

MIXED TYPE SYMMETRIC AND SELF DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS WITH SUPPORT FUNCTIONS

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Abstract: In this paper, a pair of mixed type symmetric dual multiobjective variational problems containing support functions is formulated. This mixed formulation unifies two existing pairs Wolfe and Mond-Weir type symmetric dual multiobjective variational problems containing support functions. For this pair of mixed type nondifferentiable multiobjective variational problems, various duality theorems are established under convexity-concavity and pseudoconvexity-pseudoconcavity of certain combination of functionals appearing in the formulation. A self duality theorem under additional assumptions on the kernel functions that occur in the problems is validated. A pair of mixed type nondifferentiable multiobjective variational problem with natural boundary values is also formulated to investigate various duality theorems. It is also pointed that our duality theorems can be viewed as dynamic generalizations of the corresponding (static) symmetric and self duality of multiobjective nonlinear programming with support functions.

Keywords: Efficiency, mixed type symmetric duality, mixed type self duality, natural boundary values, multiobjective nonlinear programming, convexity-concavity, pseudoconvexity-pseudoconcavity, support functions.

MSC: 90C30, 90C11, 90C20, 90C26.

1. INTRODUCTION

Following Dorn [6], symmetric duality results in mathematical programming have been derived by a number of authors, notably, Dantzig et al [7], Mond [13], Bazaraa and Goode [1]. In these researches, the authors have studied symmetric duality under the hypothesis of convexity-concavity of the kernel function involved. Mond and Cottle [14] presented self duality for the problems of [7] by assuming skew symmetric of the kernel function. Later Mond-Weir [16] formulated a different pair of symmetric dual nonlinear program with a view to generalize convexity-concavity of the kernel function to pseudoconvexity-pseudoconcavity.

Symmetric duality for variational problems was first introduced by Mond and Hanson [17] under the convexity-concavity conditions of scalar functions like $\psi(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ with $x(t) \in R^n$ and $y(t) \in R^m$. Bector, Chandra and Husain [3] presented a different pair of symmetric dual variational problems in order to relax the requirement of convexity-concavity to that of pseudoconvexity-pseudoconcavity while in [5] Chandra and Husain gave a fractional analogue.

Bector and Husain [4] were probably the first to study duality for multiobjective variational problems under appropriate convexity assumptions. Subsequently, Gulati, Husain and Ahmed [8] presented two distinct pairs of symmetric dual multiobjective variational problems and established various duality results under appropriate invexity requirements. In this reference, self duality theorem is also given under skew symmetric of the integrand of the objective functional. Husain and Jabeen [12] formulated a pair of mixed type symmetric dual variational problem in order to unify the Wolfe and Mond-Weir symmetric dual pairs of variational problems studied in [8].

The purpose of this research is to unify the formulations of the pairs of Wolfe and Mond-Weir type symmetric dual multiobjective variational problems involving support functions recently treated by Husain and Rumana [11] and study symmetric and self duality for these pairs of nondifferentiable variational problem under appropriate assumptions. Our duality results reported in this research extend the results of Husain and Rumana [11] to nondifferentiable setting by introducing support functions. The support functions which appear in the problems of facility location and related problems of decision theory are quite significant functions amongst well known nondifferentiable convex functions. The dual problems presented in this research are pretty hard to solve. So, expecting any immediate application of these problems would be premature. Unfortunately, there has not always been sufficient flow between the researchers in the multiple criteria decision making and the researchers applying it to their problems. Of course, one can find optimal control applications in varieties of contexts, which reflects the utility of our models. It is also indicated that our results can be viewed as dynamic generalizations of corresponding (static) symmetric duality results of multiobjective nonlinear programming with support functions.

