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AN ESTIMATION FROM WITHIN OF THE REACHABLE SET OF NONLINEAR R. BROCKETT INTEGRATOR WITH SMALL NONLINEARITY

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Abstract: In this paper, the nonlinear R. Brockett integrator with small nonlinear addition to the right-hand side of the corresponding differential equations is considered. More precisely, investigating the possibility to estimate from within the corresponding reachable set, we have obtained an efficient form of the ellipsoidal estimation from within. We used our previous results on the similar theme.

Keywords: Control problem, reachable set, R. Brockett integrator, estimation from within.

MSC: 49J10.

1. INTRODUCTION

The important object of interest in Optimal Control Theory is the reachable set. If linear case is in study of reachable sets, convex analysis is successfully applied, and in nonlinear case, nonlinear analysis is successfully applied.

In this paper, we obtain a non-trivial estimation from within for the reachable set of the nonlinear R. Brockett integrator with small nonlinearity. Such estimations are the object of interest for the Theory of Optimal Control and its Applications.

2. PROBLEM FORMULATION

We consider the nonlinear R. Brockett integrator with small nonlinearity of the form

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_2 x_1 - u_1 x_2 + \varepsilon g(x,u) \end{aligned} \tag{1}$$

with zero initial condition

$$x(0) = 0. \tag{2}$$

Where $x \in R^3, \varepsilon > 0, g(x,u)$ is a continuous function on $R^3 \times U$. We impose on the control vector u the constrain

$$u \in U = \left\{ u \in R^2 : \left| u_i \right| \le 1, \quad i = 1,2 \right\}. \tag{3}$$

We assume that:

1. For $x \in R^3, u \in U$ is realized the inequality

$$\left| g(x,u) \right| \le c(1 + \left| x \right|), \tag{4}$$

where $c > 0$ is a constant, $\left| x \right|$ means the norm of the vector x.

2. The function $g(x,u)$ satisfies the local Lipchitz condition on the variables (x,u) on $R^3 \times U$ and

$$g(x,u) \not\equiv 0. \tag{5}$$

Note that if $\varepsilon = 0$, the system (1) describes the dynamics of R. Brockett integrator (see [1]-[3]).

Further, we fix the constant $T > 0$. We consider all possible measurable controls $u(t) \in U, t \in \Delta$, where $\Delta = \left[0, T \right]$, and the corresponding absolutely continuous solutions $x(t, u(\cdot), \varepsilon)$ of the system (1) with the initial condition (2). The reachable set we are interested in is

$$D(T, \varepsilon) = \bigcup_{u(\cdot)} x(T, u(\cdot), \varepsilon), \tag{6}$$

where $u(\cdot)$ is an arbitrary measurable control, satisfying (3) for $t \in \Delta$. From the general theory of nonlinear controlled objects (see, for example, [4, p.264]), it follows that $\overline{D(T, \varepsilon)}$ is a compact set in R^3, where the line means the closing of the set. Our objective is to construct for such a convex compact set $K \subset R^3$, so that for sufficiently small $\varepsilon > 0$

$$K \subset D(T, \varepsilon) \text{ and } 0 \in \operatorname{Int} K, \tag{7}$$

where $\operatorname{Int} K$ means the interior of the set *K*. We denote by $L_\infty(\Delta)$ (see [5, p.31]) the Banach space of 2-dimensional measurable functions $u(t)$ on the interval $\Delta = [0, T]$, substantially bounded in modulus, i.e. $\underset{t \in \Delta}{vrai \sup} |u(t)| < +\infty$, and with the usual identification of functions, which coincide almost everywhere on Δ. We define the norm of the element $u(\cdot) \in L_\infty(\Delta)$ by the formula

$$\|u(\cdot)\| = \underset{t \in \Delta}{vrai \sup} |u(t)|.$$

For any measurable functions $u(t) \in U, t \in \Delta$ (see (3)), the corresponding solution $x(t, u(\cdot), \varepsilon)$ is uniquely defined with the initial condition (2). Note that it is possible to consider the admissible controls as the elements of $L_\infty(\Delta)$.

We will estimate the reachable set $D(T, \varepsilon)$ of the system (1), (2) from within.

