}.t_1\right\}\ \geq\ 0$$

(iv) $T'\leq T$ i.e. $T-T'\geq 0$

i.e.

$$\left[\frac{1}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)^2}\left\{\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1\left[d_1-\lambda\alpha\right]\right\}.\left(e^{-(\lambda\beta_1+(1-\lambda)\beta_2)t_1}-1\right)\right.$$
$$\left.+\frac{1}{\left(\lambda\beta_1+(1-\lambda)\beta_2\right)}\left\{\frac{\beta_2d_2r^n}{1-r}+\beta_1d_1\right\}.t_1\right]\times\left[\frac{\lambda}{d_1}-\left(\frac{1-r}{r^n}\right)\left(\frac{1-\lambda}{d_2}\right)\right] \geq 0$$

(v) $0<r<1$.

The above optimization problem can be solved by 'Interior Penalty Function Method' for the optimal values of t_1 and r. (See Appendix)

Special Case: When r is known and fixed. $(0<r<1)$

Let us take,

$$\lambda\beta_1+(1-\lambda)\beta_2=M,$$
$$\beta_2\left[\frac{d_2r^n}{1-r}-(1-\lambda)\alpha\right]+\beta_1\left[d_1-\lambda\alpha\right]=A \text{ and}$$
$$\frac{\lambda d_2r^n}{1-r}-(1-\lambda)d_1=C$$
$$\Rightarrow \frac{\beta_2d_2r^n}{1-r}+\beta_1d_1=A+\alpha M$$

Then, we can re-write our model as

$$\begin{aligned}\textit{Maximize }\ \pi(t_1)=&\left\{S\left[1-r(1-\lambda)\right]-\left(\eta+I_c\right)+\frac{C_h}{M}\right\}\times\left\{\frac{A}{M^2}(e^{-Mt_1}-1)+\frac{(A+\alpha M)}{M}t_1\right\}\\&-C_h\left\{\frac{\alpha}{M}t_1+\frac{(\beta_2-\beta_1)C}{2M}t_1^2+\frac{1}{2d_1}\left\{\frac{\lambda A}{M^2}(e^{-Mt_1}-1)+\frac{\beta_2C}{M}t_1\right\}^2\right.\\&\left.+\frac{1}{2d_2}\left(\frac{1-r}{r^n}\right)\left\{\frac{(1-\lambda)A}{M^2}(e^{-Mt_1}-1)-\frac{\beta_1C}{M}t_1\right\}^2\right\}\end{aligned} \tag{30}$$

Subject to the Constraints

(i) $\frac{A}{M^2}(e^{-Mt_1}-1)+\frac{(A+\alpha M)}{M}t_1 \geq d_1+\frac{r^n}{1-r}d_2.$

(ii) $\frac{1}{d_1}\left\{\frac{\lambda A}{M^2}(e^{-Mt_1}-1)+\frac{\beta_2C}{M}t_1\right\} > 0.$

(iii) $\left(\frac{1-r}{r^n}\right).\frac{1}{d_2}\left\{\frac{(1-\lambda)A}{M^2}(e^{-Mt_1}-1)-\frac{\beta_1C}{M}t_1\right\} \geq 0.$

(iv) $\left[\frac{A}{M^2}(e^{-Mt_1}-1)+\frac{(A+\alpha M)}{M}t_1\right]\times\left[\frac{\lambda}{d_1}-\left(\frac{1-r}{r^n}\right)\left(\frac{1-\lambda}{d_2}\right)\right] \geq 0.$

(v) $0<r<1$ and $t_1>0$.

Now, if r is fixed, then the optimal value of t_1 that maximizes π can be obtained by setting $\frac{\mathrm{d}\pi}{\mathrm{d}t_1}=0$ if $\frac{\mathrm{d}^2\pi}{\mathrm{d}t_1{}^2}<0$ at the value of t_1 for which $\frac{\mathrm{d}\pi}{\mathrm{d}t_1}=0$.

