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OPTIMALITY AND SECOND ORDER DUALITY FOR A CLASS OF QUASI-DIFFERENTIABLE MULTIOBJECTIVE OPTIMIZATION PROBLEM

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Abstract: A second order Mond-Weir type dual is presented for a non-differentiable multiobjective optimization problem with square root terms in the objective as well as in the constraints. Optimality and duality results are presented. Classes of generalized higher order η – bonvex and related functions are introduced to study the optimality and duality results. A fractional case is presented at the end.

Keywords: Higher order η – bonvexity, Strict minimizers, Second order duality.

MSC: 26A51, 90C29, 90C46.

1. INTRODUCTION

The notion of second order duality was first introduced by Mangasarian [15]. The motivation behind the construction of a second order dual was the applicability in the development of algorithms for certain problems. The second order dual has computational advantage over the first order dual as it provides a tighter bound for the value of the objective function when approximations are used. One more advantage of second order duality is that, if a feasible point for the problem is provided and first order duality does not hold then, one can use a second order dual to get a lower bound for the value of primal objective function [13]. Recently, several authors [1, 2 and 14] have studied second order duality for various classes of optimization problems.

Under the assumption that means, variances and covariances of the random variables are known, Sinha [18] established a way that a stochastic linear programming problem leads to a deterministic nonlinear programming problem, where the functions involve square roots of positive semi-definite quadratic forms. It is generally difficult to solve such problems because of non-differentiability of square root terms involved. However, it is useful to study the duality aspects of such problems, which may easily lead to the solution of these problems. First order duality for various forms of scalar as well as multiobjective optimization problems, involving square root terms of certain positive semi-definite quadratic forms have been studied by many authors (see, for example [6,7,14,16,18 and 19]). Practical applications of these problems can be found in multi-facility location problems and in portfolio selection problem.

In this paper, a second order Mond-Weir type dual for a multiobjecitve optimization problem involving square root terms in objectives as well as the constraining functions is presented, and duality results are established. For this purpose, we introduce classes of generalized higher order η – bonvex and related functions. The results of this paper are more general than the corresponding results already existing in literature [4, 15].

2. PRELIMINARIES

Let X be a nonempty subset of R^n endowed with the Euclidean norm $\|\cdot\|$.

Definition 2.1 *A function $f : X \to R$ is said to be locally Lipschitz if for each bounded subset B of X, there exists a constant $l \geq 0$ such that*

$$\left|f(x) - f(y)\right| \leq l \, \| x - y \|,$$

for all $x, y \in B$.

Definition 2.2 *The directional derivative of a function $f : X \to R$ at a point $x \in B$ in the direction $d \in R^n$ is defined as*

$$f'(x;d) = \lim_{\alpha \to 0} \frac{f(x+\alpha d) - f(x)}{\alpha}.$$

Definition 2.3 ([10, 17]) *A function $f : X \to R$ is said to be quasi-differentiable at a point x if f possesses a directional derivative at $x \in X$ for each direction $d \in R^n$ such that $f'(x;d)$ is convex with respect to d.*

It is known that if $f_i(x), i = 1,2,...,p$ are differentiable, then the functions

$$\theta_i(x) = f_i(x) + (x^t B_i x)^{1/2}, \; i = 1,2,...,p$$

are quasi-differentiable.

Let $L(x^0)$ be the set of directions, i.e.

$$L(x^0) = \left\{ d \in R^n : \theta_i'(x;d) < 0, i = 1,2,...,p \right\}$$

and $T_X(x^0)$ be the tangent cone to X at x^0, i.e.

$$T_X(x^0) = \left\{ d \in R^n : \exists\{d_k\} \to d, \alpha_k \to 0, x^0 + \alpha_k d_k \in X \right\}$$

Lemma 2.1 [3] *Let $\theta(x) = \left(\theta_1(x),...,\theta_p(x)\right)$, where $\theta_i : X \to R$, $i = 1,2,...,p$ are locally Lipschitz and possess directional derivatives at each point in each direction. If x^0 is a strict minimizer of $\theta(x)$ on X, then*

$$L(x^0) \cap T_X(x^0) = \phi .$$

Remark 2.1 [3] *Let $\delta\theta_i(x^0)$ be the sub-differential of function $\theta_i, i = 1,2,...,p$ at x^0, then we have for each $w_i \in \delta\theta_i(x^0), i = 1,2,...,p$*

$$w_i^t d \le \theta_i'(x^0;d)$$

for all $d \in R^n$.

