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SCARF'S GENERALIZATION OF LINEAR COMPLEMENTARITY PROBLEM REVISITED

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Abstract: In this paper, we revisit Scarf's generalization of the linear complementarity problem, formulate this as a vertical linear complementarity problem and obtain some new results on this generalization. Also, a neural network model for solving Scarf's generalized complementarity problems is proposed. Numerical simulation results show that the proposed model is feasible and efficient.

Keywords: Scarf's complementarity problem, Vertical linear complementarity problem, Cottle-Dantzig's algorithm, Lemke's algorithm, Neural network approach.

MSC: 90C33, 92B20.

1. INTRODUCTION

Given a matrix $M \in R^{n\times n}$ and a vector $q \in R^n$, the linear complementarity problem denoted by LCP (q, M), is to find $w \in R^n$ and $z \in R^n$ such that

$$w - Mz = q,\ w \geq 0, z \geq 0 \tag{1.1}$$

$$w^t z = 0\,. \tag{1.2}$$

LCP is normally identified as a problem of mathematical programming and provides a unifying framework for several optimization problems. For recent books on this problem and applications see Cottle, Pang and Stone [3] and the references cited therein.

Scarf [23] introduced a generalization of the linear complementarity problem to accomodate more complicated real life problems as well as to diversify the field of applications. In this paper, we consider a generalization by Scarf, known as Scarf's generalized linear complementarity problem which involves a vertical block matrix. The concept of a vertical block matrix was introduced by Cottle and Dantzig [2] in connection with vertical linear complementarity problem. Consider a rectangular matrix A of order $m \times k$ with $m \geq k$. Suppose A is partitioned row-wise into k blocks. A is said to be a vertical block matrix of type $(m_1, m_2, ..., m_k)$ if A is partitioned row-wise into k blocks $A^j \in R^{m_j \times k}$, $j = 1, 2, ..., k$ such that $\sum_{j=1}^{k} m_j = m$ in the form $A = \left[A^1 \dots A^k\right]^t$ where each $A^j = ((a^j_{rs})) \in R^{m_j \times k}$ with $\sum_{j=1}^{k} m_j = m$ Then A is called a *vertical block matrix of type* $(m_1, m_2, \dots, m_k)$. If $m_j = 1, \forall\ j = 1, \dots, k,$ then A is a square matrix. The vector $q \in R^m$ is also partitioned into k subvectors $q^j \in R^{m_j}$, $j = 1, 2, \dots, k$. We use the notation $J_1 = \{1, 2, \dots, m_1\}$ to denote the set of row indices of the first block in A and $J_r = \{\sum_{j=1}^{r-1} m_j + 1, \sum_{j=1}^{r-1} m_j + 2, \dots, \sum_{j=1}^{r} m_j\}$ to denote the set of row indices of the r^{th} block in A for $r = 2, 3, \dots, k$.

Scarf [23] introduced the following interesting generalization of the linear complementarity problem involving a vertical block matrix A of type $(m_1, m_2, \dots, m_k)$ described above. Let $M^j(x)$ where $0 \leq x \in R^k$ be k homogeneous linear functions, each of which is the maximum of a finite number of linear functions and $q = [q^1, q^2, \dots, q^k]^t \in R^k$ be a vector. Scarf poses the following problem. Under what conditions can we say that the equations

$$\begin{aligned} M^1(x) - r_1 &= q^1 \\ M^2(x) - r_2 &= q^2 \\ &\vdots \\ M^k(x) - r_k &= q^k \end{aligned}$$

have a solution in nonnegative variables x and r with $x_j r_j = 0$ for all j?

Note that the important difference of Scarf's problem and LCP (see [23]) is that each linear function is replaced by the maximum of several linear functions. Scarf pointed out that if $M^j(x)$ are the minimum rather than the maximum of linear functions, the problem could be solved by a trivial reformulation of LCP.

A slightly generalized version of Scarf's complementarity problem stated by Lemke [13] is as follows:

Given an $m \times k,\ m \geq k$ vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$ and $\bar{q} \in R^m$ where $\sum_{j=1}^{k} m_j = m$ find $x \in R^k$ such that

$$r_j(x) = \max_{i \in J_j} (A^j x + \bar{q}^j)_i \geq 0,\ j = 1, \ldots, k,\ x \geq 0 \tag{1.3}$$

$$\sum_{j=1}^{k} x_j r_j(x) = 0, \tag{1.4}$$

where A is a given $m \times k,\ m \geq k$ vertical block matrix of type $(m_1, m_2, \ldots, m_k),\ \bar{q} \in R^m$ and $\sum_{j=1}^{k} m_j = m.$

For a given a matrix $A \in R^{m \times k}$ of type $(m_1, m_2, \ldots, m_k)$ and $\bar{q} \in R^m$, let $x^* \in R^k_+$ is a solution of SCP $(\bar{q}, A).$ Then in the solution x^* if $x^*_j = 0$ then $(A^j x^* + \bar{q}^j)_i \geq 0$ for atleast one $i \in J_j$ and if $x^*_j > 0$ then $max(A^j x^* + \bar{q}^j)_i = 0.$

We refer to this generalization as Scarf's complementarity problem and denote this problem by SCP$(\bar{q}, A).$ Lemke [13] formulated the Scarf's complementarity problem as a linear complementarity problem LCP(q, M) but he remained silent about the processability of this problem by his algorithm. Lemke [13] showed that this formulation arises for calculating a vector in the core of an n person game (see [24]).

