Yugoslav Journal on Operations Research
23(2013) Number 2, 299–310
DOI: 10.2298/YJOR130108025S

ON SUFFICIENCY IN MULTIOBJECTIVE PROGRAMMING INVOLVING GENERALIZED (G,C,ρ)-TYPE I FUNCTIONS

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Received: January, 2013 / Accepted: March, 2013

Abstract: In this paper, a new class of (G,C,ρ)-type I functions and their generalizations are introduced. We consider a class of differentiable multiobjective optimization problems and establish sufficient optimality conditions. The results of the paper are more general than those existing in the literature.

Keywords: Multiobjective programming, (G,C,ρ)-convexity, efficient solution, type I functions, generalized convexity.

MSC: 90C46; 52A01

1. INTRODUCTION

Convexity plays an important role in optimization theory as it extends the validity of a local solution of a minimization problem to a global one. But in several real world problems, the notion of convexity is no longer sufficient, which motivated the introduction of various generalizations of convex functions. It has been found that only a few properties of convex functions are needed for establishing sufficiency and duality theorems. Hanson

[11] introduced the concept of differentiable invexity, which is a generalization of the concept of convexity. After the work of Hanson, other classes of differentiable nonconvex functions have been introduced to generalize the class of invex functions from different points of view, see the in [7-9, 12, 13, 16, 21, 22]. Later, Kaul and Kaur [14] presented strictly pseudoinvex, pseudoinvex and quasiinvex functions.

Hanson and Mond [12] defined two new classes of functions called type I and type II functions. Rueda and Hanson [23] have introduced pseudo type I and and quasi type I functions. Mishra [24] studied a multiple objective nonlinear programming problem by combining the concepts of type-I, pseudo-type-I, quasi-type-I, quasi-pseudo-type-I, pseudo-quasi-type-I and univex functions. More details on type-I functions can be found in Ye [35], Suneja and Srivastava [25], Mishra et al. [26-30]. Other classes of generalized type I functions have been introduced in [2, 15].

In [17] and [18], Liang et al. introduced (F,α,ρ,d)-convexity, which is uniformulation of generalized convexity and an extension of (F,ρ)-convexity [22] and generalized (F,ρ)-convexity [8]. They obtained optimality conditions and duality results for the single objective fractional problems. Yuan et al. [32] introduced (C,α,ρ,d)-convexity, which is a generalization of (F,α,ρ,d)-convexity. Chinchuluun et al. [10] and Long [19] later studied multiobjective fractional programming problems in the framework of (C,α,ρ,d)- convexity. Antczak[4] extended further Hanson's invexity to G-invexity for scalar differentiable functions. In the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions in [6].

Motivated by [4-6, 33], we consider a class of differentiable multiobjective optimization problems. We introduce some new generalizations of (G,C,ρ)-convex functions and establish sufficient optimality conditions for the optimization problem. The results of the paper extend and unify some earlier results from the literature to a more general class of functions.

2. DEFINITIONS AND PRELIMINARIES

In this section, we provide some definitions and some results which will we be used in the sequel. The following convention for vector in $\mathbb{R}^n$ will be adopted.

$$x < y \text{ if and only if } x_i < y_i, \text{ for } i = 1,...,n;$$
$$x \leqq y \text{ if and only if } x_i \leq y_i, \text{ for } i = 1,...n;$$
$$x \leqslant y \text{ if and only if } x_i \leq y_i, \text{ for } i = 1,...,n, \text{ but } x \neq y;$$

We consider the following nonlinear multiobjective programming problem:
(MOP)

$$\text{minimize } f(x) := (f_1(x),...,f_p(x)),$$

$$\text{subject to } g(x) := (g_1(x),...,g_q(x)) \leqq 0,$$

$$x \in X,$$

where X is a nonempty open subset of $\mathbb{R}^n$. Let A denote the set of all feasible points of (MOP) and $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^q$ are differentiable functions at $x_0 \in A$. The index set $P = \{1,2,...,p\}$ and $Q = \{1,2,...,q\}$. For $x_0 \in A$ the index set $J(x_0) = \{j \in Q : g_j(x_0) = 0\}$ and g_J denote the vector for active constraints.
In the sequel, we need the following vector minimization problem:
(G-MOP)

$$\text{minimize } G_f f(x) := (G_{f_1} f_1(x), ..., G_{f_p} f_p(x)),$$

$$\text{subject to } G_g g(x) := (G_{g_1} g_1(x), ..., G_{g_q} g_q(x)) \leqq G_g(0),$$

$$x \in X,$$

where $G_f : \mathbb{R} \to \mathbb{R}^p$ and $G_g : \mathbb{R} \to \mathbb{R}^q$ are vector valued functions.

