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OPTIMAL PRODUCTION POLICY FOR MULTI-PRODUCT WITH INVENTORY-LEVEL-DEPENDENT DEMAND IN SEGMENTED MARKET

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Abstract: Market segmentation has emerged as the primary means by which firms achieve optimal production policy. In this paper, we use market segmentation approach in multi-product inventory system with inventory-level-dependent demand. The objective is to make use of optimal control theory to solve the inventory-production problem and develop an optimal production policy that minimizes the total cost associated with inventory and production rate in segmented market. First, we consider a single production and inventory problem with multi-destination demand that vary from segment to segment. Further, we describe a single source production and multi destination inventory and demand problem under the assumption that firm may choose independently the inventory directed to each segment. The optimal control is applied to study and solve the proposed problem.

Keywords: Market Segmentation, Inventory-Production System, Optimal Control Problem.

MSC: 90I305.

1. INTRODUCTION

For a long time, industrial development in various sectors of economy induced strategies of mass production and marketing. Those strategies were manufacturing oriented, focusing on reduction of production costs rather than satisfaction of customers. But as production processes become more flexible, and customer's affluence led to the diversification of demand, firms that identified the specific needs of groups of customers were able to develop the right offer for one or more sub–markets and thus obtained a competitive advantage. As market–oriented thought evolved within firm, the concept of market segmentation emerged. Market segmentation is the division of a market into different groups of customers with distinctly similar needs and product/service requirements. In other words, market segmentation [1] is defined as the process of partitioning a market into groups of potential customers who share similar defined characteristics (attributes) and are likely to exhibit similar purchase behavior. Some major variables used for segmentation are Geographic variables (such as Nations, region, state, countries, cities and neighborhoods), Demographic variables (like Age, gender, income, family size, occupation, education), Psychographic variables (which includes Social class, life style, personality, value) and Behavioral variables, (User states, usage rate, purchase occasion, attitude towards product). Nevertheless, market segmentation is not well known in mathematical inventory-production models. Only a few papers on inventory-production models deal with market segmentation [2, 3]. To look forward in this direction, in [4] has been proposed a concept of market segmentation in inventory-production system for a single product and studied the optimal production rate in a segmented market.

Optimal control theory is a fruitful and interdisciplinary area of research in dynamic systems, i.e. systems that evolve over time and use mathematical optimization tool for deriving control policies over time. The application of optimal control theory in inventory-production control analysis is possible due to its dynamic behavior and optimal control models, which provide a powerful tool for understanding the behavior of inventory-production system where dynamic aspect plays an important role. It has been used in inventory-production [5-7] to derive the theoretical structure of optimal policies. Apart from inventory-production, it has been successfully applied to many areas of operational research such as Finance [8,9], Economics[10-12], Marketing [13-16], Maintenance [17] and the Consumption of Natural Resources[18-20] etc.

In this paper, we assume that firm has defined its target market in a segmented consumer population and develop an inventory-production plan to attack each segment with the objective of minimizing total cost. In addition, we shed some light on the problem in the control of a single firm with a finite production capacity (producing a single item at a time), which serves as a supplier of a common product to multiple market segments. Segmented customers place demand continuously over time with rates that vary from segment to segment. We consider demand as a function of on hand inventory and time [21]. In response to segmented customer demand, the firm must decide on how much inventory to stock and when to replenish this stock by producing. We apply optimal control theory to solve the problem, and find the optimal production and inventory policy.

The rest of the paper is organized as follows. Following this introduction, all the notations and assumptions needed in the sequel is stated in Section 2. In section 3, we describe the single source inventory problem with multi-destination demand that vary from segment to segment, and develop the optimal control theory problem so as its solution. In section 4, we introduce optimal control formulation of multi-destination demand and inventory problem and discuss its solution. Section 5 discusses the conclusion with future prospects.