From lemma 2.1, it follows that for all $w_i \in \delta\theta_i(x^0), i = 1,2,...,p$ the system

$$w_i^t d < 0, i = 1,2,...,p$$

has no solution in $T_X(x^0)$.

We shall need the following generalized Schwarz inequality in the sequel.

Lemma 2.2 [11] *Let B be an $n \times n$ symmetric positive semi-definite matrix and $x, z \in R^n$, then*

$$(x^t Bz) \le (x^t Bx)^{1/2} (z^t Bz)^{1/2},$$

where equality holds if and only if $Bx = \lambda Bz$ for some $\lambda \ge 0$. Evidently, if $(z^t Bz)^{1/2} = 1$, we have $(x^t Bz) \le (x^t Bx)^{1/2}$.

Lemma 2.3 [9] *Let $\varphi(x) = (x^t Bx)^{1/2}$. Then $\varphi(x)$ is convex and $w \in \delta\varphi(x)$ if and only if $w = Bz, z^t Bz \le 1, x^t Bz = (x^t Bx)^{1/2}$.*

Multiobjective optimization problems are encountered in many areas of human activity including engineering and management. For many interesting applications and development of multiobjective optimization, one may refer to [8]. In this paper, we study the following multiobjective optimization problem;

(MOP) Minimize $(\theta_1(x),...,\theta_p(x))$

subject to $G_j(x) \leq 0, j = 1, 2, ..., q.$

Where $\theta_i(x) = f_i(x) + (x^t B_i x)^{1/2}, i = 1, 2, ..., p$

$G_j(x) = g_j(x) + (x^t C_j x)^{1/2}, j = 1, 2, ..., q.$

$f_i : X \to R, i = 1, 2, ..., p, g_j : X \to R, j = 1, 2, ...q$ are twice differentiable; and $B_i, i = 1, 2, ..., p$ and $C_j, j = 1, 2, ..., q$ are $n \times n$ positive semi-definite symmetric matrices.

The functions $\theta_i(x), i = 1, 2, ..., p$ and $G_j(x), j = 1, 2, ..., q$ are quasi-differentiable functions. Thus (MOP) may be referred as a quasi-differentiable multiobjective optimization problem. Let S be the set of all feasible solutions of (MOP). Here minimization means finding a strict minimizer.

Definition 2.4 *A point $x^0 \in S$ is said to be strict minimizer for (MOP) if for all $x \in S$*

$$\theta(x) \nless \theta(x^0),$$

that is there exists no $x \in S$ such that

$$\theta(x) < \theta(x^0).$$

To explore the applicability of optimality and duality results several authors [6,7,14,16,18 and 19] have studied the above type of multiobjective optimization problems by weakening the convexity assumptions. We move a step further in this direction and introduce the classes of generalized higher order η-bonvex and related functions as follows:

Let $\eta, \psi : X \times X \to R^n$ be vector valued functions, $b : X \times X \to R_+$ and $\phi : R \to R$ are real valued functions.

Definition 2.5 *The function $\theta : X \to R$ is said to be generalized η-bonvex of order $m(\geq 1)$ at $x^0 \in S$ with respect to mappings b, ϕ, η and ψ if there exist a vector $r \in R^n$ and a constant $k \in R$ such that for all $x \in S$*

$$b(x, x^0)\phi[\theta(x) - \theta(x^0) + \frac{1}{2} r^t \nabla^2 \theta(x^0) r] \geq$$
$$\eta^t(x, x^0)[\nabla \theta(x^0) + \nabla^2 \theta(x^0) r] + k \| \psi(x, x^0) \|^m .$$

Remark 2.2 *If $k > 0$, then the function θ is called strongly generalized η- bonvex of order m. If $k < 0$, then the function θ is called weakly generalized η- bonvex of order m. If $k = 0$ and in addition $b = 1, \phi = I$ (identity map), we obtain the definition of η – bonvex functions [4].*

Remark 2.3 *If* $r = 0, k = 0$, *we obtain the definition of univexity [5]. If* $b = 1, k = 0, p = 0$ *and* $\phi = I$, *then the definition of generalized higher order* $\eta -$ *bonvexity reduces to the definition of invexity [12].*

We now present the following obvious implications of the above definition.