Definition 2.1. *We say that $x_0 \in A$ is an efficient solution for problem (MOP) if and only if there exists no $x \in A$ such that $f(x) \leqslant f(x_0)$, that is, $f_i(x) \leq f_i(x_0)$ for all $i \in P$ with strict inequality for at least one $i \in P$.*

Definition 2.2. *We say that $x_0 \in A$ is a weak efficient solution for problem (MOP) if and only if there exists no $x \in A$ such that $f(x) < f(x_0)$, that is, $f_i(x) < f_i(x_0)$ for all $i \in P$.*

Let X be a subset of $\mathbb{R}^n$. For our convenience, an element of $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ is represented, in the following, as the ordered pair (τ, ρ) with $\tau \in R^n$ and $\rho \in \mathbb{R}$.

Definition 2.3. *A function $C : X \times X \times \mathbb{R}^{n+1} \to \mathbb{R}$ is convex on $\mathbb{R}^{n+1}$ with respect to the third argument if and only if, for any fixed $(x, x_0) \in X \times X$, the inequality*

$$C_{(x,x_0)}(\lambda z_1 + (1-\lambda) z_2) \leq \lambda C_{(x,x_0)}(z_1) + (1-\lambda) C_{(x,x_0)}(z_2), \forall\, \lambda \in (0,1),$$

holds for all $z_1 = (\tau_1, \rho_1) \in \mathbb{R}^{n+1}$ and $z_2 = (\tau_2, \rho_2) \in \mathbb{R}^{n+1}$, where $\tau_1, \tau_2 \in \mathbb{R}^n$ and $\rho_1, \rho_2 \in \mathbb{R}$.

Definition 2.4. *Let $f = (f_1, .., f_p) : X \to \mathbb{R}^p$ be a vector-valued function defined on a nonempty set $X \subset \mathbb{R}^n$, $I_{f_i}(x)$, be the range of f_i, $i \in P$. If there exist a vector-valued function $G_f = (G_{f_1}, ..., G_{f_p}) : \mathbb{R} \to \mathbb{R}^p$ such that any of its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain and $G_{f_i}(f_i)$ is a differentiable function on X, and real numbers $\rho_i (i \in P)$ such that for any $x \in X (x \neq x_0)$, the inequality*

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(x_0)) \geq (>) C_{(x,x_0)}(\bigtriangledown(G_{f_i}(f_i))(x_0), \rho_i), \tag{2.1}$$

holds for each $i \in P$, then f is said to be (strictly) (G_f, C, ρ)-convex at $x_0 \in X$, where $\rho = (\rho_1, ...\rho_p)^T$. The function f is said to be (strictly) (G_f, C, ρ)-convex over X if, for all $x_0 \in X$, it is (strictly) (G_f, C, ρ)-convex. In particular, f is said to be strong (strictly)(G_f, C, ρ)-convex or strictly (G_f, C, ρ)-convex with respect to $\rho > 0$ or $\rho = 0$, respectively.

In order to define an analogous class of (strictly)(G_f, C, ρ)-incave functions, the function G_{f_i} of inequality in the above definition should be replaced by the function $-G_{f_i}$. That is, the inequality

$$-(G_{f_i}(f_i(x)) - G_{f_i}(f_i(x_0))) \geq (>) C_{(x,x_0)}(- \bigtriangledown (G_{f_i}(f_i))(x_0), \rho_i),$$

holds for $i \in P$.

Remark 2.1. *From the above definition, (C, α, ρ, d)-convexity defined in [32] is a special case of (G_f, C, ρ)-convexity whenever $G_f(t) = t$, $t \in \mathbb{R}$. Therefore, (F, α, ρ)-convexity [17, 18] and (F, ρ)-convexity [22] are a special case of (G_f, C, ρ)-convexity since any sublinear functional is also convex.*

Theorem 2.1. *Let $G_{f_i}(i \in P)$ be strictly increasing function defined on $I_{f_i}(X)$, G_{g_j} $(j \in Q)$ be strictly increasing function defined on $I_{g_j}(X)$. Further, let $0 \in I_{g_j}(X)$. Then x_0 is an efficient (weak) solution for (MOP) if and only if x_0 is an efficient (weak) solution for (G-MOP).*

