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VECTOR EQUILIBRIUM PROBLEMS WITH NEW TYPES OF GENERALIZED MONOTONICITY

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Abstract: In this paper, we introduce the concept of generalized relaxed α-pseudomonotonicity for vector valued bi-functions. By using the KKM technique, we obtain some substantial results of the vector equilibrium problems with generalized relaxed α-pseudomonotonicity assumptions in reflexive Banach spaces. Several examples are provided to illustrate our investigations.

Keywords: Vector equilibrium problem, generalized relaxed α-pseudomonotonicity, KKM mapping.

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1. INTRODUCTION

Equilibrium problems in the sense of Blum and Oettli [1] has vast applications in the several branches of pure and applied sciences. The equilibrium problem includes many mathematical problems as its particular cases, e.g., mathematical programming problems, complementary problems, variational inequality problems, and fixed point problems. Inspired by the notion of vector variational inequality problem introduced and studied by Giannessi [2], Chen and Yang [3], the equilibrium problem has been extended to vector equilibrium problem. The vector equilibrium problem contains vector optimization problems, vector variational inequality problems, and vector complementarity problems as a special case.

Let Y be a real Banach space and C be a nonempty subset of Y. C is called a cone if $\lambda C \subset C$, for any $\lambda \geq 0$. Further, the cone C is called convex cone if $C + C \subset C$. C is pointed cone if C is cone and $C \cap (-C) = \{0\}$.

Now, consider $C \subseteq Y$ is a pointed closed convex cone with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ is the set of interior points of C. Then, C induce a vector ordering in Y as follows:

$$\begin{aligned} x \leq y &\Leftrightarrow y - x \in C; \\ x \nleq y &\Leftrightarrow y - x \notin C; \\ x < y &\Leftrightarrow y - x \in \text{intC}; \\ x \nless y &\Leftrightarrow y - x \notin \text{intC}. \end{aligned}$$

By (Y, C), we denote an ordered space with the ordering of Y defined by set C.

Now, let $K \subseteq X$ be nonempty closed convex subset of a real reflexive Banach space X and (Y, C) be an ordered Banach space induced by the pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$. The vector equilibrium problem (for short, (VEP)) for the bi-function $f : K \times K \to Y$ is to find $\overline{x} \in K$, such that

$$f(\overline{x}, y) \nless 0, \ \forall y \in K, \tag{1.1}$$

with $f(x, x) = 0, \forall x \in K$.

In the study of vector equilibrium problems and vector variational inequalities, the generalized monotonicity plays an important role. In recent years, a number of authors have proposed many important notions of generalized monotonicities such as monotonicity, pseudomonotonicity, quasimonotonicity, relaxed monotonicity, relaxed η-α monotonicity, relaxed η-α pseudomonotonicity, see, [4, 5, 6, 7, 8]. In 2003, Fang and Huang [6] introduced the concept of relaxed η-α monotonicity and obtained the existence of solutions for the variational-like inequalities in the reflexive Banach spaces. Very recently, Bai et al. [4] introduced a new concept of relaxed η-α pseudomonotone mappings and obtained the solutions for the variational-like inequalities. In 2007, Wu and Huang [8] extended the idea of Bai et al. [4] for vector variational-like inequality problem. Wu and Huang [8] defined the concepts of relaxed η-α pseudomonotone mappings to study vector variational-like inequality problem in Banach spaces. It is well known that many results of variational inequality problems can be adapted to equilibrium problems under some suitable modifications, which is a very fundamental phenomenon in this field.

Inspired and motivated by [4, 6, 8, 9], in this paper we introduce the concept of generalized relaxed α-pseudomonotonicity for vector valued bi-functions. By using the KKM technique, we obtain the existence of solutions for (VEP) with generalized relaxed α-pseudomonotone mappings in reflexive Banach spaces.

2. PRELIMINARIES

Throughout the paper, unless otherwise specified, K is a nonempty closed convex subset of a reflexive Banach space X and (Y, C) is an ordered Banach space induced by the pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$. The following definitions and lemma will be useful in our paper.

Definition 2.1. The mapping $f : X \to Y$ is C-convex on X if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$, for all $x,\ y \in X$, $t \in [0, 1]$.