Note that when $\varepsilon = 0$, the corresponding estimate is the one obtained in the work [3] of M. S. Nikolskii.

3. SOLUTION OF THE PROBLEM

We denote by $y(T, u(\cdot))$ the solution of R. Brockett system of equations (see [1] - [3]).

$$\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= u_2 y_1 - u_1 y_2 \end{aligned} \tag{8}$$

where $y(0) = 0$ and the admissible $u(t) \in U, t \in \Delta$.

Then, it is easy to show that the solution $x(T, u(\cdot), \varepsilon)$ of the system (1), (2) allows the following representation for admissible $u(t) \in U, t \in \Delta$:

$$x(T, u(\cdot), \varepsilon) = y(T, u(\cdot)) + \varepsilon R(u(\cdot), \varepsilon), \tag{9}$$

where

$$y(T, u(\cdot)) = Au(\cdot) + \frac{1}{2} C[u(\cdot), u(\cdot)], \tag{10}$$

and A is a linear vector-bounded operator, acting from $L_\infty(\Delta)$ to R^3 by the following formulas for its component

$$A_i u(\cdot) = \int_0^T u_i(s) ds, \quad i = 1, 2; \quad A_3 u(\cdot) = 0; \tag{11}$$

$C[u(\cdot), u(\cdot)]$ is a continuous vector quadratic form (see [5], [6]) with components

$$C_i\left[u(\cdot),\, u(\cdot)\right] = 0, \quad i = 1, 2, \tag{12}$$

$$C_3\left[u(\cdot),\, u(\cdot)\right] = 2\int_0^T (u_2(s)y_1(s,\, u(\cdot)) - u_1(s)y_2(s,\, u(\cdot)))\, ds, \tag{13}$$

here (see (8))

$$y_i(t, u(\cdot)) = \int_0^t u_i(s)ds, \quad i = 1, 2, \tag{14}$$

$R(u(\cdot),\, \varepsilon)$ is a nonlinear vector function, an also (see (1), (5))

$$R(u(\cdot), \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ \int_0^T g(x(s, u(\cdot), \varepsilon), u(s))ds \end{pmatrix}. \tag{15}$$

Next, we will show that the function $\left|R(u(\cdot), \varepsilon)\right|$ is bounded for $\varepsilon \in \left[0,1\right]$ and any admissible $u(t) \in U$, $t \in \Delta$. To do that, we will first estimate the solution $x(t, u(\cdot), \varepsilon)$ of (1), (2) for $\varepsilon \in \left[0,1\right]$ and any admissible $u(t) \in U, t \in \Delta$.

We denote by $f(x, u, \varepsilon)$ the right-hand side of the differential system of equations (1), then using (3), (4), it is easy to prove that for $x \in R^3$, $u \in U$, $\varepsilon \in \left[0,1\right]$.

$$\left|f(x, u, \varepsilon)\right| \le (2 + c)(1 + |x|).$$

Therefore for all admissible controls $u(\cdot)$,

$$\left|x(t, u(\cdot), \varepsilon)\right| \le (2 + c)t + (2 + c)\int_0^t \left|x(s, u(\cdot), \varepsilon)\right| ds.$$

Hence, using the Gronwall-Bellman inequality (see [7]), we obtain for $t \in \left[0, T\right]$.

$$\left|x(t, u(\cdot), \varepsilon)\right| \le (2 + c)Te^{(2+c)t}. \tag{16}$$

Using (4), (15), (16), we obtain for $\varepsilon \in \left[0,1\right]$ and $u(\cdot)$ from the set of measurable functions (see (3))

$$u(t) \in U, \quad t \in \left[0, T\right],$$

$$\left|R(u(\cdot), \varepsilon)\right| \le cT(1 + (2 + c)\int_0^T e^{(2+c)s} ds),$$

i.e.