Definition 2.6 *The function* $\theta : X \to R$ *is said to be generalized* $\eta -$ *pseudo bonvex of order* $m(\geq 1)$ *at* $x^0 \in S$ *with respect to mappings* b, ϕ, η *and* ψ *if there exist a vector* $r \in R^n$ *and a constant* $k \in R$ *such that for all* $x \in S$

$$\eta^t(x, x^0)[\nabla \theta(x^0) + \nabla^2 \theta(x^0) r] + k \| \psi(x, x^0) \|^m \geq 0$$

implies $b(x, x^0)\phi[\theta(x) - \theta(x^0) + \frac{1}{2} r^t \nabla^2 \theta(x^0) r] \geq 0$

or equivalently $b(x, x^0)\phi[\theta(x) - \theta(x^0) + \frac{1}{2} r^t \nabla^2 \theta(x^0) r] < 0$

implies $\eta^t(x, x^0)[\nabla \theta(x^0) + \nabla^2 \theta(x^0) r] + k \| \psi(x, x^0) \|^m < 0$

Definition 2.7 *The function* $\theta : X \to R$ *is said to be generalized* η*-strictly pseudo bonvex of order* $m(\geq 1)$ *at* $x^0 \in S$ *with respect to mappings* b, ϕ, η *and* ψ *if there exist a vector* $r \in R^n$ *and a constant* $k \in R$ *such that for all* $x \in S$

$$\eta^t(x, x^0)[\nabla \theta(x^0) + \nabla^2 \theta(x^0) r] + k \| \psi(x, x^0) \|^m \geq 0$$

implies $b(x, x^0)\phi[\theta(x) - \theta(x^0) + \frac{1}{2} r^t \nabla^2 \theta(x^0) r] > 0$.

Definition 2.8 *The function* $\theta : X \to R$ *is said to be generalized* η*- quasi bonvex of order* $m(\geq 1)$ *at* $x^0 \in S$ *with respect to mappings* b, ϕ, η *and* ψ *if there exist a vector* $r \in R^n$ *and a constant* $k \in R$ *such that for all* $x \in S$

$$b(x, x^0)\phi[\theta(x) - \theta(x^0) + \frac{1}{2} r^t \nabla^2 \theta(x^0) r] \leq 0$$

implies $\eta^t(x, x^0)[\nabla \theta(x^0) + \nabla^2 \theta(x^0) r] + k \| \psi(x, x^0) \|^m \leq 0$.

3. OPTIMALITY

We now derive the following necessary optimality conditions for (MOP).

Theorem 3.1 *If* x^0 *is a strict minimizer for (MOP) and assume that* G_I *satisfies the Abadie constraint qualification at* x^0, *where* $I = \{ j : G_j(x^0) = 0 \}$. *Then, there exist* $y \in R_+^q, v_j \in R^n$, $j = 1, 2, \ldots q$, $\lambda \in R_+^p$, $\sum_{i=1}^{p} \lambda_i = 1$ *and* $z_i \in R^n, i = 1, 2, \ldots p$ *such that*

$$\sum_{i=1}^{p}\lambda_i \nabla f_i(x^0)+\sum_{i=1}^{p}\lambda_i B_i z_i+\sum_{j=1}^{q}y_j\nabla g_j(x^0)+\sum_{j=1}^{q}y_j C_j v_j=0 \quad (3.1)$$

$$y_j G_j(x^0)=0,\ j=1,2,...,q \quad (3.2)$$

$$z_i^t B_i z_i \le 1,\ i=1,...p \quad (3.3)$$

$$(v_j^t C_j v_j)\le 1,\ j=1,2,...,q \quad (3.4)$$

$$x^{0t}B_i z_i=(x^{0t}B_i x^0)^{1/2},\ i=1,...,p \quad (3.5)$$

$$v_j^t C_j x^0=(x^{0t}C_j x^0)^{1/2},\ j=1,2,...,q \quad (3.6)$$

Proof Since $f_i, i=1,2,...,p$ are differentiable functions, $B_i, i=1,2,...,p$ are positive semi-definite matrices, we have from [9] that the functions $\theta_i(x)=f_i(x)+(x^t B_i x)^{1/2}$, $i=1,2,...,p$ are quasi-differentiable, hence locally Lipschitz and have directional derivatives $\theta_i'(x;d)$ for all $d\in R^n$, $i=1,2,...,p$. Therefore $\theta_i, i=1,2,...,p$ satisfy the conditions of Lemma 2.1.