2. NOTATIONS AND PRELIMINARIES

The following notation will be used for vectors in R^n .

$$\begin{aligned}
 x < y, & \quad \Leftrightarrow \quad x_i < y_i, \quad i = 1, 2, \dots, n. \\
 x \leq y, & \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \dots, n. \\
 x \leq y, & \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \dots, n, \text{ but } x \neq y \\
 x \not\leq y, & \text{ is the negation of } x \leq y
 \end{aligned}$$

Let $I = [a, b]$ be the real interval, and $\phi^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$, $i = 1, 2, \dots, p$ be a scalar function and twice differentiable function where $x : I \rightarrow R^n$ and $y : I \rightarrow R^n$ with derivatives \dot{x} and \dot{y} . In order to consider each $\phi^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ denote the first partial derivatives of ϕ^i with respect to $t, x(t), \dot{x}(t), y(t), \dot{y}(t)$ respectively, by $\phi_t^i, \phi_x^i, \phi_{\dot{x}}^i, \phi_y^i, \phi_{\dot{y}}^i$, that is,

$$\begin{aligned}
 \phi_t^i &= \frac{\partial \phi^i}{\partial t} \\
 \phi_x^i &= \left[\frac{\partial \phi^i}{\partial x_1}, \frac{\partial \phi^i}{\partial x_2}, \dots, \frac{\partial \phi^i}{\partial x_n} \right], & \phi_{\dot{x}}^i &= \left[\frac{\partial \phi^i}{\partial \dot{x}_1}, \frac{\partial \phi^i}{\partial \dot{x}_2}, \dots, \frac{\partial \phi^i}{\partial \dot{x}_n} \right] \\
 \phi_y^i &= \left[\frac{\partial \phi^i}{\partial y_1}, \frac{\partial \phi^i}{\partial y_2}, \dots, \frac{\partial \phi^i}{\partial y_n} \right], & \phi_{\dot{y}}^i &= \left[\frac{\partial \phi^i}{\partial \dot{y}_1}, \frac{\partial \phi^i}{\partial \dot{y}_2}, \dots, \frac{\partial \phi^i}{\partial \dot{y}_n} \right].
 \end{aligned}$$

The twice partial derivatives of ϕ^i , $i = 1, 2, \dots, p$ with respect to $t, x(t), \dot{x}(t), y(t)$ and $\dot{y}(t)$, respectively are the matrices

$$\begin{aligned}
 \phi_{xx}^i &= \left(\frac{\partial^2 \phi^i}{\partial x_k \partial x_s} \right)_{n \times n}, & \phi_{x\dot{x}}^i &= \left(\frac{\partial^2 \phi^i}{\partial x_k \partial \dot{x}_s} \right)_{n \times n}, & \phi_{xy}^i &= \left(\frac{\partial^2 \phi^i}{\partial x_k \partial y_s} \right)_{n \times n} \\
 \phi_{x\dot{y}}^i &= \left(\frac{\partial^2 \phi^i}{\partial x_k \partial \dot{y}_s} \right)_{n \times n}, & \phi_{\dot{x}y}^i &= \left(\frac{\partial^2 \phi^i}{\partial \dot{x}_k \partial y_s} \right)_{n \times n}, & \phi_{\dot{x}\dot{y}}^i &= \left(\frac{\partial^2 \phi^i}{\partial \dot{x}_k \partial \dot{y}_s} \right)_{n \times n} \\
 \phi_{yy}^i &= \left(\frac{\partial^2 \phi^i}{\partial y_k \partial y_s} \right)_{n \times n}, & \phi_{y\dot{y}}^i &= \left(\frac{\partial^2 \phi^i}{\partial y_k \partial \dot{y}_s} \right)_{n \times n}, & \phi_{\dot{y}\dot{y}}^i &= \left(\frac{\partial^2 \phi^i}{\partial \dot{y}_k \partial \dot{y}_s} \right)_{n \times n}
 \end{aligned}$$