In section 2, we provide the necessary definitions and notations used in this paper. Some results on solutions of a Scarf's Complementarity Problem are presented in section 3. In section 4, we present equivalent formulation of Scarf's Complementarity Problem SCP$(\bar{q}, A)$ as a vertical linear complementarity problem. A neural network model for Scarf's complementarity problem is presented in section 5. Finally, section 6 presents concluding remarks.

2. PRELIMINARIES

Let A be a vertical block matrix of type $(m_1, m_2, \ldots, m_k).$ A submatrix of size k of A is called a *representative submatrix* if its j^{th} row is drawn from the j^{th} block A^j of A. Clearly, a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$ has at most $\prod_{j=1}^{k} m_j$ distinct representative submatrices. An example of Scarf's Complementarity Problem is given below.

Example 1 *Consider the vertical block matrix* A *of type* $(2,2,1)$ *given below where*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & 1 \\ 4 & 0 & 1 \\ 4 & 2 & 7 \\ 1 & 1 & 3 \end{bmatrix} \text{ and } \bar{q} = \begin{bmatrix} -1 \\ -1 \\ -4 \\ -4 \\ 2 \end{bmatrix}.$$

The vertical block matrix A has four representative matrices namely $A_1 = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 0 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 2 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 0 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 2 & 7 \\ 1 & 1 & 3 \end{bmatrix}$. The $\bar{q}$ vector corresponding to representative submatrices A_1, A_2, A_3 and A_4 are $\bar{q}_1 = \bar{q}_2 = \bar{q}_3 = \bar{q}_4 = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$.

The solutions corresponding to representative submatrices A_1, A_2, A_3 and A_4 are given as follows.

$x_1 = 1,\ x_2 = 0, x_3 = 0$ for LCP $(\bar{q}_1, A_1)$

$x_1 = 1,\ x_2 = 0, x_3 = 0$ and $x_1 = 0, x_2 = 2, x_3 = 0$ for LCP $(\bar{q}_2, A_2)$

$x_1 = 1,\ x_2 = 1, x_3 = 0$ for LCP $(\bar{q}_3, A_3)$

$x_1 = \frac{5}{7},\ x_2 = \frac{4}{7}, x_3 = 0$ for LCP $(\bar{q}_4, A_4)$

Note that only the solution $x_1 = 0, x_2 = 2, x_3 = 0$ obtained from LCP $(\bar{q}_2, A_2)$ produces a solution for SCP $(\bar{q}, A)$.

Another generalization of the linear complementarity problem by Cottle and Dantzig [2] appears in the literature in connection with a vertical block matrix. This generalization is presented below.

Given a vertical block matrix $A \in R^{m\times k}, (m \geq k)$ of type $(m_1, m_2, \ldots, m_k)$ and $q \in R^m$ where $\sum_{j=1}^{k} m_j = m$ the vertical linear complementarity problem is to find $w \in R^m$ and $z \in R^k$ such that

$$w - Az = q,\ w \geq 0,\ z \geq 0 \tag{2.1}$$

$$z_j \prod_{i=1}^{m_j} w_i^j = 0, \; j = 1, 2, \ldots, k. \tag{2.2}$$

Cottle-Dantzig's [2] generalization was designated later by the name *vertical linear complementarity problem* [3] and this problem is denoted as VLCP(q, A). Lemke's algorithm for the LCP(q, M) (see [3]) has been extended with some modifications to the VLCP(q, A) by Cottle and Dantzig in [2].

Different classes of matrices in LCP literature:
We say that $M \in R^{n \times n}$ is

- P-matrix if all its principal minors are positive.
- *copositive* (C_0) (*strictly copositive* (C)) if $x^t M x \geq 0 \;\forall\; x \geq 0$ $(x^t M x > 0$ $\forall\; 0 \neq x \geq 0)$.
- *copositive-plus* (C_0^+) if $M \in C_0$ and the implication $[x^t M x = 0, x \geq 0] \Rightarrow$ $(M + M^t)x = 0$ holds.
- L_1*-matrix* if for every $0 \neq y \geq 0,\; y \in R^n$ $\exists$ an i such that $y_i > 0$ and $(My)_i \geq 0$.
- L_2 *-matrix* if for each $0 \neq \xi \geq 0,\; \xi \in R^n$ satisfying $\eta = M\xi \geq 0$ and $\eta^t \xi = 0$ $\exists$ a $0 \neq \hat{\xi} \geq 0$ satisfying $\hat{\eta} = -M^t \hat{\xi},\; \eta \geq \hat{\eta} \geq 0,\; \xi \geq \hat{\xi} \geq 0.$
- L *-matrix* if it is in both L_1 and L_2.

If $M \in L$ then Lemke's algorithm can process LCP(q, M).
Different classes of vertical block matrix:
A vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$ is called a *vertical block* $P\left(C_0, C, C_0^+, L_1\right)$*-matrix* if all its representative submatrices are P $(C_0, C,\; C_0^+,\; L_1)$-matrices.

A vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$ is called a vertical block matrix with copositive-plus property if every representative submatrix is copositive-plus.

3. ON SOLUTION OF A SCARF'S COMPLEMENTARITY PROBLEM

First we observe the following result on solutions of Scarf's complementarity problem.
Theorem 3.1 *SCP*$(\bar{q}, A)$ *has a solution if and only if there exists a representative submatrix* A_R *and a corresponding subvector* $\bar{q}_R$ *of* $\bar{q}$ *so that LCP*$(\bar{q}_R, A_R)$ *is solvable with a solution* x *that satisfies* $\max_{i \in J_j} (\bar{q}^j + A^j x)_i = 0$ *if* $x_j > 0$ *and* $\max_{i \in J_j} (\bar{q}^j + A^j x)_i \geq 0$ *if* $x_j = 0.$

Proof. Suppose SCP$(\bar{q}, A)$ has a solution $x \in R_{+}^{k}$. Then $\max_{i \in J_j}(\bar{q}^j + A^j x)_i \geq 0$ *for* $j=1,\ldots,k$. Let $i_1,\ldots,i_p$ be the row indices for which $\max_{i \in J_j}(\bar{q}^j + A^j x)_i = 0$ and $i_{p+1}, i_{p+2},\ldots,i_k$ be the row indices for which $\max_{i \in J_j}(\bar{q}^j + A^j x)_i \geq 0$. Let A_R be the corresponding representative submatrix, i.e., the submatrix which corresponds to the rows of the blocks of A that results in the maximum and $\bar{q}_R$ be the corresponding subvector of $\bar{q}$. Consequently, x is a solution of LCP$(\bar{q}_R, A_R)$. Conversely, let x be a solution of LCP$(\bar{q}_R, A_R)$ as defined above, that satisfies $\max_{i \in J_j}(\bar{q}^j + A^j x)_i = 0$ if $x_j > 0$ and $\max_{i \in J_j}(\bar{q}^j + A^j x)_i \geq 0$ if $x_j = 0$. Therefore, clearly x is also a solution of SCP$(\bar{q}, A)$. This completes the proof.

Note that conditions stated in the above theorem cannot be relaxed. The following example shows that even though LCP$(\bar{q}_R, A_R)$ involving both representative submatrices have a solution but SCP$(\bar{q}, A)$ has no solution.

Example 1 *Consider the vertical block matrix A of type* $(2,1)$ *as given below*
$A = \begin{bmatrix} 0 & 8 \\ 0 & -5 \\ 4 & 0 \end{bmatrix}$ and $\bar{q} = \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$. The above vertical block matrix A has two representative matrices namely $A_1 = \begin{bmatrix} 0 & 8 \\ 4 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & -5 \\ 4 & 0 \end{bmatrix}$.

The $\bar{q}$ vector corresponding to A_1 and A_2 are $\bar{q}_1 = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$ and $\bar{q}_2 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ respectively.

The solutions corresponding to representative submatrices A_1 and A_2 are as follows.

$x_1 = 0.5, x_2 = 0.5$ for LCP $(\bar{q}_1, A_1)$ and $x_1 = 0.5, x_2 = 1$ for LCP $(\bar{q}_2, A_2)$.

But it may be noted that though both the representative submatrices have solutions, none is a solution of the SCP $(\bar{q}, A)$.

However in the above example, VLCP$(\bar{q}, A)$ has two solutions namely $x_1 = 0.5, x_2 = 0.5$ and $x_1 = 0.5, x_2 = 1$.

Scarf [23] proves the following theorem on existence of solution of a SCP.

Theorem 3.2 *(Scarf [23]) Suppose* $A = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^k \end{bmatrix}$ *where* $M_j x = \max_{i \in J_j} (A^j x)_i$. *A solution to SCP* $(\bar{q}, A)$ *exists for all* $\bar{q}$ *if*

$$x \geq 0 \text{ and } \sum_j x_j \max_{i \in J_j} (A^j x)_i \leq 0, \text{ i.e., } x^t Mx \leq 0 \Rightarrow x = 0.$$

Theorem 3.3 *Suppose* A *is a vertical block* $P-matrix$ *of type* $(m_1, m_2, \ldots, m_k)$. *Then SCP* $(\bar{q}, A)$ *has a solution for all* $\bar{q}$.

Proof. Suppose R is a representative submatrix of A, where A is a vertical block $P-matrix$. Then by definition, R is a $P-matrix$. Therefore $x_j (Rx)_j \leq 0 \; \forall \; j \Rightarrow x = 0$, i.e., $\sum_{j=1}^{k} x_j (Rx)_j \leq 0 \Rightarrow x = 0$. Since $x_j (Rx)_j \leq 0 \; \forall \; j \Rightarrow x = 0$, for each of the representative submatrices, it follows that

$$x \geq 0 \text{ and } \sum_j x_j \max_{i \in J_j} (A^j x)_i \leq 0, \text{ i.e., } x^t Mx \leq 0 \Rightarrow x = 0,$$

where M is as defined in Theorem 3.2. This completes the proof.

Theorem 3.4 *Suppose* A *is a vertical block strictly copositive matrix. Then a solution to SCP* $(\bar{q}, A)$ *exists for all* $\bar{q}$.