From Lemma 2.1 and Abadie constraint qualification, it follows that the system

$$\rho_i^t d<0,\quad i=1,2,...,p$$

$$w_j^t d\le 0,\quad j\in I,$$

is inconsistent for all $\rho_i\in\delta\theta_i(x^0), i=1,2,...,p$ and $w_j\in\delta G_j(x^0),\ j\in I$. Therefore by basic alternative theorem [3], there exists $\lambda_i\ge 0, i=1,2,...,p$ not all zero and $y_j\ge 0, j\in I$ such that:

$$\sum_{i=1}^{p}\lambda_i\rho_i+\sum_{j\in I}y_j w_j=0 \quad (3.7)$$

for all $(\rho_1,...,\rho_p)=\rho\in\delta\theta(x^0)$ and $w_j\in\delta G_j(x^0),\ j\in I$. Setting $y_j=0$ for all *j* not in *I*, we can rewrite (3.7) as

$$\sum_{i=1}^{p}\lambda_i\rho_i+\sum_{j=1}^{q}y_j w_j=0 \quad (3.8)$$

$$y_j G_j(x^0)=0,\quad j=1,2,...,q \quad (3.9)$$

But $\delta\theta_i(x^0), i=1,2,...,p$ is the set

$$\left\{\nabla f_i(x^0)+B_i z_i : z_i^t B_i z_i \le 1,\ x^{0t}B_i z_i=(x^{0t}B_i x^0)^{1/2}\right\}, \tag{3.10}$$

for some $z_i \in R^n$. Similarly $\delta G_j(x^0)$ is the set

$$\left\{\nabla g_j(x^0)+C_j v_j : v_j^t C_j v_j \le 1,\ v_j^t C_j x^0=(x^{0t}C_j x^0)^{1/2}, j\in I\right\}, \tag{3.11}$$

Hence from (3.10) and (3.11), we have

$$\sum_{i=1}^{p}\lambda_i\nabla f_i(x^0)+\sum_{i=1}^{p}\lambda_i B_i z_i+\sum_{j=1}^{q}y_j\nabla g_j(x^0)+\sum_{j=1}^{q}y_j C_j v_j=0$$

$$y_j G_j(x^0)=0,\ j=1,2,...,q$$

$$z_i^t B_i z_i \le 1,\ i=1,...p$$

$$(v_j^t C_j v_j)\le 1,\ j=1,2,...,q$$

$$x^{0t}B_i z_i=(x^{0t}B_i x^0)^{1/2}, i=1,...,p$$

$$v_j^t C_j x^0=(x^{0t}C_j x^0)^{1/2},\ j=1,2,...,q\ .$$

4. DUALITY

We now propose the following second order Mond-Weir type dual for (MOP).

(MD)Maximize $(f_1(u)+z_1^t B_1 u-\frac{1}{2}r^t\nabla^2 f_1(u)r,...,$

$$f_p(u)+z_p^t B_p u-\frac{1}{2}r^t\nabla^2 f_p(u)r)$$

subject to

$$\sum_{i=1}^{p}\lambda_i(\nabla f_i(u)+B_i z_i+\nabla^2 f_i(u)r)+\sum_{j=1}^{q}y_j(\nabla g_j(u)+C_j v_j+\nabla^2 g_j(u)r)=0 \tag{4.1}$$

$$y_j(g_j(u)+v_j^t C_j u-\frac{1}{2}r^t\nabla^2 g_j(u)r)\ge 0 \tag{4.2}$$

$$z_i^t B_i z_i \le 1,\ i=1,...p \tag{4.3}$$

$$(v_j^t C_j v_j)\le 1,\ j=1,2,...,q \tag{4.4}$$

$$y_j\ge 0, j=1,...,q,\ \lambda_i\ge 0, i=1,...,p, \sum_{i=1}^{p}\lambda_i=1$$

Theorem 4.1 (Weak duality) *Let* x *be feasible for (MOP) and* (u,λ,y,z,v,r) *be feasible for (MD). Further suppose that*

i. $(f_i(.)+\cdot^t B_i z_i)$, $i=1,2,...,p$ *be generalized* η *– pseudo bonvex of order m at u with respect to* b_i, ϕ_i, η *and* ψ, *where* $b_i > 0$ *for all* $i=1,2,...,p$.

ii. $y_j(g_j(\cdot)+\cdot^t C_j v_j)$, $j=1,2,...,q$ *be generalized* η *– quasi bonvex of order m at u with respect to* $\bar{b}_j, \bar{\phi}_j$, η *and* ψ.

iii. $a \le 0 \Rightarrow \bar{\phi}_j(a) \le 0,\ j=1,...,q$ *and*

$a < 0 \Rightarrow \phi_i(a) < 0, i=1,...,p$

iv. $\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \bar{k}_j \ge 0$.