2. SETS AND SYMBOLS

2.1. Assumptions

The time horizon is finite. The model is developed for multi-product in segmented market. The production and demand are function of time. The holding cost rate is function of inventory level & production cost rate, which depends on the production rate. The functions $h_{ij}(I_{ij}(t))$ (in case of single source $h_j(I_j(t))$) are convex. All functions are assumed to be non- negative, continuous and differentiable. This allows us to derive the most general and robust conclusions. Further, we will consider more specific cases, which for we obtain some important results.

2.2. Sets

- Period set [0, T].
- Segment set with cardinality *n* and indexed by *i*.
- Product set with cardinality *m* and indexed by *j*.

2.3. Parameters

T	Length of planning period
$P_j(t)$	Production rate for j^{th} product
$I_j(t)$	Inventory level for j^{th} product
$I_{ij}(t)$	Inventory level for j^{th} product in i^{th} segment
$D_{ij}(t,I_{ij}(t))$	Demand rate for j^{th} product in i^{th} segment
$h_j(I_j(t))$	Holding cost rate for j^{th} product (single source inventory)
$h_{ij}(I_{ij}(t))$	Holding cost rate for j^{th} product in i^{th} segment
$K_j(P_j(t))$	Cost rate corresponding to the production rate for j^{th} product
ρ	Constant non-negative discount rate

3. SINGLE SOURCE PRODUCTION AND INVENTORY WITH MULTI DESTINATION DEMAND PROBLEM

Many manufacturing enterprises use an inventory-production system to manage fluctuations in consumers demand for a product. Such a system consists of a manufacturing plant and a finished goods warehouse used to store those products which were manufactured but not immediately sold. Here, we assume that once a product is made, it is put inventory into single warehouse and that demand for all products comes from each segment. Here, we assume that there are *m* products and *n* segments (i.e. $j = 1\ to\ m$ and $i = 1\ to\ n$). Therefore, the inventory evolution in segmented market is described by the following differential equation:

$$\frac{d}{dt} I_j(t) = P_j(t) - \sum_{i=1}^{n} D_{ij}(t, I_j(t)), \quad I_j(0) = I_{j0} \quad \forall j \tag{1}$$

So far, a firm wants to minimize the cost during planning period in segmented market. Therefore, the objective functional for all segments is defined as

$$\underset{P_j(t) \geq \sum_{i=1}^{m} D_{ij}(t, It, I(t))}{Min} \ J = \int_0^T e^{-\rho t} \sum_{j=1}^{m} \left[K_j(P_j(t)) + h_j(I_j(t)) \right] dt \tag{2}$$

subject to the equation (1). This is the optimal control problem with *m*-control variables (rate of production) with *m*-state variable (inventory states). Since total demand occurs at rate $\sum_{i-1}^{n} D_{ij}(t, I(t))$ and production occurs at controllable rate $P_j(t)$ for all products, it follows that $I_j(t)$ evolves according to the above state equation (1). The constraints $P_j(t) \geq \sum_{i=1}^{n} D_{ij}(t, I(t))$ and $I_j(0) = I_{j0} \geq 0$ ensure that shortages are not allowed.

Using the maximum principle [10], the necessary conditions for (P_j^*, I_j^*) to be an optimal solution of the above problem are that there should exist a piecewise continuously differentiable function λ and piecewise continuous function μ, called the adjoint and Lagrange multiplier function, respectively such that

$$H(t, I^*, P^*, \lambda) \geq H(t, I^*, P, \lambda), \ for\ all\ P_j(t) \geq \sum_{i=1}^{n} D_{ij}(t, I(t)) \tag{3}$$

$$\frac{d}{dt}\lambda_j(t) = -\frac{\partial}{\partial I_j} L(t, I, P, \lambda, \mu) \tag{4}$$

$$I_j(0) = I_{j0}, \lambda_j(T) = \beta_j \tag{5}$$

$$\frac{\partial}{\partial P_j} L(t, I, P, \lambda, \mu) = 0 \tag{6}$$

$$P_j(t) - \sum_{i=1}^{n} D_{ij}(t) \geq 0, \ \mu_j(t) \geq 0, \quad \mu_j(t)\left[P_j(t) - \sum_{i=1}^{n} D_{ij}(t) \right] = 0 \tag{7}$$