3. SUFFICIENT OPTIMALITY CONDITIONS

We assume throughout the paper that G_f is a vector objective function and that G_g is the constraint vector in (G-MOP). The definition of type I for single objective and constraint vector function [12] can be generalized easily to a multiple objective and constraint vector.
Throughout this paper, the following notation will be used

$$\rho = (\rho^1, \rho^2), \text{ where } \rho^1 = ({\rho_1}^1, ..., {\rho_p}^1) \in \mathbb{R}^p \text{ and } \rho^2 = ({\rho_1}^2, ..., {\rho_q}^2) \in \mathbb{R}^q.$$

$$C_{(x,x_0)}(\nabla G_f(f(x)), \rho^1) := (C_{(x,x_0)}(\nabla G_{f_1}(f_1(x)), \rho_1^1), ..., C_{(x,x_0)}(\nabla G_{f_p}(f_p(x)), \rho_p^1)).$$

$$C_{(x,x_0)}(\nabla G_g(g(x)), \rho^2) := (C_{(x,x_0)}(\nabla G_{g_1}(g_1(x)), \rho_1^2), ..., C_{(x,x_0)}(\nabla G_{g_q}(g_q(x)), \rho_q^2)).$$

We are now ready to present the new classes of functions.

Definition 3.5. *(f, g) is said to be (G, C, ρ)-type I at x_0, if for all $x \in A$ we have*

$$G_f(f(x)) - G_f(f(x_0)) \geqq C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0), \rho^1), \tag{3.1}$$

$$-G_g(g(x_0)) \geqq C_{(x,x_0)}(\bigtriangledown(G_g(g))(x_0), \rho^2). \tag{3.2}$$

Remark 3.2. *Let $G_f(t) = t$, $t \in \mathbb{R}$. Then, the above definition is a generalization of (G, C, ρ)-convexity defined in [33] and (C, α, ρ, d)-type I convexity defined in [34].*

Definition 3.6. *(f,g) is said to be pseudoquasi (G,C,ρ)-type I at x_0 , if for all $x \in A$, we have*

$$G_f(f(x)) < G_f(f(x_0)) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) < 0, \tag{3.3}$$

$$-G_g(g(x_0)) \leqq 0 \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_g(g))(x_0),\rho^2) \leqq 0. \tag{3.4}$$

If in the above definition, inequality (3.3) is satisfied as

$$G_f(f(x)) \leqq G_f f(x_0) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) < 0, \tag{3.5}$$

then, we say that (f,g) is strictly pseudoquasi (G,C,ρ)-type I at x_0.

Definition 3.7. *(f,g) is said to be weak strictly-pseudoquasi (G,C,ρ)-type I at x_0 , if for all $x \in A$, we have*

$$G_f(f(x)) \leqslant G_f(f(x_0)) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) < 0, \tag{3.6}$$

$$-G_g(g(x_0)) \leq 0 \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_g(g))(x_0),\rho^2) \leq 0. \tag{3.7}$$

Definition 3.8. *(f,g) is said to be stong-pseudoquasi (G,C,ρ)-type I at x_0, if for all $x \in A$, we have*

$$G_f(f(x)) \leqslant G_f f(x_0) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) \leqslant 0, \tag{3.8}$$

$$-G_g(g(x_0)) \leqq 0 \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_g(g(x_0)),\rho^2) \leqq 0. \tag{3.9}$$

If in the above definition, inequality (3.8) is satisfied as

$$G_f(f(x)) < G_f(f(x_0)) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) \leqslant 0, \tag{3.10}$$

then we say that (f,g) is weak pseudoquasi (G,C,ρ)-type I at x_0.