Proof. Suppose A is a vertical block strictly copositive matrix. Then each representative submatrix R of A is strictly copositive. By definition $x^t Rx > 0 \; \forall \; 0 \neq x \geq 0$. Therefore, $x \geq 0, \, x^t Rx \leq 0 \Rightarrow x = 0$. Let M be the representative submatrix as defined in Theorem 3.2. Therefore, $x \geq 0, \, x^t Mx \leq 0 \Rightarrow x = 0$. So, by Theorem 3.2, a solution to SCP$(\bar{q}, A)$ exists for all $\bar{q}$. This completes the proof.

Remark 3.1 *We observe in the above two theorems that if* A *is a vertical block* P *or vertical block strictly copositive matrix then a solution to SCP* $(\bar{q}, A)$ *exists for all* $\bar{q}$. *i.e., the vertical block matrix* A *has* Q*-property.*

Now, we consider vertical block matrices with some special structures.

Example 2 *Consider a vertical block* P*-matrix* A *of type* $(2,1)$ *given below where*

$A = \begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 1 & 2 \end{bmatrix}$ and $\bar{q} = \begin{bmatrix} -6 \\ -4 \\ -4 \end{bmatrix}$. The above vertical block matrix A has two representative submatrices namely, $A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

Note that both A_1 and A_2 are P-matrices. The $\bar{q}$ vector corresponding to A_1 and A_2 are $\bar{q}_1 = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$ and $\bar{q}_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$ respectively.

The unique solutions corresponding to representative submatrices A_1 and A_2 are as follows.

$x_1 = 2, x_2 = 1$ for LCP $(\bar{q}_1, A_1)$ and $x_1 = 0, x_2 = 2$ for LCP $(\bar{q}_2, A_2)$.

It is easy to see that the unique solution corresponding to SCP $(\bar{q}, A)$ is given by $x_1 = 0,\ x_2 = 2$.

It is well known that LCP$(\bar{q}_R, A_R)$ has a unique solution where $A_R \in R^{n\times n}$ is a $P-matrix$. This result also holds true for VLCP$(\bar{q}, A)$ where A is a vertical block $P-matrix$ of type $(m_1, m_2, \ldots, m_k)$. Now we raise the following question - Is this also true for SCP $(\bar{q}, A)$, where A is a vertical block $P-matrix$ of type $(m_1, m_2, \ldots, m_k)$? Note that this issue is not addressed in Scarf [23], Lemke [13] and Eaves [4]. We pose this as an open problem.

3.1 Open problem

Is it true that A is a vertical block $P-matrix$ of type $(m_1, m_2, \ldots, m_k)$ if and only if SCP $(\bar{q}, A)$ has a unique solution for every $\bar{q} \in R^m$?

4. AN EQUIVALENT FORMULATION OF SCP

In this section we show that Scarf's complementarity problem may be formulated as vertical linear complementarity problem. This formulation is advantageous compare to LCP formulation presented by Lemke [13] from algorithmical point of view. If Lemke's algorithm is applied on Lemke's LCP formulation of Scarfs problem, it will execute some trivial pivots. However, if we apply Cottle-Dantzig algorithm on the VLCP formulation presented here, then the trivial pivots may be skipped.

Formulation I: Scarf's complementarity problem SCP$(\bar{q}, A)$ can be posed as a VLCP. Let $w \in R^m$ and $u, v, z, x, r \in R^k$. Also let us define a matrix $F \in R^{m\times k}$ as follows:

$$F = \begin{bmatrix} e^1 & 0 & \dots & 0 \\ 0 & e^2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & e^k \end{bmatrix}$$

where e^j is a column vector of 1's of order m_j.

Lemke [13] formulates this problem as an equivalent linear complementarity problem LCP(q^*, M) where

$$q^* = \begin{bmatrix} -\bar{q} \\ -e_k \\ 2e_k \end{bmatrix} \text{and } M = \begin{bmatrix} 0 & -A & F \\ F^t & 0 & 0 \\ -F^t & 0 & 0 \end{bmatrix}$$

Note that M is a square matrix of order $(m+2k)$, $e_k \in R^k$ is a column vector of all 1's of order k.

Now we show that Scarf's problem can be represented as a VLCP.

Let I be the identity matrix of order k and $e \in R^k$ be a unit vector of all 1's. Note that for a given x, the scalar

$$r_j(x) = \max_{i \in J_j} (A^j x + \bar{q}^j)_i, \ j = 1, \dots, k, \ x \geq 0$$

is equivalent to the system

$$w^j = -\bar{q}^j - A^j x + r_j e^j, w^j \geq 0, w^j \ngtr 0. \tag{4.1}$$

The system (4.1) along with the Scarf's complementarity condition

$$\sum_{j=1}^{k} x_j r_j(x) = 0, \ j = 1, 2, ..., k \tag{4.2}$$

give rise to the following vertical linear complementarity problem. The **equivalent VLCP** is

$$\begin{bmatrix} w \\ u \\ v \end{bmatrix} = \begin{bmatrix} -\bar{q} \\ -e \\ 2e \end{bmatrix} + \begin{bmatrix} 0 & -A & F \\ I & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ x \\ r \end{bmatrix}, \ \begin{bmatrix} w \\ u \\ v \end{bmatrix}, \begin{bmatrix} z \\ x \\ r \end{bmatrix} \geq 0$$

$$z_j \prod_{i=1}^{m_j} w_i^j = 0, j = 1, 2, \dots, k,$$

$$u^t x = 0, \quad v^t r = 0.$$

Here

$$A_1 = \begin{bmatrix} 0 & -A & F \\ I & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \in R^{(m+2k)\times 3k} \text{ and } \hat{q} = \begin{bmatrix} -\bar{q} \\ -e \\ 2e \end{bmatrix} \in R^{m+2k}.$$

Note that A_1 is a vertical block matrix of type $(m_1,\ldots,m_k,1,\ldots,1,1,\ldots,1)$. We call this problem as VLCP$(\hat{q}, A_1)$.