where, $H(t, I, P, \lambda)$ and $L(t, I, P, \lambda, \mu)$ are Hamiltonian function and Lagrangian function, respectively. In the present problem, Hamiltonian function and Lagrangian function are defined as

$$H = \sum_{j=1}^{m} \left[-K_j(P_j(t)) - h_j(I_j(t)) + \lambda_j(t) \left(P_j(t) - \sum_{i=1}^{n} D_{ij}(t, I(t)) \right) \right] \tag{8}$$

$$L(t, I, P, \lambda, \mu) = \sum_{j=1}^{m} \left[-K_j(P_j(t)) - h_j(I_j(t)) + (\lambda_j(t) + \mu_j(t)) \left(P_j(t) - \sum_{i=1}^{n} D_{ij}(t, I(t)) \right) \right] \tag{9}$$

A simple interpretation of the Hamiltonian is that it represents the overall profit of the various policy decisions with both the immediate and the future effects taken into account; and the value of $\lambda_j(t)$ at time t describes the future effect on profits upon making a small change in $I_j(t)$. Hence, the Hamiltonian H for all segments is strictly concave in $P_j(t)$, in accordance with Mangasarian sufficiency theorem [4, 10]; and there exist a unique Production rate. From equations (4) and (6), we have following equations respectively.

$$\frac{d}{dt}\lambda_j(t) = \rho\lambda_j(t) + \left\{ \begin{array}{l} \dfrac{\partial h_j(I_j(t))}{\partial I_j} \\ +(\lambda_j(t) + \mu_j(t)) \left(\dfrac{\partial}{\partial I_j} \left(\sum_{i=1}^{n} D_{ij}(t, I_j(t)) \right) \right) \end{array} \right\} \tag{10}$$

$$\lambda_j(t) + \mu_j(t) = \frac{d}{dP_j} K_j(P_j(t)) \tag{11}$$

Now, consider equation (7). Then, for any t, we have either $P_j(t) - \sum_{i=1}^{n} D_{ij}\left(t, I(t)\right) = 0$ or $P_j(t) - \sum_{i=1}^{n} D_{ij}\left(t, I(t)\right) > 0 \quad \forall \ j$.

3.1. Case 1:

Let S be a subset of planning period $[0, T]$, when $P_j(t) - \sum_{i=1}^{n} D_{ij}\left(t, I(t)\right) = 0$. Then $\frac{d}{dt} I_j(t) = 0$ on S. In this case I_j^* is obviously constant on S and the optimal production rate is given by the following equation:

$$P_j^*(t) = \sum_{i=1}^{n} D_{ij}(t, I(t)) \quad \forall\, t \in S \tag{12}$$

By using equations (10) and (11), we have

$$\frac{d}{dt}\lambda_j(t) = \rho\lambda_j(t) + \left\{ \frac{\partial h_j\left(I_j^*(t)\right)}{\partial I_j} + \frac{d}{dP_j}K_j(P_j(t)) \left(\frac{\partial}{\partial I_j}\left(\sum_{i=1}^{n} D_{ij}(t, I_j(t)) \right) \right) \right\} \tag{13}$$

After solving the above equations, we get an explicit form of the adjoint function $\lambda_j(t)$. From equation (10), we can obtain the value of Lagrange multiplier $\mu_j(t)$.

3.2. Case2:

$P_j(t) - \sum_{i=1}^{n} D_{ij}(t, I_j(t)) > 0 \quad \forall\, t \in [0,T]/S$. Then $\mu_j(t) = 0 \quad \forall\, t \in [0,T]/S$. In this case, equations (10) and (11) become

$$\frac{d}{dt}\lambda_j(t) = \rho\lambda_j(t) + \left\{ \frac{\partial h_j\left(I_j(t)\right)}{\partial I_j} + \lambda_j(t) \left(\frac{\partial}{\partial I_j}\left(\sum_{i=1}^{n} D_{ij}(t, I_j(t)) \right) \right) \right\} \tag{14}$$

$$\lambda_j(t) = \frac{d}{dP_j}K_j(P_j(t)) \tag{15}$$

Combining these equations with the state equation, we have the following second order differential equation:

$$\frac{d}{dt}P_j(t)\frac{d^2}{dP_j^2}K_j(P_j) - \left[\rho + \frac{\partial}{\partial I_j}\left(\sum_{i=1}^{n} D_{ij}(t, I_j(t)) \right) \right] \frac{d}{dP_j}K_j(P_j) = \frac{\partial h_j(t, I_j(t))}{\partial I_j} \tag{16}$$

and $I_j(0) = I_{j0}$, $\frac{d}{dP_j}K_j(P_j(T)) = \beta_j$. For the purpose of illustration, let us assume the following forms of the exogenous functions $K_j(P_j(t)) = k_j P_j^2(t)/2$, $h_j(t, I_j(t)) = h_j \frac{I_j^2(t)}{2}$,and $D_{ij}(t, I_j(t)) = a_{ij}(t) + b_{ij}\alpha_{ij}I_j(t)$,where $k_j, h_j, \alpha_{ij}, b_{ij}$ are positive constants for all $j = 1, \ldots m$.

For these functions, the necessary conditions for (P_j^*, I_j^*) to be an optimal solution of problem (2) with equation (1) become

$$\frac{d^2}{dt^2}I_j(t) - \rho\frac{d}{dt}I_j(t) - \left(\frac{h_j}{k_j} + (\rho + \sum_{i=1}^{n} b_{ij}\alpha_{ij})\sum_{i=1}^{n} b_{ij}\alpha_{ij} \right) I_j(t) = \eta_j(t) \tag{17}$$

with $I_j(0) = I_{j0}$, $\frac{d}{dP_j} K_j(P_j(T)) = \beta_j$

where $\eta_j(t) = \left(\rho + \sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)\sum_{i=1}^{n} a_{ij}(t) - \sum_{i=1}^{n} \dot{a}_{ij}(t)$ .This problem is a two-point boundary value problem.

Proposition: The optimal solution (P_j^*, I_j^*) to the problem is given by

$$I_j^*(t) = c_{1j}e^{m_{1j}t} + c_{2j}e^{m_{2j}t} + Q_j(t) \tag{18}$$

And the corresponding P_j^* is given by

$$\begin{aligned} P_j^*(t) = {} & c_{1j}\left(m_{1j} + \sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)e^{m_{1j}t} + c_{2j}\left(m_{2j} + \sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)e^{m_{2j}t} \\ & + \frac{d}{dt}Q_j(t) + \left(\sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)Q_j(t) + \sum_{i=1}^{n} a_{ij}(t) \end{aligned} \tag{19}$$

Where the constants c_{1j}, c_{2j}, m_{1j} *and* m_{2j} are given in the proof, and $Q_j(t)$ is a particular solution of the equation (17).

Proof: The solution of two point boundary value problem (17) is given y standard method. Its characteristic equation $m_j^2 - \rho m_j - \left(\frac{h_j}{k_j} + \left(\rho + \sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)\sum_{i=1}^{n} b_{ij}\alpha_{ij}\right) = 0$, has two real roots of opposite sign, given by

$$m_{1j} = \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\left(\frac{h_j}{k_j} + \left(\rho + \sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)\sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)}\right) < 0$$

$$m_{2j} = \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\left(\frac{h_j}{k_j} + \left(\rho + \sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)\sum_{i=1}^{n} b_{ij}\alpha_{ij}\right)}\right) > 0$$

And therefore $I_j^*(t)$ is given by (18), where $Q_j(t)$ is the particular solution. Then initial and terminal condition used to determined the values of constant a_{1j} and a_{2j} are as follows

$c_{1j} + c_{2j} + Q_j(0) = I_{j0}$,

$c_{1j}(m_{1j})e^{m_{1j}T} + c_{2j}(m_{2j})e^{m_{2j}T} + \left(\frac{d}{dt}Q_j(T) + \sum_{i=1}^{n} b_{ij}\alpha_{ij}Q_j(T) + \sum_{i=1}^{n} a_{ij}(T)\right) = 0$