$$\left|R(u(\cdot),\varepsilon)\right| \le k_1\,, \tag{17}$$

where

$$k_1 = cTe^{(2+c)T}.$$

Using the local Lipchitz condition in (x,u) of the function $g(x,u)$ $R^3 \times U$, the inequality (17), we can prove the continuity in $(u(\cdot),\varepsilon)$ of the nonlinear mapping $R(u(\cdot),\varepsilon)$ on the set of all admissible controls $u(\cdot),\varepsilon \in [0,1]$ in the norm of $L_\infty(\Delta)\times[0,1]$. From formula (11), we see that $AL_\infty \neq R^3$, i.e. the linear operator does not realize a covering of the space R^3.
We consider the admissible control $\tilde{u}(t) \in U, t \in \Delta$, with the components

$$\tilde{u}_i(t) = \alpha(t), \quad i = 1,2, \tag{18}$$

where

$$\alpha(t) = \frac{1}{2} \quad for \quad t \in \left[0,\frac{T}{2}\right], \quad \alpha(t) = -\frac{1}{2} \quad for \quad t \in \left(\frac{T}{2},T\right]. \tag{19}$$

It is easy to see that (see (11)-(14)) for $\tilde{u}(t), t \in \Delta$, and its corresponding solution $\tilde{x}(t)$ of system (1), (2) the equalities

$$A\tilde{u}(\cdot) = 0, \quad C\left[\tilde{u}(\cdot),\tilde{u}(\cdot)\right] = 0, \quad \tilde{x}(T) = \varepsilon R(\tilde{u}(\cdot),\varepsilon) \tag{20}$$

are realized. We consider for $t \in \Delta$ and $\mu \in (0,1]$ the admissible control

$$u_\mu(t) = \mu\tilde{u}(t) + \mu^{3/2}v(t) \tag{21}$$

where the measurable function $v(t) \in R^2$ satisfies on Δ the inequality

$$\left|v(t)\right| \le \frac{1}{2}\,. \tag{22}$$

From (9), (10), (20) and (21), we obtain the following formula for the solution $x_\mu(t)$ of the system (1), (2), that corresponds to the control $u_\mu(\cdot)$:

$$x_\mu(T,u_\mu(\cdot),\varepsilon) = \mu^{3/2}Av(\cdot) + \mu^{5/2}C\left[\tilde{u}(\cdot),v(\cdot)\right] + \frac{\mu^3}{2}C\left[v(\cdot),v(\cdot)\right] + \varepsilon R(u_\mu(\cdot),\varepsilon), \tag{23}$$

where $C[\xi,\eta]$ is a continuous symmetric bilinear form (here ξ,η are arbitrary elements of $L_\infty(\Delta)$). Note that (see (12)-(14), (18)-(20))

$$C_i\left[\xi,\eta\right] = 0, \quad i = 1,2 \tag{24}$$

For arbitrary ξ,η from $L_\infty(\Delta)$ and

$$C_3\left[\tilde{u}(\cdot),v(\cdot)\right]=\int_0^T\left[\alpha(s)\int_0^s(v_1(r)-v_2(r))dr-\int_0^s\alpha(r)dr(v_1(s)-v_2(s))\right]ds\,, \tag{25}$$

where $v_i(t), i=1,2$ are the components of the vector function $v(t)$. Further (see(19)), $\int_0^T\alpha(\tau)d\tau=0$, therefore

$$\int_r^T\alpha(\tau)d\tau=-\int_0^r\alpha(\tau)d\tau \tag{26}$$

for $r\in\Delta$.
Changing the order of integration in the first iterated integral (25) and using the formula (26), we can rewrite (25) in the following form:

$$C_3\left[\tilde{u}(\cdot),v(\cdot)\right]=-2\int_0^T\left[\int_0^s\alpha(r)dr(v_1(s)-v_2(s))\right]ds. \tag{27}$$

We consider now the linear operator N, acting from $L_\infty(\Delta)$ to R^3 by the formula

$$Nv(\cdot)=Av(\cdot)+C\left[\tilde{u}(\cdot),v(\cdot)\right]. \tag{28}$$

Note that the linear operator N is closely connected with the formula (23). From the formulas (11), (24), (25), it follows that

$$N_iv(\cdot)=\int_0^T v_i(s)ds,\quad i=1,2, \tag{29}$$

$$N_3v(\cdot)=C_3\left[\tilde{u}(\cdot),v(\cdot)\right], \tag{30}$$

where $N_iv(\cdot)$ is a i-th component of the vector $Nv(\cdot)$.
We denote by W the set of functions $v(\cdot)\in L_\infty(\Delta)$, which are almost everywhere refered to the zero vector for $t\in\left[\frac{T}{2},T\right]$. On the set W (see (12) (18), (19), (27)-(30))