Noting that

$$\frac{d}{dt} \phi_y^i = \phi_{yt}^i + \phi_{yy}^i \dot{y} + \phi_{y\dot{y}}^i \ddot{y} + \phi_{yx}^i \dot{x} + \phi_{y\dot{x}}^i \ddot{x}$$

and hence

$$\frac{\partial}{\partial y} \frac{d}{dt} \phi_y = \frac{d}{dt} \phi_{yy}, \quad \frac{\partial}{\partial \dot{y}} \frac{d}{dt} \phi_y = \frac{d}{dt} \phi_{y\dot{y}} + \phi_{yy}, \quad \frac{d}{d\ddot{y}} \frac{d}{dt} \phi_y^i = \phi_{y\ddot{y}}^i$$

$$\frac{\partial}{\partial x} \frac{d}{dt} \phi_y = \frac{d}{dt} \phi_{yx}, \quad \frac{\partial}{\partial \dot{x}} \frac{d}{dt} \phi_y = \frac{d}{dt} \phi_{y\dot{x}} + \phi_{yx}, \quad \frac{\partial}{\partial \ddot{x}} \frac{d}{dt} \phi_y = \phi_{y\ddot{x}}$$

In order to establish our main results, the following concepts are needed.

Definition 1. (Support function): Let K be a compact set in R^n , then the support function of K is defined by

$$s(x(t)|K) = \max \{x(t)^T v(t) : v(t) \in K, t \in I\}$$

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis i.e., there exists $z(t) \in R^n$, $t \in I$, such that

$$s(y(t)|C) - s(x(t)|C) \geq (y(t) - x(t))^T z(t)$$

From [15], subdifferential of $s(x(t)|K)$ is given by

$$\partial s(x(t)|K) = \{z(t) \in K, t \in I \mid x(t)^T z(t) = s(x(t)|K)\}.$$

For any set $\Gamma \subset R^n$, the normal cone to Γ at a point $x(t) \in \Gamma$ is defined by

$$N_\Gamma(x(t)) = \{y(t) \in R^n \mid y(t)(z(t) - x(t)) \leq 0, \forall z(t) \in \Gamma\}$$

It can be verified that for a compact convex set K , $y(t) \in N_K(x(t))$ if and only if

$$s(y(t)|K) = x(t)^T y(t), \quad t \in I$$

Definition 2. (Skew Symmetric function): The function $h : I \times R^n \times R^n \times R^n \times R^n \rightarrow R$ is said to be skew symmetric if for all x and y in the domain of h if

$$h(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) = -h(t, y(t), \dot{y}(t), x(t), \dot{x}(t)), \quad t \in I$$

where x and y (piecewise smooth are on I) are of the same dimension.

Now consider the following multiobjective variational problem (VPO):

$$(\mathbf{VP}_0) \quad \text{Minimize } \int_I F(t, x, \dot{x}) dt$$

Subject to

$$x(a) = \alpha, \quad x(b) = \beta$$

$$g(t, x, \dot{x}) \leq 0, \quad t \in I,$$

where $F : I \times R^n \times R^n \rightarrow R^p$ and $g : I \times R^n \times R^n \rightarrow R^m$.

Definition 3. (Efficient Solution): A feasible solution \bar{x} is efficient for (VPo) if there exists no other feasible x for (VP) such that for some $i \in P = \{1, 2, \dots, p\}$,

$$\int_I F^i(t, x, \dot{x}) dt < \int_I F^i(t, \bar{x}, \dot{\bar{x}}) dt \quad \text{for all } i \in P.$$

and

$$\int_I F^j(t, x, \dot{x}) dt \leq \int_I F^j(t, \bar{x}, \dot{\bar{x}}) dt \quad \text{for all } j \in P, \quad j \neq i.$$

3. STATEMENT OF THE PROBLEMS

For $N = \{1, 2, \dots, n\}$ and $M = \{1, 2, \dots, m\}$, let $J_1 \subset N, L_1 \subset M, J_2 = N \setminus J_1$ and $L_2 = M \setminus L_1$. Let $|J_1|$ denote the number of elements in the subset J_1 . The other symbols $|J_2|, |L_1|$ and $|L_2|$ are similarly defined. Let $x^1 : I \rightarrow R^{|J_1|}$ and $x^2 : I \rightarrow R^{|J_2|}$, then any $x : I \rightarrow R^n$ can be written as $x = (x^1, x^2)$. Similarly for $y^1 : I \rightarrow R^{|L_1|}$ and $y^2 : I \rightarrow R^{|L_2|}$ can be written as $y = (y^1, y^2)$. Let $f : I \times R^{|J_1|} \times R^{|J_2|} \rightarrow R^p$ and $g : I \times R^{|J_1|} \times R^{|J_2|} \rightarrow R^m$ be twice continuously differentiable functions.