Putting $r_{1j} = I_{j0} - Q_j(0)$ and $r_{2j} = -\left(\frac{d}{dt}Q_j(T) + \sum_{i=1}^{n} b_{ij}\alpha_{ij}Q_j(T) + \sum_{i=1}^{n} a_{ij}(T)\right)$, we obtain the following system of two linear equations with two unknowns

$$\begin{aligned} & c_{1j} + c_{2j} = r_{1j}, \\ & c_{1j}(m_{1j})^{m_{1j}T} + c_{2j}(m_{2j})e^{m_{2j}T} = r_{2j} \end{aligned} \tag{20}$$

The value of P_j^* is deduced using the value of I_j^* and the state equation.

4. SINGLE SOURCE PRODUCTION AND MULTI DESTINATION INVENTORY AND DEMAND PROBLEM

We consider the single source production and multi destination demand-inventory system. Hence, the inventory evolution in each segment is described by the following differential equation:

$$\frac{d}{dt}I_{ij}(t) = \gamma_{ij}\, P_j(t) - D_{ij}(t, I_{ij}(t)) \quad \forall i, j \tag{21}$$

Here, $\gamma_{ij} > 0$, $\sum_{i=1}^{n} \gamma_{ij} = 1$ with the conditions $I_{ij}(0) = I_{ij}^0$ and $\gamma_{ij}\, P_j(t) \geq D_{ij}(t, I_{ij}(t))$. We called $\gamma_{ij} > 0$ the segment production spectrum and $\gamma_{ij}\, \mathrm{P}_j(t)$ define the relative segment production rate of j^{th} product towards i^{th} segment.

We develop a marketing-production model in which firm seeks to minimize its all cost by properly choosing production and market segmentation. Therefore, we defined the cost minimization objective function as follows:

$$\underset{\gamma_{ij}P_j(t) \geq \sum_{i=1}^{m} D_{ij}(t, I_j(t))}{Min} \; J = \int_0^T e^{-\rho t} \sum_{j=1}^{m} \left[K_j(P_j(t)) + \sum_{i=1}^{n} h_{ij}(I_{ij}(t)) \right] dt \tag{22}$$

Subject to the equation (21), this is the optimal control problem (production rate) with *m* control variable with *nm* state variable (stock of inventory in n segments).

To solve the optimal control problem expressed in equation (21) and (22), the following Hamiltonian and Lagrangian are defined as

$$H = \sum_{j=1}^{m} \left[-K_j(P_j(t)) - \sum_{i=1}^{n} h_{ij}(I_{ij}(t)) + \lambda_{ij}(t)\left(\gamma_{ij}\, P_j(t) - D_{ij}(t, I_{ij}(t))\right) \right] \tag{23}$$

$$L\left(t, I, P, \lambda, \mu\right) = \sum_{j=1}^{m} \left[-K_j(P_j(t)) - \sum_{i=1}^{n} h_{ij}(I_{ij}(t)) + (\lambda_{ij}(t) + \mu_{ij}(t))(\gamma_{ij}\, P_j(t) - D_{ij}(t, I_{ij}(t))) \right] \tag{24}$$

Equations (4), (6) and (21) yield

$$\frac{d}{dt}\lambda_{ij}(t) = \rho\lambda_{ij}(t) - \left\{ \begin{aligned} & -\frac{\partial h_{ij}(I_{ij}(t))}{\partial I_{ij}} \\ & -(\lambda_{ij}(t) + \mu_{ij}(t))\frac{\partial D_{ij}(t, I_{ij}(t))}{\partial I_{ij}} \end{aligned} \right\} \quad \forall i,\ j \tag{25}$$

$$\sum_{i=1}^{n}\left(\lambda_{ij}(t) + \mu_{ij}(t)\right)\gamma_{ij} = \frac{d}{dP_j}K_j(P_j(t)) \tag{26}$$

In the next section of the paper, we consider only the case when $\gamma_{ij}\, P_j(t) - D_{ij}(t, I_{ij}(t)) > 0 \ \ \forall i, j$.