Here,

$$\begin{aligned}\frac{\mathrm{d}\pi}{\mathrm{d}t_1} &= \left\{S[1-r(1-\lambda)]-(\eta+I_c)+\frac{C_h}{M}\right\}\times\left\{\frac{A}{M}(1-e^{-Mt_1})+\alpha\right\}\\ &\quad -\frac{C_h}{M^2}\left\{\alpha M+(\beta_2-\beta_1)CMt_1+\frac{1}{d_1}\left\{-\frac{\lambda A}{M}(1-e^{-Mt_1})+\beta_2 Ct_1\right\}\times\left\{-\lambda Ae^{-Mt_1}+\beta_2 C\right\}\right.\\ &\qquad \left.+\frac{1}{d_2}\left(\frac{1-r}{r^n}\right)\left\{\frac{(1-\lambda)A}{M}(1-e^{-Mt_1})+\beta_1 Ct_1\right\}\times\left\{(1-\lambda)Ae^{-Mt_1}+\beta_1 C\right\}\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2\pi}{\mathrm{d}t_1^2} &= \left\{S[1-r(1-\lambda)]-(\eta+I_c)+\frac{C_h}{M}\right\}.Ae^{-Mt_1}\\ &\quad -\frac{C_h}{M^2}\left\{(\beta_2-\beta_1)CM+\frac{1}{d_1}\left\{-\frac{\lambda A}{M}(1-e^{-Mt_1})+\beta_2 Ct_1\right\}.\lambda AMe^{-Mt_1}+\frac{1}{d_1}\left\{-\lambda Ae^{-Mt_1}+\beta_2 C\right\}^2\right.\\ &\quad \left.+\frac{1}{d_2}\left(\frac{1-r}{r^n}\right)\left\{\frac{(1-\lambda)A}{M}(1-e^{-Mt_1})+\beta_1 Ct_1\right\}\times\left\{-(1-\lambda)AMe^{-Mt_1}\right\}+\frac{1}{d_2}\left(\frac{1-r}{r^n}\right)\left\{(1-\lambda)Ae^{-Mt_1}+\beta_1 C\right\}^2\right\}\\ &= -\frac{C_h}{M^2}\left[(\beta_2-\beta_1)CM+\frac{1}{d_1}\left\{-\lambda Ae^{-Mt_1}+\beta_2 C\right\}^2+\frac{1}{d_2}\left(\frac{1-r}{r^n}\right)\left\{(1-\lambda)Ae^{-Mt_1}+\beta_1 C\right\}^2\right]\\ &\quad +Ae^{-Mt_1}\left\{S[1-r(1-\lambda)]-(\eta+I_c)+\frac{C_h}{M}\left[1-\frac{\lambda}{d_1}\left\{\frac{\lambda A}{M}(e^{-Mt_1}-1)+\beta_2 Ct_1\right\}-\frac{(1-\lambda)}{d_2}\left(\frac{1-r}{r^n}\right)\left\{\frac{(1-\lambda)A}{M}(e^{-Mt_1}-1)-\beta_1 Ct_1\right\}\right]\right\}\end{aligned}$$

where

$$\frac{1}{d_1}\left\{\frac{\lambda A}{M}(e^{-Mt_1}-1)+\beta_2 Ct_1\right\} > 0 \qquad \text{(using constraint (ii))}$$

and

$$\frac{1}{d_2}.\left(\frac{1-r}{r^n}\right)\left\{\frac{(1-\lambda)A}{M}(e^{-Mt_1}-1)-\beta_1 Ct_1\right\} \geq 0. \qquad \text{(using constraint (iii))}$$