Definition 2.2. A mapping $f : K \to Y$ is said to be completely continuous if for any sequence $\{x_n\} \in K$, $x_n \rightharpoonup x_0 \in K$ weakly, then $f(x_n) \to f(x_0)$.

Lemma 2.1. ([10, 3]) Let (Y, C) be an ordered Banach space induced by the pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$. Then for any $x, y, z \in Y$, the following holds:

$$z \not< x \geq y \text{ implies y} \not> \text{z};$$
$$z \not> x \leq y \text{ implies y} \not< \text{z}.$$

Definition 2.3. Let $f : K \to 2^X$ be a set-valued mapping. Then f is said to be KKM mapping if for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of K we have $co\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^{n} f(y_i)$, where $co\{y_1, y_2, \ldots, y_n\}$ denotes the convex hull of $y_1, y_2, \ldots, y_n$.

Lemma 2.2. [11] Let M be a nonempty subset of a Hausdorff topological vector space X and let $f : M \to 2^X$ be a KKM mapping. If $f(y)$ is closed in X for all $y \in M$ and compact for some $y \in M$, then $\bigcap_{y \in M} f(y) \neq \emptyset$.

3. (VEP) WITH GENERALIZED RELAXED α-PSEUDOMONOTONICITY

In this section, we define the definition of generalized relaxed α-pseudomonotone mappings to study (VEP).

Definition 3.4. A mapping $f : K \times K \to Y$ is said to be generalized relaxed α-pseudomonotone if there exists a function $\alpha : X \times X \to Y$ with

$$\lim_{t \to 0} \frac{\alpha(ty + (1-t)x, x)}{t} = 0,$$

such that

$$f(x, y) \not< 0 \;\Rightarrow\; f(y, x) \leq \alpha(y, x), \forall x, y \in K. \tag{3.1}$$

Remark 3.1. (i) If there exists a function $\alpha : X \to Y$ with $\alpha(tx) = k(t)\alpha(x)$ for all $t > 0$ and $x \in X$, where k is function from $(0, \infty)$ to $(0, \infty)$ with $\lim_{t \to 0} \frac{k(t)}{t} = 0$, such that for every pair of points $x, y \in K$, we have

$$f(x, y) \not< 0 \;\Rightarrow\; f(y, x) \leq \alpha(y - x), \tag{3.2}$$

then f is said to be weakly relaxed α-pseudomonotone (see [9]).

(ii) If $\alpha \equiv 0$ then from (3.1) it follows that
$f(x, y) \not< 0 \Rightarrow f(y, x) \leq 0, \forall x, y \in K$, and f is said to be pseudomonotene.

(iii) If $Y = \mathbb{R}$, and $\alpha(tx) = k(t)\alpha(x)$ for all $t > 0$ and $x \in X$, where k is function from $(0, \infty)$ to $(0, \infty)$ with $\lim_{t \to 0} \frac{k(t)}{t} = 0$, then (3.2) reduces to that for any $x, y \in K$,

$$f(x, y) \geq 0 \;\Rightarrow\; f(y, x) \leq \alpha(y - x), \tag{3.3}$$

and f is said to be weakly relaxed α-pseudomonotone (see [9]).

So from the above definitions, it follows that pseudomonotonicity $\Rightarrow$ weakly relaxed α-pseudomonotonicity $\Rightarrow$ generalized relaxed α-pseudomonotonicity. But the converse implications are not true in general, which is shown by the following examples. The Example 3.1 shows that weakly relaxed α-pseudomonotone may not be pseudomonotone, and The Example 3.2 shows that generalized relaxed α-pseudomonotone mapping neither pseudomonotone nor relaxed α-pseudomonotone mapping.

Example 3.1. [9] Let $X = \mathbb{R}$, $K = [1, 10]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and

$$f(x, y) = \begin{cases} \begin{pmatrix} x - y - 1 \\ x - y - 1 \end{pmatrix}, & x \geq y; \\ \\ \begin{pmatrix} x + y - 1 \\ x + y - 1 \end{pmatrix}, & x < y. \end{cases}$$

Then f is weakly relaxed α-pseudomonotone mapping with $\alpha(x) = \begin{pmatrix} 20x^2 \\ 20x^2 \end{pmatrix}$, $\forall x \in X$, but f is not pseudomonotone.