We make use of the idea similar to Lemke [13] to prove the following lemma.

Lemma 4.1 *Consider the problem $SCP(\bar{q}, A)$ and $VLCP(\hat{q}, A_1)$ as defined earlier. $SCP(\bar{q}, A)$ has a solution if and only if $VLCP(\hat{q}, A_1)$ has a solution.*

Proof. Let $(\bar{\eta}, \bar{\xi}), \bar{\eta} \in R^{(m+2k)}, \bar{\xi} \in R^{3k}$ be a solution to VLCP$(\hat{q}, A_1)$ where $\bar{\eta} = \begin{bmatrix} w^t & u^t & v^t \end{bmatrix}^t$ *and* $\bar{\xi} = \begin{bmatrix} z^t & x^t & r^t \end{bmatrix}^t$.

Note that $u_j + v_j = 1$ and $z_j \geq 0 \Rightarrow \prod_{i \in J_j} w_i^j = 0 \Rightarrow w^j \not> 0$. Since $r_j > 0 \Rightarrow v_j = 0 \Rightarrow u_j = 1$ $\Rightarrow x_j = 0$ and $x_j > 0 \Rightarrow u_j = 0 \Rightarrow v_j = 1 \Rightarrow r_j = 0.$ Hence $(w, \begin{bmatrix} x^t & r^t \end{bmatrix}^t)$ solves SCP$(\bar{q}, A)$.

Conversely, suppose $(w, \begin{bmatrix} x^t & r^t \end{bmatrix}^t)$ solves SCP$(\bar{q}, A)$. This implies $w^j \not> 0$ $\Rightarrow \prod_{j \in J_j} w_i^j = 0.$ Hence $z_j \prod_{j \in J_i} w_i^j = 0$ is satisfied.

Choose z_j in the following manner:

$$z_j = \begin{cases} 1 & \text{if } x_j > 0 \text{ and } r_j = 0 \\ 2 & \text{if } x_j = 0 \text{ and } r_j > 0 \\ z_j^*, \quad 1 < z_j^* < 2 & \text{if } x_j = 0 \text{ and } r_j = 0. \end{cases}$$

From the choice of z_j it follows that $u^t x = 0,$ $v^t r = 0.$ Therefore $(w, [x^t \; r^t]^t)$ solves VLCP$(\hat{q}, A_1)$. This completes the proof.

Formulation II: We now show that SCP$(\bar{q}, A)$ can be formulated as a vertical linear complementarity problem with a smaller size.

For a given x, (4.1) and (4.2) are equivalent to the assertion that

$$w_j = -\bar{q}^j - A^j x + r_j e^j, x \geq 0, r_j \geq 0, w^j \geq 0, w^j \not> 0, j = 1, ..., k, \tag{4.3}$$

$$\sum_{j=1}^{k} x_j r_j = 0, j = 1, ..., k \tag{4.4}$$

where the scalar $r_j = r_j(x)$. Now we define $r_j = -1 + u_j, r_j \geq 0, j = 1, 2, ..., k.$

We can rewrite (4.3) and (4.4) as

$$w_j = (-\bar{q}^j - e^j) - A^j x + u_j e^j, \quad x \geq 0, u_j \geq 0, w^j \geq 0, w^j \not> 0, j = 1, 2, ..., k \tag{4.5}$$

$$r_j = -1 + u_j, \quad r_j \geq 0, j = 1, 2, ..., k \tag{4.6}$$

$$u_j \prod_{i=1}^{m_j} w_i^j = 0, \quad j = 1, 2, ..., k \tag{4.7}$$

$$\prod_{j=1}^{k} x_j r_j = 0, \quad j = 1, 2, ..., k. \tag{4.8}$$

Let $u = \begin{bmatrix} u^1 \\ \vdots \\ u_k \end{bmatrix}$ and $r = \begin{bmatrix} r^1 \\ \vdots \\ r_k \end{bmatrix}$. We note that (4.5) through (4.8) give us the VLCP(q, A_2)

where $A_2 = \begin{bmatrix} F & -A \\ I & 0 \end{bmatrix}$ is of order $(m+k) \times 2k$ *and* $q^* = \begin{bmatrix} -\bar{q} - e_m \\ -e_k \end{bmatrix}$.

In the above formulation $e_m \in R^m$ and $e_k \in R^k$ are vectors of all 1's and I is the identity matrix of order k.