We state the following pair of mixed type multiobjective symmetric dual variational problems with support functions involving vector functions f and g .

(Mix SP): Minimize: $\int_I \left(H^1(t, x^1, x^2, y^1, y^2, \dot{x}^1, \dot{x}^2, \dot{y}^1, \dot{y}^2, z^1, z^2, \lambda), \dots, \right.$

$$\left. H^p(t, x^1, x^2, y^1, y^2, \dot{x}^1, \dot{x}^2, \dot{y}^1, \dot{y}^2, z^1, z^2, \lambda) \right) dt$$

Subject to:

$$x^1(a) = 0 = x^1(b), \quad y^1(a) = 0 = y^1(b), \tag{1}$$

$$x^2(a) = 0 = x^2(b), \quad y^2(a) = 0 = y^2(b). \tag{2}$$

$$\sum_{i=1}^p \lambda^i \left[f_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right] \leq 0, \quad t \in I, \tag{3}$$

$$\sum_{i=1}^p \lambda^i \left[g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t) - Dg_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right] \leq 0, \quad t \in I, \quad (4)$$

$$\int_I y^2(t)^T \left[\sum_{i=1}^p \lambda^i \left(g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t) - Dg_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right) \right] \geq 0, \quad (5)$$

$$(x^1(t), x^2(t)) \geq 0, \quad t \in I, \quad (6)$$

$$z_i^1(t) \in K_i^1 \text{ and } z_i^2(t) \in K_i^2, \quad (7)$$

$$\lambda > 0, \quad \lambda^T e = 1, \quad e^T = (1, \dots, 1). \quad (8)$$

where

$$\begin{aligned} H^i &= f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) + s(x^1(t)|C_i^1) + s(x^2(t)|C_i^2) \\ &\quad - y^1(t) \sum_{i=1}^p \lambda^i \left[f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right] \\ &\quad - z_i^1(t) y^1(t) - z_i^2(t) y^2(t) \end{aligned}$$

$$\text{(Mix SD): Maximize } \int_I \left(G^1(t, u^1, u^2, v^1, v^2, \dot{u}^1, \dot{u}^2, \dot{v}^1, \dot{v}^2, w^1, w^2, \lambda), \dots, \right.$$

$$\left. G^p(t, u^1, u^2, v^1, v^2, \dot{u}^1, \dot{u}^2, \dot{v}^1, \dot{v}^2, w^1, w^2, \lambda) \right) dt$$

Subject to:

$$u^1(a) = 0 = u^1(b), \quad v^1(a) = 0 = v^1(b), \quad (9)$$

$$u^2(a) = 0 = u^2(b), \quad v^2(a) = 0 = v^2(b). \quad (10)$$

$$\sum_{i=1}^p \lambda^i \left[f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right] \geq 0, \quad t \in I, \quad (11)$$

$$\sum_{i=1}^p \lambda^i \left[g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) + \omega_i^2(t) - Dg_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] \geq 0, \quad t \in I, \quad (12)$$

$$\int_I u^2(t)^T \left[g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) + \omega_i^2(t) - Dg_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] \leq 0, \quad t \in I, \quad (13)$$

$$(v^1(t), v^2(t)) \geq 0, \quad t \in I, \quad (14)$$

$$\omega_i^1(t) \in C_i^1 \text{ and } \omega_i^2(t) \in C_i^2, \quad (15)$$

$$\lambda > 0, \quad \lambda^T e = 1, \quad e^T = (1, \dots, 1), \quad (16)$$

where,

$$G^i = f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) + s(v^1(t)|K_i^1) + s(v^2(t)|K_i^2) + u^1(t)\omega_i^1(t) + u_i^2(t)\omega^2(t) - u^1(t) \sum_{i=1}^p \lambda^i [f_{u^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + \omega_i^1(t) - Df_{u^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1)]$$

4. MIXED TYPE MULTIOBJECTIVE SYMMETRIC DUALITY

In this section, we present various duality results for a pair of mixed type multiobjective symmetric problems, (Mix SP) and (Mix SD) under pseudo-concavity-pseudo-concavity assumptions.