Example 3.2. Let $X = \mathbb{R}$, K is any nonempty closed convex subset of $\mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and

$$f(x,y) = \begin{pmatrix} (\sin x - \sin y)^2 \\ (\sin x - \sin y)^2 \end{pmatrix}.$$

Then f is generalized relaxed α-pseudomonotone mapping with

$$\alpha(x,y) = \begin{pmatrix} 2(\sin x - \sin y)^2 \\ 2(\sin x - \sin y)^2 \end{pmatrix}, \forall x, y \in X.$$

But f is not pseudomonotone, in fact
$f(x,y) = \begin{pmatrix} (\sin x - \sin y)^2 \\ (\sin x - \sin y)^2 \end{pmatrix} \not< 0$, for $x \neq y \Rightarrow f(y,x) = \begin{pmatrix} (\sin y - \sin x)^2 \\ (\sin y - \sin x)^2 \end{pmatrix} \not\leq 0$, for $x \neq y$. Again, f is not weakly relaxed α-pseudomonotone mapping as the function α is not a function of $(y - x)$.

Theorem 3.1. Let K be a nonempty closed convex subset of a reflexive Banach space E and (Y, C) is an ordered Banach space induced by the pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$. Suppose $f : K \times K \to Y$ be hemicontinuous in the first argument and generalized relaxed α-pseudomonotone. Let the following condition hold:

(i) for fixed $z \in K$, the mapping $x \mapsto f(z,x)$ is C-convex.

Then $\overline{x} \in K$ is a solution of (1.1) if and only if

$$f(y, \overline{x}) \leq \alpha(y, \overline{x}), \forall y \in K. \tag{3.4}$$

Proof. Assume that $\overline{x}$ is a solution of (VEP) (1.1), i.e., $f(\overline{x}, y) \not< 0, \forall y \in K$.
Since f is generalized relaxed α-pseudomonotone, we have $f(y, \overline{x}) \leq \alpha(y, \overline{x}), \forall y \in K$, which proved (3.4).
Conversely, suppose there exists an $\overline{x} \in K$ satisfying (3.4). Choose any point $y \in K$ and consider $x_t = ty + (1-t)\overline{x}, t \in [0,1]$. Therefore from (3.4), we have

$$f(x_t, \overline{x}) \leq \alpha(x_t, \overline{x}). \tag{3.5}$$

Now condition (i) implies,

$$0 = f(x_t, x_t)) \leq t f(x_t, y) + (1-t) f(x_t, \overline{x})$$

$$\Rightarrow t[f(x_t, \overline{x}) - f(x_t, y)] \leq f(x_t, \overline{x}). \tag{3.6}$$

From (3.5) and (3.6), we have

$$f(x_t, \overline{x}) - f(x_t, y) \leq \frac{\alpha(x_t, \overline{x})}{t}, \forall y \in K.$$

Since f is hemicontinuous in the first argument and taking $t \to 0$, we get

$$f(\overline{x}, y) \geq 0 \Rightarrow f(\overline{x}, y) \not< 0, \forall y \in K.$$

Hence $\overline{x}$ is a solution of (VEP). □

Theorem 3.2. Let K be a nonempty closed bounded convex subset of a reflexive Banach space E and (Y, C) is an ordered Banach space induced by the pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$. Suppose $f : K \times K \to Y$ be hemicontinuous in the first argument and generalized relaxed α-pseudomonotone. Let the following conditions hold:

(i) for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is C-convex and completely continuous;

(ii) $\alpha : X \times X \to Y$ is completely continuous in the second argument.

Then the (VEP) has a solution.

Proof. Consider the set valued mappings $F : K \to 2^X$ and $G : K \to 2^X$ such that
$$F(y) = \{x \in K : f(x, y) \not< 0\},\ \forall y \in K,$$
and
$$G(y) = \{x \in K : f(y, x) \leq \alpha(y, x)\},\ \forall y \in K.$$
Now, $\overline{x} \in K$ solves (VEP) if and only if $\overline{x} \in \bigcap_{y \in K} F(y)$. Thus it suffices to prove $\bigcap_{y \in K} F(y) \neq \emptyset$. First to show that F is a KKM mapping.