Definition 3.9. *(f,g) is said to be sub-strictly-pseudoquasi (G,C,ρ)-type I at x_0, if for all $x \in A$, we have*

$$G_f(f(x)) \leqq G_f(f(x_0)) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) \leqslant 0, \tag{3.11}$$

$$-G_g(g(x_0)) \leq 0 \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_g(g))(x_0),\rho^2) \leqq 0. \tag{3.12}$$

Definition 3.10. *(f,g) is said to be weak quasistrictly-pseudo (G,C,ρ)-type I at x_0, if for all $x \in A$, we have*

$$G_f(f(x)) \leqslant G_f(f(x_0)) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) \leq 0, \tag{3.13}$$

$$-G_g(g(x_0)) \leqq 0 \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_g(g))(x_0),\rho^2) \leqslant 0. \tag{3.14}$$

Definition 3.11. *(f,g) is said to be weak quasisemi-pseudo (G,C,ρ)-type I at x_0, if for all $x \in A$, we have*

$$G_f(f(x)) \leqslant G_f(f(x_0)) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0)),\rho^1) \leq 0, \tag{3.15}$$

$$-G_g(g(x_0)) \leq 0 \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_g(g(x_0),\rho^2) < 0. \tag{3.16}$$

Definition 3.12. *(f,g) is said to be weak strictly-pseudo (G,C,ρ)-type I at x_0, if for all $x \in A$, we have*

$$G_f(f(x)) \leqslant G_f(f(x_0)) \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_f(f))(x_0),\rho^1) < 0, \tag{3.17}$$

$$-G_g(g(x_0)) \leqq 0 \Rightarrow C_{(x,x_0)}(\bigtriangledown(G_g(g))(x_0)),\rho^2) < 0. \tag{3.18}$$

Now, we establish a sufficient optimality condition for a feasible point to be an efficient solution for (G-MOP).

Theorem 3.2. *Let x_0 be a feasible solution for (G-MOP), and let there exist vector $u \in R^p$ and vector $v \in R^q$ such that*

$$\sum_{i=1}^{p} u_i \nabla(G_{f_i}(f_i))(x_0) + \sum_{j \in J(x_0)} v_j \nabla(G_{g_j}(g_j))(x_0) = 0, \tag{3.19}$$

$$v_j G_{g_j} g_j(x_0) = 0, \forall j \in Q, \tag{3.20}$$

$$v_j \geq 0, \forall j \in Q, \tag{3.21}$$

$$u_i > 0, \forall i \in P. \tag{3.22}$$

If (f,g_J) is strong pseudoquasi (G,C,ρ)-type I at x_0 such that

$$\sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2 \geq 0, \tag{3.23}$$

$$C_{(x,x_0)}(0,r) < 0 \Rightarrow r < 0, \forall x \in X, \tag{3.24}$$

then, x_0 is an efficient solution for (G-MOP).

Proof : Suppose that x_0 is not an efficient solution for (G-MOP). Then, there exists $x \in A$, such that

$$G_{f_i}(f_i(x)) \leq G_{f_i}(f_i(x_0)), \forall i \in P, \tag{3.25}$$

with strict inequality for at least one $i \in P$. Also,

$$g_j(x_0) = 0, \forall j \in J(x_0). \tag{3.26}$$

Since (f,g_J) is strong pseudoquasi (G,C,ρ)-type I at x_0, from (3.25) and (3.26), it follows that

$$C_{(x,x_0)}(\bigtriangledown(G_{f_i}(f_i))(x_0),\rho_i^1) \leq 0, \forall i \in P, \tag{3.27}$$

with strict inequality for at least one $i \in P$, and

$$C_{(x,x_0)}(\bigtriangledown(G_{g_j}(g_j))(x_0),{\rho_j}^2) \leq 0, \forall j \in J(x_0). \tag{3.28}$$

Let

$$\tau = \sum_{i=1}^{p} u_i + \sum_{j \in J(x_0)} v_j.$$

Multiplying (3.27) and (3.28) with $\frac{1}{\tau}u_i$ and $\frac{1}{\tau}v_j$, respectively, and then adding the inequalities, we have

$$\sum_{i=1}^{p} \frac{1}{\tau} u_i C_{(x,x_0)}(\bigtriangledown(G_{f_i}(f_i))(x_0), \rho_i{}^1) + \sum_{j \in J(x_0))} \frac{1}{\tau} v_j C_{(x,x_0)}(\bigtriangledown(G_{g_j}(g_j))(x_0), \rho_j{}^2) < 0.$$

Using the convexity of C, we get

$$C_{(x,x_0)}(\frac{1}{\tau}(\sum_{i=1}^{p} u_i \bigtriangledown (G_{f_i}(f_i))(x_0) + \sum_{j \in J(x_0)} v_j \bigtriangledown (G_{g_j}(g_j))(x_0)), \sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2) < 0.$$

From (3.19), it follows that

$$C_{(x,x_0)}(0, \sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2) < 0.$$

Therefore, from (3.24), it follows that

$$\sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2 < 0.$$

Which is a contradiction to (3.23). Hence, x_0 is an efficient solution for (G-MOP). This complete the proof.