4.1 Case 2:

$\gamma_{ij}\, P_j(t) - D_{ij}\left(t, I_{ij}(t)\right) > 0 \ \ \forall i, j \ \ for\, t \in [0,T]/S$, then $\mu_{ij}(t) = 0\, on\ t \in [0,T]/S$. In this case, the equations (25) and (26) become

$$\frac{d}{dt}\lambda_{ij}(t) = \rho\lambda_{ij}(t) + \left\{ -\frac{\partial h_{ij}(I_{ij}(t))}{\partial I_{ij}} + \lambda_{ij}(t)\frac{\partial D_{ij}(t, I_{ij}(t))}{\partial I_{ij}} \right\} \tag{27}$$

$$\sum_{i=1}^{n}\gamma_{ij}\lambda_{ij}(t) = \frac{d}{dP_j}K_j(P_j(t)) \tag{28}$$

Combining the above equations with the state equation, we have the following second order differential equation:

$$\begin{aligned} & \frac{d}{dt}P_j(t)\frac{d^2}{dP_j^{\,2}}K_j(P_j) - \rho\frac{d}{dP_j}K_j(P_j) \\ & \quad = \sum_{i=1}^{n}\gamma_i \left(\sum_{i=i}^{n}\frac{\partial h_{ij}(t, I_{ij}(t))}{\partial I_{ij}} + \sum_{i=1}^{n}\lambda_{ij}\frac{\partial}{\partial I_{ij}(t)}D_{ij}(t, I_{ij}(t)) \right) \end{aligned} \tag{29}$$

And $I_{ij}(0) = I_{ij}^0\ \forall i, j$, $\sum_{i=1}^{n}\gamma_{ij}\lambda_{ij}(T) = \frac{d}{dP_j}K_j\left(P_j(T)\right)$, $\lambda_{ij}(T) = \beta_{ij}\ \forall i, j$. For illustration purpose, let us assume the following forms of the exogenous functions $K_j(P_j) = k_j P_j^{\,2}/2$, $h_{ij}(t, I_{ij}(t)) = \frac{h_{ij}\, I_{ij}^2(t)}{2}$ and $D_{ij}(t, I_j(t)) = a_{ij}(t) + b_{ij}\alpha_{ij}I_j(t)$ where k_j , h_{ij} , α_{ij} , b_{ij} , are positive constants.

For these functions, the necessary conditions for (P_j^{*}, I_{ij}^{*}) to be an optimal solution of problem (19) with equation (18) become

$$I_{ij}''(t) + (q_{ij} - \rho)I_{ij}'(t) - \rho q_{ij}I_{ij}(t) = \eta_{ij}(t) \qquad \forall i, j \tag{30}$$

With $I_{ij}(0) = I_{ij}^{0} \quad \forall i, \quad \sum_{i=1}^{n} \beta_{ij}\gamma_{ij} = e^{-\rho t}\frac{d}{dP_j}K_j(P_j(T)),$ because of $\lambda_{ij}(T) = \beta_{ij} \; \forall i, j$

Where $\eta_{ij}(t) = (\rho - 2q_{ij})g_{ij}(t) + \frac{\gamma_{ij}\sum_{i=1}^{n}\gamma_{ij}}{k_j}\left(\sum_{i=1}^{n} h_{ij}I_{ij}(t) + \sum_{i=1}^{n}\lambda_{ij}q_{ij}\right) - \frac{dg_{ij}(t)}{dt}$. This problem is also a system of two-point boundary value problem.

The above system of two point boundary value problem (30) is solved by the same method that we used in (17).

5. CONCLUSION

The concept of market segmentation was developed in economic theory to show how a firm selling a homogenous product in a market characterized by heterogeneous demand could minimize the cost. In this paper, we have introduced market segmentation concept in the production inventory system for multi product and its optimal control formulation. We have used maximum principle to determine the optimal production rate policies that minimize the cost associated with inventory and production rate. The resulting analytical solution yield good insight on how production planning task can be carried out in segmented market environment. In the present paper, we have assumed that the segmented demand for each product is a function of time and inventory. A natural extension of the analysis developed here is that items can be taken as deteriorating.

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