Lemma 4.2 *SCP*$(\bar{q}, A)$ *has a solution if and only if VLCP*(q^*, A_2) *has a solution.*

Proof. Let $(\bar{r}, \bar{x})$ be a solution SCP$(\bar{q}, A)$. Let $\bar{u}_j = \bar{r}_j + 1$ and $\bar{w}^j = (-\bar{q}^j - e^j) - A^j \bar{x} + \bar{u}_j e^j$. It is easy to verify that $\left(\begin{bmatrix} \bar{w} \\ \bar{r} \end{bmatrix}, \begin{bmatrix} \bar{u} \\ \bar{x} \end{bmatrix} \right)$ solves VLCP(q^*, A_2).

Conversely, let $\left(\begin{bmatrix} \bar{w} \\ \bar{r} \end{bmatrix}, \begin{bmatrix} \bar{u} \\ \bar{x} \end{bmatrix} \right)$ solve VLCP(q^*, A_2). Note that $\bar{u}_j = \bar{r}_j + 1, \ 1 \leq j \leq k.$ Hence $\bar{u}_j > 0$ which implies $\bar{w}^j \not> 0$. Thus $(\bar{r}, \bar{x})$ solves SCP$(\bar{q}, A)$. This completes the proof.

4.1 VLCP Formulation of Scarf's Complementarity Problem presented as an LCP

In [16], an equivalent LCP(q, M) of order m is constructed from VLCP(q, A) by copying $A_{\cdot j}$, m_j times for $j=1,2,\ldots,k$. In [16], it is shown that VLCP(q, A) has a solution if and only if LCP(q, M) has a solution.

Now consider the VLCP$(\overline{q}, A_1)$ where A_1 is a vertical block matrix of type $(m_1,\ldots,m_k,1,\ldots,1,1,\ldots,1)$ constructed from Scarf's complementarity problem. The equivalent matrix M_1 and $\overline{q}$ in the equivalent LCP$(\overline{q}, M_1)$ are given by

$$M_1 = \begin{bmatrix} 0 & -A & F \\ F^t & 0 & 0 \\ -F^t & 0 & 0 \end{bmatrix} \in R^{(M+2k)\times(m+2k)} \text{ and } \overline{q} = \begin{bmatrix} q \\ -e \\ 2e \end{bmatrix} \in R^{m+2k}.$$

Note that incidently the matrix M in Lemke's equivalent LCP [13] is the same as above. We prove the following lemma.

Lemma 4.3 *Let* $M = \begin{bmatrix} 0 & -A & F \\ F^t & 0 & 0 \\ -F^t & 0 & 0 \end{bmatrix}$ *be the equivalent LCP matrix. Then* $M \in L_1$.

Proof. We show that $M \in L_1$. Let $\beta_1 = \{1,\ldots,m\}$, $\beta_2 = \{(m+1),\ldots,(m+k)\}$ and $\beta_3 = \{(m+k),\ldots,(m+2k)\}$. Suppose $\overline{\xi} = \begin{bmatrix} z^t & x^t & r^t \end{bmatrix}^t$.

Case I: $0 \neq z \geq 0$, $0 \neq x \geq 0$ and $0 \neq r \geq 0$. In this case for any index $k \in \beta_2$, $x_k > 0$ and $(M\overline{\xi})_k \geq 0$.

Case II: $0 \neq z \geq 0$, $0 \neq x \geq 0$ and $r = 0$. In this case for any index $k \in \beta_2$, $x_k > 0$ and $(M\overline{\xi})_k = \geq 0$.

Case III: $0 \neq z \geq 0$, $0 \neq r \geq 0$ and $x = 0$. In this case for any index $k \in \beta_1$, $z_k > 0$ and $(M\overline{\xi})_k = \geq 0$.

Case IV: $0 \neq x \geq 0$, $0 \neq r \geq 0$ and $z = 0$. In this case for any index $k \in \beta_2 \cup \beta_3$, $\begin{bmatrix} x^t \\ r^t \end{bmatrix}^t_k > 0$ and $(M\overline{\xi})_k \geq 0$.

The other cases **(V)** $0 \neq z \geq 0, x = 0, r = 0$, **(VI)** $z = 0$, $0 \neq x \geq 0$, $r = 0$ and **(VII)** $z = 0$, $x = 0$, $0 \neq r \geq 0$ are easy to verify. Thus $M \in L_1$. This completes the proof.

Remark 4.1 *We observe that vertical block matrix of type* $(m_1, m_2, \ldots, m_k)$ *are encountered in both VLCP and SCP. Cottle-Dantzig's algorithm (a generalization of Lemke's algorithm) can process the VLCP*(q, A) *where* A *is a vertical block matrix of* *type* $(m_1, m_2, \ldots, m_k)$ *and it belongs to some known processable class. In view of Lemma 4.1 and 4.2, it is clear that we can make use of Cottle-Dantzig's algorithm to process a subclass of Scarf's complementarity problem* $SCP(\bar{q}, A)$. *In Lemma 4.3, the observaion that* M *is semimonotone (which is a special structure) may be useful from computational point of view.*

5. NEURAL NETWORK MODEL FOR SCARF'S PROBLEM

The dynamic system presented below is in the form of first-order ordinary differential equations. It is expected that for an initial state, the dynamic system will approach its static state (or equilibrium point) which corresponds the solution of the underlying Scarf's complementarity problem.