Then

$$f(x)+(x^t B_i x)^{1/2} \nleq f(u)+u^t B_i z_i - \frac{1}{2} r^t \nabla^2 f(u) r \qquad (4.5)$$

Proof Let x be any feasible solution for (MOP) and (u,λ,y,z,v,r) be any feasible solution for (MD). Then we have

$$y_j(g_j(x)+(x^t C_j x)^{1/2}) \le y_j(g_j(u)+v_j^t C_j u - \frac{1}{2} r^t \nabla^2 g_j(u) r),\quad j=1,...,q$$

Using relation (4.4) and Lemma 2.2, we have

$$y_j(g_j(x)+v_j^t C_j x) \le y_j(g_j(u)+v_j^t C_j u - \frac{1}{2} r^t \nabla^2 g_j(u) r),\quad j=1,...,q\ ,$$

which can be rewritten as

$$y_j(g_j(x)+v_j^t C_j x) - y_j(g_j(u)+v_j^t C_j u) + \frac{1}{2} r^t \nabla^2 y_j g_j(u) r \le 0,\ j=1,2,...,q \qquad (4.6)$$

Since $a \le 0 \Rightarrow \bar{\phi}_j(a) \le 0$ and $\bar{b}_j(x,u) \ge 0, j=1,...,q;$ (4.6) yields

$$\bar{b}_j(x,u)\bar{\phi}_j[y_j(g_j(x)+v_j^t C_j x) - y_j(g_j(u)+v_j^t C_j u) + \frac{1}{2} r^t \nabla^2 y_j g_j(u) r)] \le 0$$

On using generalized η – quasi bonvexity of order m at u for $y_j(g_j(\cdot)+\cdot^t C_j v_j)$ with respect to $\bar{b}_j, \bar{\phi}_j$, η and ψ, $j=1,2,...,q$, we have

$$\eta^t(x,u)[\nabla y_j g_j(u) + y_j C_j v_j + \nabla^2 y_j g_j(u) r] + \bar{k}_j \, \| \psi(x,u) \|^m \le 0,\ j=1,...,q\ .$$

The above inequality yields

$$\eta^t(x,u)[\sum_{j=1}^{q}(\nabla y_j g_j(u) + y_j C_j v_j + \nabla^2 y_j g_j(u)r)] \leq -(\sum_{j=1}^{q}\bar{k}_j) \| \psi(x,u) \|^m \tag{4.7}$$

On using (4.1), the inequality (4.7) yields

$$\eta^t(x,u)[\sum_{i=1}^{p}\lambda_i(\nabla f_i(u) + B_i z_i + \nabla^2 f_i(u)r] \geq (\sum_{j=1}^{q}\bar{k}_j) \| \psi(x,u) \|^m \tag{4.8}$$

Contrary to the result of the theorem, let

$$f_i(x) + (x^t B_i x)^{1/2} < f_i(u) + u^t B_i z_i - \frac{1}{2} r^t \nabla^2 f_i(u) r \text{ , } i = 1,..., p$$

Using lemma 2.2, we have

$$f_i(x) + x^t B_i z_i < f_i(u) + u^t B_i z_i - \frac{1}{2} r^t \nabla^2 f_i(u) r \text{ ,} i = 1,..., p \tag{4.9}$$

Since $a < 0 \Rightarrow \phi_i(a) < 0$ and $b_i > 0$ for all $i = 1,..., p,$ the inequalities in (4.9) lead to

$$b_i(x,u)\phi_i[(f_i(x) + x^t B_i z_i) - f_i(u) - u^t B_i z_i + \frac{1}{2} r^t \nabla^2 f_i(u) r] < 0, i = 1,.., p$$

From generalized η – pseudo bonvexity of order m for $(f_i(\cdot) + \cdot^t B_i z_i)$ at u with respect to b_i, ϕ_i , η and ψ , $i = 1, 2,..., p$, we have

$$\eta^t(x,u)[\nabla f_i(u) + B_i z_i + \nabla^2 f_i(u)r] + k_i \| \psi(x,u) \|^m < 0, i = 1,..., p$$

Since $\lambda_i \geq 0, i = 1, 2,..., p$ and $\sum_{i=1}^{p}\lambda_i = 1$, we obtain

$$\eta^t(x,u)[\sum_{i=1}^{p}\lambda_i(\nabla f_i(u) + B_i z_i + \nabla^2 f_i(u)r)] + (\sum_{i=1}^{p}\lambda_i k_i) \| \psi(x,u) \|^m < 0 .$$

Using hypothesis (iv), the above inequality yields

$$\eta^t(x,u)[\sum_{i=1}^{p}\lambda_i(\nabla f_i(u) + B_i z_i + \nabla^2 f_i(u)r)] < (\sum_{j=1}^{q}\bar{k}_j) \| \psi(x,u) \|^m \text{ ,}$$

a contradiction to (4.8).