If possible let F is not a KKM mapping. Then there exists $\{x_1, x_2, \ldots, x_m\} \subset K$ such that

$co\{x_1, x_2, \ldots, x_m\} \nsubseteq \bigcup_{i=1}^{m} F(x_i)$, that means there exists a $x_0 \in co\{x_1, x_2, \ldots, x_m\}$, $x_0 = \sum_{i=1}^{m} t_i x_i$ where $t_i \geq 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} t_i = 1$, but $x_0 \notin \bigcup_{i=1}^{m} F(x_i)$.

Hence, $f(x_0, x_i) < 0$; for $i = 1, 2, \ldots, m$.

From (i), it follows that
$$0 = f(x_0, x_0) \leq \sum_{i=1}^{m} t_i f(x_0, x_i) < 0,$$
which is a contradiction. Hence F is KKM mapping.

From the generalized relaxed α-pseudomonotonicity of f, it follows that $F(y) \subset G(y),\ \forall y \in K$. Therefore G is also a KKM mapping.

Since K is closed, bounded and convex, we know that K is weakly compact. From the assumptions, we know that $G(y)$ is weakly closed for all $y \in K$. In fact, since $x \mapsto f(z, x)$ is C-convex and completely continuous, and α is completely continuous in the second argument. $G(y)$ is weakly closed for all $y \in K$ and so $G(y)$ is weakly compact in K for all $y \in K$. Hence, from Lemma 2.2 and Theorem 3.1 follows that
$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$
So there exists $\overline{x} \in K$ such that $f(\overline{x}, y) \not< 0, \forall y \in K$, i.e. (VEP) (1.1) has a solution. □

Theorem 3.3. Let K be a nonempty closed unbounded convex subset of a reflexive Banach space E and (Y, C) is an ordered Banach space induced by the pointed closed convex cone C with $\operatorname{int} C \neq \emptyset$. Suppose $f : K \times K \to Y$ be hemicontinuous in the first argument and generalized relaxed α-pseudomonotone. Let the following conditions hold:

(i) for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is C-convex and completely continuous;

(ii) $\alpha : X \times X \to Y$ is completely continuous in the second argument;

(iii) f is weakly coercive that is there exists $x_0 \in K$ such that $f(x, x_0) < 0$, whenever $\|x\| \to +\infty$ and $x \in K$.

Then (VEP) has a solution.

Proof. For $r > 0$, assume $B_r = \{y \in K : \|y\| \leq r\}$.
Consider the problem: find $x_r \in K \cap B_r$ such that

$$f(x_r, y) \not< 0, \ \forall y \in K \cap B_r. \tag{3.7}$$

By Theorem 3.2, we know that problem (3.7) has solution $x_r \in K \cap B_r$.
Choose $\|x_0\| < r$ with x_0 as in condition (iii). Then $x_0 \in K \cap B_r$ and

$$f(x_r, x_0) \not< 0. \tag{3.8}$$

If $\|x_r\| = r$ for all r, we may choose r large enough so that by the assumption (iii) imply that $f(x_r, x_0) < 0$, which contradicts (3.8).
Therefore, there exists r such that $\|x_r\| < r$. For any $y \in K$, we can choose $0 < t < 1$ small enough such that $x_r + t(y - x_r) \in K \cap B_r$.
From (3.8), it follows that

$$\begin{aligned} 0 &\not> f(x_r, x_r + t(y - x_r)) \\ &\leq tf(x_r, y) + (1 - t)f(x_r, x_r) \\ &= tf(x_r, y). \end{aligned}$$

Hence, $$0 \not> f(x_r, x_r + t(y - x_r)) \leq tf(x_r, y) \tag{3.9}$$

From Lemma 2.1 and (3.9), it follows that

$$f(x, y) \not< 0, \ \forall y \in K.$$

□

4. CONCLUSIONS

In this work, the concept generalized relaxed α-pseudomonotonicity has been introduced for bi-functions. It has been shown that the generalized relaxed α-pseudomonotonicity is a proper generalization of monotonicity and pseudomonotonicity. The existence of solutions of vector equilibrium problems have been studied with generalized relaxed α-pseudomonotone mappings. The use of generalized monotonicity and generalized convexity in the study of equilibrium problems and variational inequality problems will orient the future study of the research.

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