We can weaken the strict inequality requirement that $u_i > 0$ for all $i \in P$ in the above theorem, but we require different convexity conditions on (f, g_J). This is given by the following two theorems.

Theorem 3.3. *Let x_0 be a feasible solution for (G-MOP) and let there exist vector $u \in R^p$ and vector $v \in R^q$ such that*

$$\sum_{i=1}^{p} u_i \nabla(G_{f_i}(f_i))(x_0) + \sum_{j \in J(x_0)} v_j \nabla(G_{g_j}(g_j))(x_0) = 0, \tag{3.29}$$

$$v_j G_{g_j} g_j(x_0) = 0, \forall j \in Q, \tag{3.30}$$

$$v_j \geq 0, \forall j \in Q, \tag{3.31}$$

$$u_i \geq 0, \forall i \in P, \tag{3.32}$$

with strict inequality for at least one $i \in P$.
If (f, g_J) is weak strictly pseudoquasi (G, C, ρ)-type I at x_0 such that

$$\sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2 \geq 0, \tag{3.33}$$

$$C_{(x,x_0)}(0,r) < 0 \Rightarrow r < 0, \forall x \in X, \tag{3.34}$$

then, x_0 is an efficient solution for (G-MOP).

Proof: Suppose that x_0 is not an efficient solution for (G-MOP). Then, there exists $x \in A$, such that

$$G_{f_i} f_i(x) \leq G_{f_i}(f_i(x_0)), \forall i \in P, \tag{3.35}$$

with strict inequality for at least one $i \in P$. Also

$$g_j(x_0) = 0, \forall j \in J(x_0). \tag{3.36}$$

Since (f, g_J) is weak strictly pseudoquasi (G,C,ρ)-type I at x_0, from (3.35) and (3.36), it follows that

$$C_{(x,x_0)}(\bigtriangledown(G_{f_i}(f_i))(x_0), \rho_i^1) < 0, \forall i \in P, \tag{3.37}$$

$$C_{(x,x_0)}(\bigtriangledown(G_{g_j}(g_j))(x_0), {\rho_j}^2) \leq 0, \forall j \in J(x_0). \tag{3.38}$$

Now, the proof is similar to that of Theorem 3.1.

Theorem 3.4. *Let x_0 be a feasible solution for (G-MOP) and let there exist vector $u \in R^p$ and vector $v \in R^q$ such that*

$$\sum_{i=1}^{p} u_i \nabla(G_{f_i}(f_i))(x_0) + \sum_{j \in J(x_0)} v_j \nabla(G_{g_j}(g_j))(x_0) = 0, \tag{3.39}$$

$$v_j G_{g_j}(g_j(x_0)) = 0, \forall j \in Q, \tag{3.40}$$

$$v_j \geq 0, \forall j \in J(x_0), \tag{3.41}$$

$$(u_i, v_j) \geq 0, \forall i \in P, \forall j \in Q. \tag{3.42}$$

If (f, g_J) is weak quasistrictly pseudo (G,C,ρ)-type I at x_0 such that

$$\sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2 \geq 0, \tag{3.43}$$

$$C_{(x,x_0)}(0,r) < 0 \Rightarrow r < 0, \forall x \in X, \tag{3.44}$$

then, x_0 is an efficient solution for (G-MOP).

Proof: Suppose that x_0 is not an efficient solution for (G-MOP). Then, there exists $x \in A$, such that

$$G_{f_i} f_i(x) \leq G_{f_i} f_i(x_0), \forall i \in P, \tag{3.45}$$

with strict inequality for at least one $i \in P$. Also

$$g_j(x_0) = 0, \forall j \in J(x_0) \tag{3.46}$$

Since (f, g_J) is weak quasistrictly pseudo (G, C, ρ)-type I at x_0, from (3.45) and (3.46), it follows that

$$C_{(x,x_0)}(\bigtriangledown(G_{f_i}(f_i))(x_0), \rho_i^1) \leq 0, \forall i \in P, \tag{3.47}$$

$$C_{(x,x_0)}(\bigtriangledown(G_{g_j}(g_j))(x_0), {\rho_j}^2) \leq 0, \forall j \in J(x_0), \tag{3.48}$$

with strict inequality for at least one $j \in J(x_0)$.
Now, the proof is similar to that of Theorem 3.1.