Hence $f(x) + (x^t B_i x)^{1/2} \not< f(u) + u^t B_i z_i - \frac{1}{2} r^t \nabla^2 f(u) r.$

Theorem 4.2 (strong duality) *Let x^0 be a strict minimizer for (MOP) and assume that Abadie constraint qualification holds at x^0. Then, there exist $\lambda^0 \in R^p_+$, $y^0 \in R^q_+, z_i^{\,0} \in R^n, v_j^{\,0} \in R^n$ such that $(x^0, \lambda^0, y^0, z^0, v^0, r^0 = 0)$ is feasible for (MD) and the corresponding values of (MOP) and (MD) are equal. Further, if the assumptions of* *weak duality Theorem 4.1 hold, then* $(x^0, \lambda^0, y^0, z^0, v^0, r^0 = 0)$ *is a strict maximizer for (MD).*

Proof Since x^0 is a strict minimizer for (MOP) and Abadie constraint qualification is satisfied at x^0, then by Theorem 3.1 there exist $\lambda^0 \in R_+^p$, $y^0 \in R_+^q$, $z_i^0 \in R^n$, $v_j^0 \in R^n$, such that

$$\sum_{i=1}^{p} \lambda_i^0 \nabla f_i(x^0) + \sum_{i=1}^{p} \lambda_i^0 B_i z^0{}_i + \sum_{j=1}^{q} y_j^0 \nabla g_i(x^0) + \sum_{j=1}^{q} y_j^0 C_j v_j^0 = 0$$

$$y_j^0 (g_j(x^0) + (x^{0t} C_j x^0)^{1/2}) = 0, j = 1,...,q$$

$$z^{0t}{}_i B_i z^0{}_i \le 1, i = 1,...p$$

$$(v^{0t}{}_j C_j v_j^0) \le 1, j = 1,2,...,q$$

$$x^{0t} B_i z^0{}_i = (x^{0t} B_i x^0)^{1/2}, i = 1,..., p$$

$$v_j^{0t} C_j x^0 = (x^{0t} C_j x^0)^{1/2}, j = 1,2,...,q$$

$$y^0 \ge 0, \lambda^0 \ge 0, \sum_{i=1}^{p} \lambda_i^0 = 1.$$

Hence $(x^0, \lambda^0, y^0, z^0, v^0, r^0 = 0)$ is feasible for (MD) and the corresponding values of objective functions are equal. Weak duality Theorem 4.1 implies that $(x^0, \lambda^0, y^0, z^0, v^0, r^0 = 0)$ is a strict maximizer for (MD).

Theorem 4.3 (strict converse duality) *Let* x^0 *and* $(u^0, \lambda^0, y^0, v^0, z^0, r^0)$ *be strict extrema for (MOP) and (MD) respectively, such that*

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(x^0) + (x^{0t} B_i x^0)^{1/2}) = \sum_{i=1}^{p} \lambda_i^0 (f_i(u^0) + u^{0t} B_i z_i^0 - \frac{1}{2} r^{0t} \nabla^2 f_i(u^0) r^0) \tag{4.10}$$

Further, suppose that

i. $y_j^0 (g_j(\cdot) + \cdot^t C_j v_j^0)$ *be generalized* η *– quasi bonvex of order m with respect to* $\overline{b}_j, \overline{\phi}_j$, η *and* ψ, $j = 1,2,...,q$ *at* u^0.

ii. $\sum_{i=1}^{p} \lambda_i^0 (f_i(\cdot) + \cdot^t B_i z_i^0)$ *be generalized* η *– strict pseudo bonvex of order m with respect to* b, ϕ, η *and* ψ *at* u^0.

iii. $a \le 0 \Rightarrow \overline{\phi}_j(a) \le 0, j = 1,...,q$ *and* $\phi(a) > 0 \Rightarrow a > 0$.

iv. $k+\sum_{j=1}^{q}\bar{k}_j \geq 0$.

Then $x^0 = u^0$, that is, u^0 is a strict minimizer for (MOP).

Proof Suppose that $x^0 \neq u^0$.