Also, from constraints (i) and (iv)

$$C=\lambda d_2\left(\frac{r^n}{1-r}\right)-(1-\lambda)d_1\geq 0\cdot$$

Thus, the value of t_1 (say t_1') obtained by setting $\frac{\mathrm{d}\pi}{\mathrm{d}t_1}=0$ which also satisfies the constraints for the feasibility of the model is an optimal value if $\left.\frac{\mathrm{d}^2\pi}{\mathrm{d}t_1^2}\right|_{t_1=t_1'}<0$ i.e. if

$$Ae^{-Mt_1'}\left\{S[1-r(1-\lambda)]-(\eta+I_c)+\frac{C_h}{M}\left[\begin{array}{l}1-\frac{\lambda}{d_1}\left\{\frac{\lambda A}{M}(e^{-Mt_1'}-1)+\beta_2 Ct_1'\right\}-\frac{(1-\lambda)}{d_2}\left(\frac{1-r}{r^n}\right)\\ \left\{\frac{(1-\lambda)A}{M}(e^{-Mt_1'}-1)-\beta_1 Ct_1'\right\}\end{array}\right]\right\}<0 \text{ and } \beta_2>\beta_1.$$

4. NUMERICAL EXAMPLE AND CONCAVITY OF THE FUNCTION

Example1. Let the values of our model parameters in their appropriate units be $n=2,\ \lambda=0.9,\ \eta=\$160,\ C_h=\$20,\ I_c=\$10,\ S=\$200,\ \alpha=2100,\ \beta_1=0.2,\ \beta_2=0.3,$ $d_1=1500,\ d_2=1000.$

Then, using the interior penalty function method for our problem, we get optimal production-run time, $t_1^*=13.10636$; optimal discount, $r^*=0.3459169$; Maximum profit, $\pi^*(t_1^*,r^*)=\$179118.50$; Total Production of the cycle = 23977 units; $T^*=14.38612$ and $T'^*=13.10636$.

We now generate a graph of $\pi(t_1,r)$ based on the parameter values taken in numerical example 1 to depict the concavity of the function (Fig. 2).

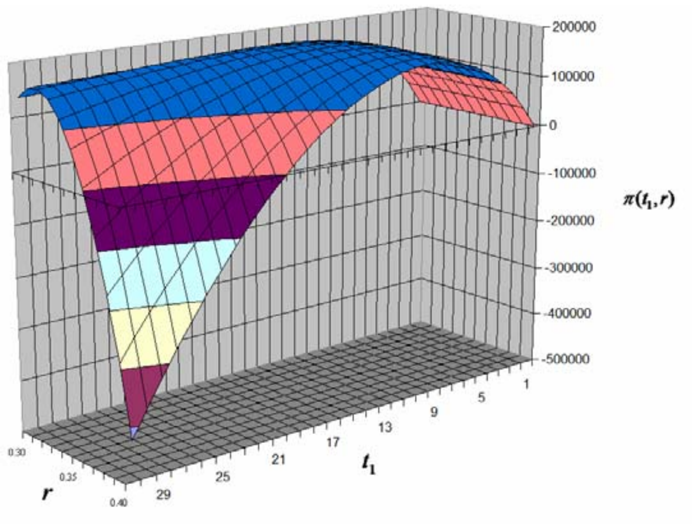

Fig. 2. Concavity of the Profit Function $\pi(t_1,r)$

Example 2. Using the same values of parameters as mentioned in example 1 and taking a fixed value of $r = 0.35$, and solving for the optimal value of t_1 we get $t_1^* = 8.549684$; Maximum profit, $\pi^*(t_1^*) = \$169640.00$; Total Production of the cycle = 16113units; $T^* = 9.667719$ and $T'^* = 8.549684$. (Fig. 3).