Neogy, Das and Das [18] used neural network approaches to solve vertical linear complementarity problems. In this paper, we propose the following neural network dynamics to solve Scarf's Complementarity Problem.

5.1 Proposed Neural Netwok Dynamics

We propose the following recurrent neural network model, which is described by the following non-linear dynamic system

$$\frac{dx_j}{dt} = \max_{i=1,2,\ldots,m_j} \left\{ \bar{q} + A(x + k\frac{dx}{dt}) \right\}, x_j > 0. \tag{5.1}$$

Theorem 5. *If the neural network whose dynamics is described by the differential equation (5.1) converges to a stable state, then the converged state is a solution for the Scarf's Complimentarity Problem.*

Proof. Consider the Scarf's Complimentarity Problem involving a vertical block matrix A of type $(m_1, \ldots, m_k)$. Equation (5.1) can be written as

$$\frac{dx_j}{dt} = \max_{i=1,2,\ldots,m_j} \left\{ \bar{q} + A(x + dx) \right\}_i^j, x_j > 0 \tag{5.2}$$

$$\frac{dx_j}{dt} = \max[-\max_{i=1,2,\ldots,m_j} \{(\bar{q} + A(x+dx))_1^j, \ldots, (\bar{q} + A(x+dx))_{m_j}^j\}, 0], \; if \;\; x_j = 0. \tag{5.3}$$

Note that equation (5.3) ensures that x will be bounded from below by 0. Let $\lim_{t\to\infty} x(t) = x^*$. By stability of convergence $\frac{dx^*}{dt} = 0$. So equations (5.2) and (5.3) become

$$-\max_{i=1,2,\ldots,m_j} \left\{ \bar{q} + Ax^*)_i^j \right\} \leq 0 \tag{5.4}$$

$$x_j^* \left[-\max_{i=1,2,\ldots,m_j} \left\{ (\bar{q} + Ax^*)_i^j \right\} \right] = 0 \tag{5.5}$$

Therefore, we get the inequalities (1.3) and (1.4) involving Scarf's problem. So, by definition, x^* is a solution of SCP $(\overline{q}, A).$ This completes the proof.

5.1.1 Matlab Code for Solving the Neural Network Dynamics

Euler's method was used for solving the differential equation (5.1). The following code describes the matlab implementation of the proposed neural network dynamics. For ease of computation, the cofficient k is taken as time step dt .

```
while ||dx|| > ε
     for j = 1 : k1              % k1 is number of blocks
          for i = Σ_{r=1}^{j-1} m_r + 1 : Σ_{r=1}^{j} m_r ;
               M_{j.} = max_i {q + A*(x + dx)}_i^j ;
          end
     end
     for j = 1 : k1
          dx_j = dt*{-q - M*(x + dx)};
     end
     dx = max(x+dx,0)-x;         % to make x≥0
     x = x+dx;
   end
```

Example 5. 1 *Consider the Scarf's Problem, where* A *is a vertical block matrix of type* $(2,2,1)$ *, and* $\overline{q} \in R^5$ *as given below*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & 1 \\ 4 & 0 & 1 \\ 4 & 2 & 7 \\ 1 & 1 & 3 \end{bmatrix} \text{ and } \overline{q} = \begin{bmatrix} -1 \\ - \\ -4 \\ -4 \\ 2 \end{bmatrix}$$

Solution to the above problem obtained using Scarf's Complemetarity Algorithm is $(0, 2, 0).$ The dynamics converges to above solution in just 75 iterations, where step length dt and convergence criteria ε are taken as 0.1 and 10^{-6} respectively. The convergence of the dynamics to the solution satisfying the complementarity condition is given in the following figure.

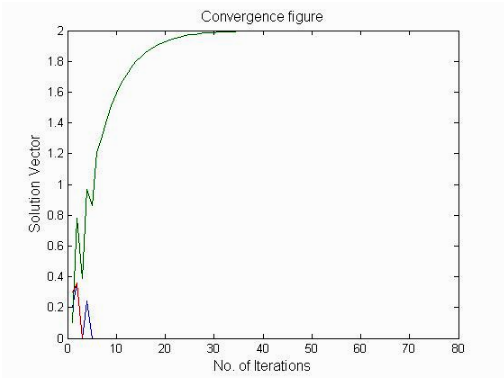

Figure 1: Example No. 5.1

Example 2 *Consider another Scarf's Problem, where* A *is a vertical block matrix of type* $(3,1,2,2)$ *, and* $\bar{q} \in R^8$ *as given below*

$$A = \begin{bmatrix} 2 & 4 & 5 & -7 \\ -4 & -3 & 5 & 9 \\ 3 & 1 & -5 & 4 \\ 2 & -3 & 4 & 2 \\ -3 & 5 & 7 & 2 \\ 2 & 5 & -3 & 4 \\ 2 & -2 & 0 & 5 \\ 4 & 5 & 2 & 4 \end{bmatrix} \text{ and } \bar{q} = \begin{bmatrix} -5 \\ -8 \\ 1 \\ 2 \\ 3 \\ -4 \\ -3 \\ -7 \end{bmatrix}$$

Solution to the above problem, obtained using Scarf's Complemetarity Algorithm, is $(0, 0, 0, 0.6)$. The dynamics converges to above solution in just 37 iterations, where step length dt and convergence criteria ε are taken as 0.1 and 10^{-6} respectively. The convergence of the dynamics to the solution satisfying the complementarity condition is given in the following figure.