Since x^0 is feasible for (MOP) and $(u^0,\lambda^0,y^0,v^0,z^0,r^0)$ is feasible for (MD), we have for $j=1,...,q$,

$$y_j^0(g_j(x^0)+(x^{0t}C_jx^0)^{1/2})-y_j^0(g_j(u^0)+v_j^{0t}C_ju^0-\frac{1}{2}r^{0t}\nabla^2 g_j(u^0)r^0)\leq 0.$$

Using Lemma 2.2, for $j=1,...,q$, we have

$$y_j^0(g_j(x^0)+v_j^{0t}C_jx^0)-y_j^0(g_j(u^0)+v_j^{0t}C_ju^0-\frac{1}{2}r^{0t}\nabla^2 g_j(u^0)r^0)\leq 0 \tag{4.11}$$

Since $\bar{b}_j(x^0,u^0)\geq 0$, inequalities in (4.11) along with hypothesis (iii) yields

$$\bar{b}_j(x^0,u^0)\bar{\phi}_j[y_j^0(g_j(x^0)+v_j^{0t}C_jx^0)-y_j^0(g_j(u^0)+v_j^{0t}C_ju^0)+\frac{1}{2}r^{0t}\nabla^2 y_j^0 g_j(u^0)r^0)]\leq 0$$

On using generalized η – quasi bonvexity of order m at u^0 for $y_j^0(g_j(\cdot)+\cdot^t C_jv_j^0)$ with respect to $\bar{b}_j,\bar{\phi}_j$, η and ψ, $j=1,2,...,q$, we have

$$\eta^t(x^0,u^0)[\nabla y_j^0 g_j(u^0)+y_j^0C_jv_j^0+\nabla^2 y_j^0 g_j(u^0)r^0]+\bar{k}_j\,\|\psi(x^0,u^0)\|^m\leq 0,\ j=1,...,q$$

The above inequality yields

$$\eta^t(x^0,u^0)[\sum_{j=1}^{q}(\nabla y_j^0 g_j(u^0)+y_j^0C_jv_j^0+\nabla^2 y_j^0 g_j(u^0)r^0)]\leq -(\sum_{j=1}^{q}\bar{k}_j)\,\|\psi(x^0,u^0)\|^m .$$

Using the dual constraint (4.1) in the above inequality, we have

$$\eta^t(x^0,u^0)[\sum_{i=1}^{p}\lambda_i^0(\nabla f_i(u^0)+B_iz_i^0+\nabla^2 f_i(u^0)r^0]\geq(\sum_{j=1}^{q}\bar{k}_j)\,\|\psi(x^0,u^0)\|^m$$

Using hypothesis (iv), we have

$$\eta^t(x^0,u^0)[\sum_{i=1}^{p}\lambda_i^0(\nabla f_i(u^0)+B_iz_i^0+\nabla^2 f_i(u^0)r^0]+k\,\|\psi(x^0,u^0)\|^m\geq 0 .$$

Now generalized η – strict pseudo bonvexity of order m at u^0 for the function $\sum_{i=1}^{p}\lambda_i^0(f_i(\cdot)+\cdot^t B_i z_i^0)$ with respect to b,ϕ, η and ψ implies

$$b(x^0,u^0)\phi[\sum_{i=1}^{p}\lambda_i^0(f_i(x^0)+x^{0t}B_i z_i^0)-\sum_{i=1}^{p}\lambda_i^0(f_i(u^0)+u^{0t}B_i z_i^0)$$

$$+\frac{1}{2}r^{0t}(\sum_{i=1}^{p}\lambda_i^0\nabla^2 f_i(u^0))r^0]>0\,.$$

Since $b(x^0,u^0)>0$, the above inequality along with hypothesis (iii) yields

$$\sum_{i=1}^{p}\lambda_i^0(f_i(x^0)+x^{0t}B_i z_i^0)>\sum_{i=1}^{p}\lambda_i^0(f_i(u^0)+u^{0t}B_i z_i^0-\frac{1}{2}r^{0t}\nabla^2 f_i(u^0)r)\,,$$

which on using Lemma 2.2 contradicts (4.10).

5. A FRACTIONAL CASE

We now consider the following quasi-differentiable multiobjective fractional programming problem (MOFP) in which the components of the objective functions are the ratios of the functions that are the sums of differentiable terms and square root terms of certain positive semi-definite quadratic forms, whereas the constraining functions are the same as those for (MOP).