$$N_iv(\cdot)=\int_0^{T/2} v_i(s)ds,\quad i=1,2, \tag{31}$$

$$N_3v(\cdot)=-\int_0^{T/2} s(v_1(s)-v_2(s))\,ds. \tag{32}$$

From formulas (31), (32), it follows that on W

$$Nv(\cdot) = \int_0^{T/2} B(s)v(s)\,ds, \tag{33}$$

where

$$B(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -s & s \end{pmatrix}. \tag{34}$$

We fix an arbitrary vector $\varphi \in R^3$ and consider for $s \in [0, T/2]$ the function

$$v_\varphi(s) = B^*(s)\varphi, \tag{35}$$

where * denotes the transpose of a matrix. Assuming $v_\varphi(s) = 0$ for $s \in \left(\frac{T}{2}, T\right]$, from (33), (35), we obtain that

$$Nv_\varphi(\cdot) = G\varphi, \tag{36}$$

where

$$G = \int_0^{T/2} B(s)B^*(s)\,ds, \tag{37}$$

φ - an arbitrary vector from R^3. From the formulas (33), (36), we obtain that

$$G = \begin{pmatrix} \beta & 0 & -\beta^2/2 \\ 0 & \beta & \beta^2/2 \\ -\beta^2/2 & \beta^2/2 & \frac{2}{3}\beta^3 \end{pmatrix},$$

where $\beta = \frac{T}{2}$. Hence, the matrix G is nondegenerate and

$$NW = R^3,\ NL_\infty(\Delta) = R^3. \tag{38}$$

From formulas (27)-(30), it follows that on the functions $v(\cdot) \in L_\infty(\Delta)$ the linear operator N is described by the following formula (cf. (33), (34)):

$$Nv(\cdot) = \int_0^T B(s)v(s)\,ds, \tag{39}$$

where $s \in \Delta$

$$B(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2\int_0^s \alpha(r)dr & 2\int_0^s \alpha(r)dr \end{pmatrix}.$$

We consider now for $s \in \Delta$ the multivalued mapping

$$\Omega(s) = B(s)\sigma , \tag{40}$$

where $\sigma = \left\{ v \in R^2 : |v| \le \frac{1}{2} \right\}$, and its integral (see [5],[8]) and formula (39), we have

$$P = \int_0^T \Omega(s)\, ds . \tag{41}$$

On the basis of the general theory of multivalued maps (see [5],[8]) and formula (39), we have

$$P = NV , \tag{42}$$

where V is the set of measurable functions on Δ satisfying the inequality (22). Next, using the theory of multivalued maps, it is proved that P is the convex compact set containing the point *0.* With the help of (38), it is easy to show that

$$0 \in Int\ P \tag{43}$$

Using the formulas (11), (12), (24), (28)-(30), we rewrite the formula (23) in the form

$$x_\mu(T, u_\mu(\cdot), \varepsilon) = \Lambda(\mu)\left(Nv(\cdot) + \frac{\mu^{1/2}}{2} C\left[v(\cdot), v(\cdot)\right] + \varepsilon\Lambda^{-1}(\mu) R(u_\mu(\cdot), \varepsilon) \right), \tag{44}$$

where

$$\Lambda(\mu) = \begin{pmatrix} \mu^{3/2} & 0 & 0 \\ 0 & \mu^{3/2} & 0 \\ 0 & 0 & \mu^{5/2} \end{pmatrix}. \tag{45}$$

Recall that in (44), $v(t)$ is an arbitrary measurable function on Δ satisfying the inequality (22).