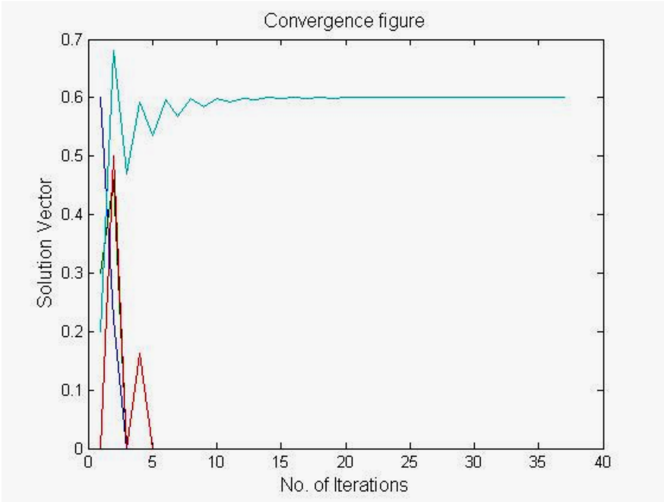

Figure 2: Example No. 5.2

Example 3 *Consider another Scarf's Problem of higher dimension, where* A *is a vertical block matrix of type* $(2,3,4,2,1,3)$*, and* $\bar{q} \in R^{15}$ *as given below*

$$A = \begin{bmatrix} 4 & -3 & 4 & -3 & 1 & 5 \\ 3 & 0 & -2 & 0 & 5 & -4 \\ -3 & 2 & 0 & 4 & -2 & 0 \\ 2 & 1 & 0 & 3 & 5 & 3 \\ 3 & -2 & 0 & 5 & 0 & 2 \\ 5 & 7 & -5 & 0 & 4 & 6 \\ 3 & 0 & -5 & 3 & 7 & 9 \\ 1 & 2 & 0 & -5 & 0 & -2 \\ 2 & -3 & 0 & 5 & 2 & 9 \\ -3 & 0 & 2 & 4 & 0 & 5 \\ 0 & -3 & 4 & 5 & -2 & -3 \\ -5 & 0 & 3 & 3 & 2 & 2 \\ 3 & -5 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 3 & -2 & 7 \\ 2 & 0 & 3 & -5 & 6 & 9 \end{bmatrix} \text{ and } \bar{q} = \begin{bmatrix} -5 \\ -10 \\ -3 \\ -3 \\ -7 \\ -3 \\ 7 \\ -2 \\ -5 \\ -3 \\ -3 \\ 7 \\ -3 \\ 9 \\ -7 \end{bmatrix}$$

Solution to the above problem, obtained using Scarf's Complemetarity Algorithm, is $(1.7, 0, 0, 0.6, 0, 0)$. The dynamics converges to above solution in 40 iterations, where step length dt and convergence criteria ε are taken as 0.1 and 10^{-6}

respectively. The convergence of the dynamics to the solution satisfying the complementarity condition is given in the following figure.

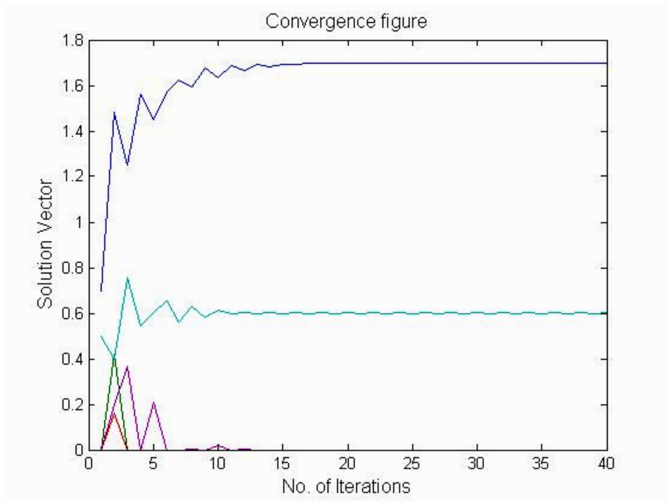

Figure 3: Example No. 5.3

6. CONCLUDING REMARK ON SCARF'S COMPLEMENTARITY PROBLEM

In this paper, we consider Scarf's complementarity problem and observe that even though LCP involving representative submatrices has a solution, SCP$(\overline{q}, A)$ has no solution. It is well known that if A is a P matrix or strictly copositive matrix, then $A \in Q$. We show that if A is a vertical block matrix P-matrix or vertical block strictly copositive matrix of type $(m_1, m_2, \ldots, m_k)$ then SCP$(\overline{q}, A)$ has a solution for all $\overline{q}$, i.e., A has Q-property. This generalizes the LCP result in the setting of Scarf's complementarity problem. We observe that a Scarf's complementarity problem can be reformulated as a vertical linear complementarity problem. This extends the application of Cottle-Dantzig's algorithm to Scarf's complementarity problem. Finally, we present a neural network model to solve a Scarf's complementarity problem. The proposed neural network model is tried for a large number of test problems and converges to a solution for all such test problems.

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