(MOFP) Maximize $\left(\dfrac{f_1(x)-(x^t B_1 x)^{1/2}}{h_1(x)+(x^t D_1 x)^{1/2}},\dots,\dfrac{f_p(x)-(x^t B_p x)^{1/2}}{h_p(x)+(x^t D_p x)^{1/2}}\right)$

subject to $G_j(x)\le 0,\ j=1,2,\dots,q$,

where $G_j(x)=g_j(x)+(x^t C_j x)^{1/2},\ j=1,2,\dots,q.$

$f_i:X\to R,\ h_i:X\to R,\ i=1,2,\dots,p$ and $g_j:X\to R,\ j=1,2,\dots q$ are twice differentiable; and $B_i, D_i, i=1,2,\dots,p$ and $C_j, j=1,2,\dots,q$ are $n\times n$ positive semi-definite symmetric matrices. Let S be the set of all feasible solutions of (MOFP). We also assume that $f_i(x)-(x^t B_i x)^{1/2}\ge 0$ and $h_i(x)+(x^t D_i x)^{1/2}>0$, $i=1,2,\dots,p$. Here minimization means finding strict minimizer.

We present the following two duality models for (MOFP):

(MD1) Minimize $(\sigma_1,\dots,\sigma_p)$

subject to $\sum_{i=1}^{p}\lambda_i[\nabla f_i(u)-\sigma_i\nabla h_i(u)-B_i z_i-\sigma_i D_i w_i+\nabla^2 f_i(u)r-\sigma_i\nabla^2 h_i(u)r]$

$$-\sum_{j=1}^{q} y_j(\nabla g_j(u)+C_j v_j+\nabla^2 g_j(u)r)=0 \quad (5.1)$$

$$f_i(u)-\sigma_i h_i(u)-u^t B_i z_i-\sigma_i u^t D_i w_i-\frac{1}{2}r^t\nabla^2(f_i(u)-\sigma_i h_i(u))r$$

$$-\sum_{j=1}^{m} y_j(g_j(u)+v_j^t C_j u)+\frac{1}{2}r^t\nabla^2(\sum_{j=1}^{m} y_j g_j(u))r\le 0\,,\, i=1,2,...,p \quad (5.2)$$

$$z_i^t B_i z_i\le 1,\; w_i^t B_i w_i\le 1,\; i=1,...p$$

$$(v_j^t C_j v_j)\le 1,\; j=1,2,...,q$$

$$\sigma_i=\frac{f_i(u)-u^t B_i z_i}{h_i(u)+u^t D_i w_i}\ge 0,\quad i=1,2,...,p \quad (5.3)$$

$$y_j\ge 0,\, j=1,...,q,\; \lambda_i\ge 0, i=1,...,p,\; \sum_{i=1}^{p}\lambda_i=1$$

(MD2) Minimize $(\sigma_1,...,\sigma_p)$

subject to $\sum_{i=1}^{p}\lambda_i[\nabla f_i(u)-\sigma_i\nabla h_i(u)-B_i z_i-\sigma_i D_i w_i+\nabla^2 f_i(u)r-\sigma_i\nabla^2 h_i(u)r]$

$$-\sum_{j=1}^{q} y_j(\nabla g_j(u)+C_j v_j+\nabla^2 g_j(u)r)=0 \quad (5.4)$$

$$f_i(u)-\sigma_i h_i(u)-u^t B_i z_i-\sigma_i u^t D_i w_i-\frac{1}{2}r^t\nabla^2(f_i(u)-\sigma_i h_i(u))r\le 0,$$

$i=1,2,...,p$

$$y_j(g_j(u)+v_j^t C_j u)-\frac{1}{2}r^t\nabla^2 y_j g_j(u)r\ge 0 \quad (5.5)$$

$$z_i^t B_i z_i\le 1,\; w_i^t B_i w_i\le 1,\; i=1,...p$$

$$(v_j^t C_j v_j)\le 1,\; j=1,2,...,q$$

$$\sigma_i=\frac{f_i(u)-u^t B_i z_i}{h_i(u)+u^t D_i w_i}\ge 0,\quad i=1,2,...,p$$

$$y_j\ge 0,\, j=1,...,q,\; \lambda_i\ge 0, i=1,...,p,\; \sum_{i=1}^{p}\lambda_i=1\,.$$

Duality results between (MOFP) and its corresponding two duals can be established on the same lines as those obtained in the case of multiobjective optimization problem (MOP).

6. CONCLUSION

In this paper, we have studied a second order Mond-Weir type dual for a quasi-differentiable programming problem with square root terms in the objective as well as in the constraining functions. For this purpose, we have introduced the notion of generalized higher order η – bonvexity. We have also considered a fractional case. The results can easily be extended to second order Mangasarian type dual. It would be interesting to extend the results for other classes of optimization problems, viz. minimax programming problem and minimax fractional programming problem.

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