Remark 3.3. *Similarly, we can prove more results like Theorem 3.1-3.3 by varying the convexity condition on (f, g_J) and changing the sign of u and v.*

It is obvious that the Theorem 3.1 and 3.2 hold for weak efficient solutions too. However, it is important to know that convexity assumptions of Theorem 3.1 and 3.2 can be weakened for weak efficient solutions.

Theorem 3.5. *Let x_0 be a feasible solution for (G-MOP) and let there exist vector $u \in R^p$ and vector $v \in R^q$ such that the triple (x, u, v) satisfies system (3.19-3.22) of the Theorem 3.2. If (f, g_J) is weak pseudoquasi (G, C, ρ)-type I at x_0 such that*

$$\sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2 \geq 0, \tag{3.49}$$

$$C_{(x,x_0)}(0, r) < 0 \Rightarrow r < 0, \forall x \in X, \tag{3.50}$$

then, x_0 is a weak efficient solution for (G-MOP).

Proof: Suppose that x_0 is not a weak efficient solution for (G-MOP). Then, there exists $x \in A$, such that

$$G_{f_i} f_i(x) < G_{f_i} f_i(x_0), \forall i \in P, \tag{3.51}$$

with strict inequality for at least one $i \in P$. Also,

$$g_j(x_0) = 0, \forall j \in J(x_0). \tag{3.52}$$

Since (f, g_J) is weak pseudoquasi (G, C, ρ)-type I at x_0, from (3.51)and(3.52), it follows that

$$C_{(x,x_0)}(\bigtriangledown(G_{f_i}(f_i(x_0)), \rho_i^1) \leq 0, \forall i \in P, \tag{3.53}$$

with strict inequality for at least one $i \in P$,

$$C_{(x,x_0)}(\bigtriangledown(G_{g_j}(g_j(x_0)), {\rho_j}^2) \leq 0 \forall j \in J(x_0). \tag{3.54}$$

Now, the proof is similar to that of Theorem 3.1.

Theorem 3.6. *Let x_0 be a feasible solution for (G-MOP) and let there exist vector $u \in R^p$ and vector $v \in R^q$ such that the triple (x,u,v) satisfies system (3.19-3.22) of the Theorem 3.2. If (f,g_J) is pseudoquasi (G,C,ρ)-type I at x_0 such that*

$$\sum_{i=1}^{p} u_i \rho_i^1 + \sum_{j \in J(x_0)} v_j \rho_j^2 \geq 0, \tag{3.55}$$

$$C_{(x,x_0)}(0,r) < 0 \Rightarrow r < 0, \forall x \in X, \tag{3.56}$$

then, x_0 is a weak efficient solution for (G-MOP).

Proof: Suppose that x_0 is not a weak efficient solution for (G-MOP). Then, there exists $x \in A$, such that

$$G_{f_i} f_i(x) < G_{f_i} f_i(x_0), \forall i \in P, \tag{3.57}$$

with strict inequality for at least one $i \in P$. Also,

$$g_j(x_0) = 0, \forall j \in J(x_0). \tag{3.58}$$

Since (f,g_J) is pseudoquasi (G,C,ρ)-type I at x_0, from (3.57) and (3.58), it follows that

$$C_{(x,x_0)}(\bigtriangledown(G_{f_i}(f_i(x_0)), \rho_i^1) < 0, \forall i \in P, \tag{3.59}$$

$$C_{(x,x_0)}(\bigtriangledown(G_{g_j}(g_j(x_0)), \rho_j{}^2) \leq 0, \forall j \in J(x_0). \tag{3.60}$$

Now, the proof is similar to that of Theorem 3.2.

Remark 3.4. *The importance of the Theorems (3.4) and (3.5) lies in the fact that a similar result does not necessarily hold for efficient solutions.*

4. CONCLUSION AND FUTURE DIRECTIONS

In this paper, we have introduced a new class of (G,C,ρ)-type I function and their generalizations. For a class of differentiable multiobjective programming problems, we have established sufficient optimality conditions. The results of the paper may be utilized to formulate Mond-Weir and Wolfe type dual problems and establish duality theorems.

Acknowledgement: The authors wish to express their deep gratitude to Dr. S. K. Mishra for his motivation and guidance throughout the preparation of the paper. The First author is supported by the Council of Scientific and Industrial Research, New Delhi, India, through grant no. 09/013(0474)/2012-EMR-1.

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