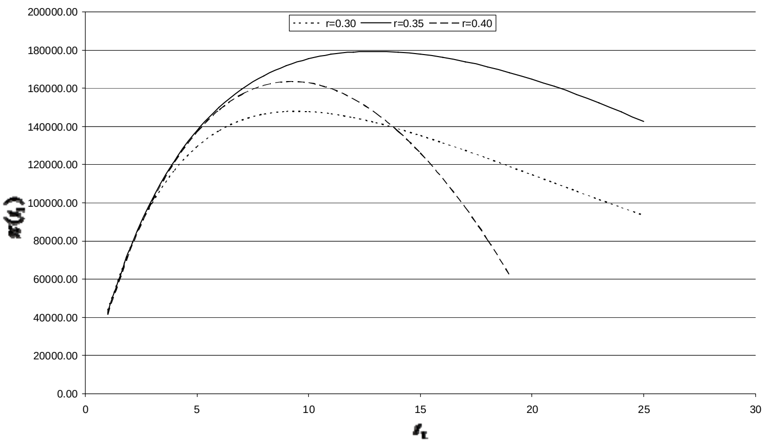

Figure 3. Concavity of the Profit Function $\pi(t_1)$ when r is fixed

5. CONCLUSION

This paper presents an inventory model to calculate the optimal production-run time for a system which manufactures both perfect and imperfect quality items and is volume flexible. The production rate of the system depends on the on-hand inventory level. The proposed model considers the demand rate of perfect quality items to be constant and that of imperfect quality items to depend on the discount rate offered in the selling price to influence the sales of imperfect quality items. Thus, the model also aims at finding the optimal value of the discount rate along with the production-run time that maximizes the total net profit. This paper also examines the model under the condition of a fixed discount rate. Finally, it includes some numerical examples to demonstrate the applicability of the proposed model.

The future research on this model aims at extending it by allowing for partial or complete backordering.

6. APPENDIX

Primal problem (General form):

Minimize $\Pi(X)$ = - Maximize $\Pi(X)$

such that $G_j(X) \le 0, \quad j = 1,2,.....,m.$

where $\Pi(X)$, $G_j(X)$ are continuous functions of $X \in R^n$

The above problem can be re-written as the following unconstrained optimization problem.

Interior penalty method: This method generally deals with an unconstrained minimization problem. The general form of the problem equivalent to the Primal Problem is:

Minimize $\chi_k(X, r_k) = \Pi(X) - r_k \sum_{j=1}^{m} \frac{1}{G_j(X)}$,

where r_k is a positive penalty parameter.

If χ_k is minimized for a sequence of decreasing values of r_k, the following theorem proves that the unconstrained minima $X_k^*(k = 1,2,....,m)$ converges to the solution X^* of the primal problem stated above.

***Theorem**: If the primal problem has a solution, the unconstrained minima X_k^* of $\chi_k(X, r)$ for a sequence of values $r_1 > r_2 > \cdots > r_k$, converge to the optimal solution of the primal problem.*

The iterative procedure:

Step 1: Start with an initial feasible point X_1 satisfying all the constraints with strict inequality sign, i.e., $G_j(X_1) < 0$ for $j = 1,2,.....,m$ and a suitable initial value of r_1 where $r_1 = -\frac{\Pi(X_1)}{\sum_{j=1}^{m} \frac{1}{G_j(X_1)}}$. Set $k = 1$.

Step 2: Minimize $\chi_k(X_k, r_k)$ by using any method of unconstrained minimization (we may use here the *Devidon Fletcher– Powell Method*) and obtain the solution X_k^*.

Step 3: Test whether $\left|\frac{\Pi(X_k^*) - \Pi(X_{k+1}^*)}{\Pi(X_k^*)}\right| \le \in_1$ $\left|X_k^* - X_{k-1}^*\right| < \in_2$ where $\in_1$ and $\in_2$ are arbitrarily small positive numbers. If it is satisfied, then terminate the process; otherwise, go to the next step.

Step 4: Find the value of the next penalty parameter r as $r_{k+1} = cr_k$ where $0 < c < 1$.

Step 5: Set the new value of $k = k + 1$, take the new starting point as $X_1 = X_k^*$ and then go to step 2.

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