For the vector quadratic form $C\left[v(\cdot), v(\cdot)\right]$ (see (12)-(14)), it is easy to prove the uniform estimation for all admissible $v(\cdot)$.

$$\left| C\left[v(\cdot), v(\cdot)\right] \right| \le \frac{1}{2}\int_0^T s\, ds = \frac{T^2}{4} \tag{46}$$

Let us denote

$$\eta = \max_{s\in[0,T/2]} \left\| B^*(s) \right\|,$$

where

$$\left\| B^*(s) \right\| = \max_{|y|\le 1} \left| B^*(s) y \right|,\ y \in R^3 .$$

It is clear that $\eta > 0$ and it is easy to see that the continuous function $v_\varphi(s)$ (see(35)) for $s \in [0, T/2]$ and $|\varphi| \le \frac{1}{2\eta}$ satisfy the condition

$$\left| v_\varphi(s) \right| \le \frac{1}{2} \tag{47}$$

Assuming $v_\varphi(s) = 0$ for $s \in \left(\frac{T}{2}, T \right]$, with the help of formulas (35)-(37), (40),(41),(47) and the definition of the integral of multivalued mapping (see, for example [5],[8]), it is easy to justify the inclusion

$$\frac{1}{2\eta} GS \subset P ,$$

where

$$S = \left\{ y \in \mathrm{R}^3 : |y| \le 1 \right\} .$$

Below, we will show that the set GS with an appropriate matrix multiplier determines an estimation from within of $D(T,\varepsilon)$ for sufficiently small $\varepsilon \ge 0$. We introduce (see (44), (45)) the map

$$F(v(\cdot),\mu,\varepsilon) = Nv(\cdot) + \frac{\mu^{1/2}}{2} C\left[v(\cdot), v(\cdot) \right] + \varepsilon R_1(v(\cdot),\mu,\varepsilon), \tag{48}$$

where $v(\cdot)$ is an arbitrary measurable function satisfying the inequality (22),

$$\mu \in (0,1], \quad \varepsilon \in [0,1] \ \text{ and } \ R_1(v(\cdot),\mu,\varepsilon) = \Lambda^{-1}(\mu) R(u_\mu(\cdot),\varepsilon). \tag{49}$$

We denote (see (35)) $F_1(\varphi,\mu,\varepsilon) = F(v_\varphi(\cdot),\mu,\varepsilon)$, where

$$\varphi \in \frac{1}{2\eta} S, \quad S = \left\{ y \in R^3 : |y| \le 1 \right\}, \quad \mu \in (0,1] \ \text{and} \ \varepsilon \in [0,1] .$$

Now, we obtain the representation

$$F_1(\varphi,\mu,\varepsilon) = G\left(\varphi + \frac{\mu^{1/2}}{2} G^{-1} C\left[v_\varphi(\cdot), v_\varphi(\cdot)\right] + \varepsilon G^{-1} R_1(v_\varphi(\cdot),\mu,\varepsilon)\right), \qquad (50)$$

where $\varphi \in \frac{1}{2\eta} S$, $\mu \in (0,1]$ and $\varepsilon \in [0,1]$. We denote

$$F_2(\varphi,\mu,\varepsilon) = \varphi + \frac{\mu^{1/2}}{2} G^{-1} C\left[v_\varphi(\cdot)\right], v_\varphi(\cdot)\right] + \varepsilon G^{-1} R_1(v_\varphi(\cdot),\mu,\varepsilon), \qquad (51)$$

where $\varphi \in \frac{1}{2\eta} S$, $\mu \in (0,1]$ and $\varepsilon \in [0,1]$. Fix $\mu_0 \in (0,1]$ such that (see (46), (50), (51))

$$\frac{\mu_0^{1/2}}{2}\left\|G^{-1}\right\|\frac{T^2}{4} \le \frac{1}{6\eta}.$$

Fix $\varepsilon_0 \in (0,1]$, such that (see (17),(45),(48)-(51))

$$\varepsilon_0 \left\|G^{-1}\right\| \mu_0^{-5/2} k_1 \le \frac{1}{6\eta}$$

Then (see the Corollary of the Topological Theorem in [4], p. 276, 277), for $\varepsilon \in [0,\varepsilon_0]$ we obtain the inclusion

$$F_2(\frac{1}{2\eta} S, \mu_0, \varepsilon_0) \supset \frac{1}{6\eta} S.$$

From the above, it follows that, (see (44), (50), (51)) for $\varepsilon \in [0,\varepsilon_0]$ the required inclusion

$$\frac{1}{6\eta}\Lambda(\mu_0) GS \subset D(T,\varepsilon)\ .$$

is realized.

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