Theorem 1. (Weak Duality): Let $(x^1(t), x^2(t), y^1(t), y^2(t), z^1(t), z^2(t), \lambda)$ be feasible for (Mix SP) and $(u^1(t), u^2(t), v^1(t), v^2(t), \omega(t)^1, \omega^2(t), \lambda)$ be feasible for (Mix SD).

Assume that

(H₁): for each $i \int_I \{f^i(t, \dots, y^1(t), \dot{y}^1(t))\} dt$ be convex in x^1, \dot{x}^1 for fixed y^1, \dot{y}^1 and $\int_I \{f^i(t, x^1(t), \dot{x}^1(t), \dots)\} dt$ be concave in y^1, \dot{y}^1 on I for fixed x^1, \dot{x}^1 .

(H₂): $\sum_{i=1}^p \lambda_i \int_I (g_{u^2}^i(t, \dots, y^2(t), \dot{y}^2(t)) + (\)^T \omega_i^2(t)) dt$ pseudo-convex in x^2, \dot{x}^2 for fixed y^2, \dot{y}^2

and

$\sum_{i=1}^p \lambda_i \int_I (g_{y^2}^i(t, x^2(t), \dot{x}^2(t), \dots) - (\)^T z_i^2(t)) dt$ pseudo-concave in y^2, \dot{y}^2 for fixed x^2, \dot{x}^2 .

Then,

$$\int_I H dt \not\leq \int_I G dt .$$

where $H = (H^1, H^2, \dots, H^i, \dots, H^p)$ and $G = (G^1, G^2, \dots, G^i, \dots, G^p)$.

Proof: Using the convexity of each $\int_I f^i(t, \dots, y, \dot{y}) dt$ in (x, \dot{x}) for fixed (y, \dot{y}) , we have

$$\begin{aligned}
& \int_I f^i(t, x^1, \dot{x}^1, v^1, \dot{v}^1) dt - \int_I f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) dt \\
& \geq \int_I \left[(x^1(t) - u^1(t))^T f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + (\dot{x}^1(t) - \dot{u}^1(t))^T f_{\dot{u}^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right] dt \\
& = \int_I \left[(x^1(t) - u^1(t))^T \left\{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right\} \right] dt \\
& \quad + (x^1(t) - u^1(t))^T f_{\dot{u}^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \Big|_{t=b}^{t=a}
\end{aligned}$$

Using (1) and (9), this yields

$$\begin{aligned}
& \int_I f^i(t, x^1, \dot{x}^1, v^1, \dot{v}^1) dt - \int_I f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) dt \\
& \geq \int_I \left[(x^1(t) - u^1(t))^T \left\{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right\} \right] dt
\end{aligned} \tag{17}$$

Also by concavity of $\int_I \left\{ f^i(t, x^1(t), \dot{x}^1(t), \dots) \right\} dt$, we have

$$\begin{aligned}
& - \int_I f^i(t, x^1, \dot{x}^1, v^1, \dot{v}^1) dt - \int_I f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) dt \\
& \geq - \int_I \left[(v^1(t) - y^1(t))^T f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + (\dot{v}^1(t) - \dot{y}^1(t))^T f_{\dot{y}^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right] dt \\
& = - \int_I \left[(v^1(t) - y^1(t))^T \left\{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right\} \right] dt \\
& \quad + (v^1(t) - y^1(t))^T f_{\dot{y}^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \Big|_{t=b}^{t=a},
\end{aligned}$$

which by using (2) and (10), gives

$$\begin{aligned}
& - \int_I f^i(t, x^1, \dot{x}^1, v^1, \dot{v}^1) dt - \int_I f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) dt \\
& \geq - \int_I \left[(v^1(t) - y^1(t))^T \left\{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right\} \right] dt
\end{aligned} \tag{18}$$

The addition of (17) and (18) implies

$$\begin{aligned}
& \int_I f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) dt - \int_I f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) dt \\
& \geq \int_I \left[(x^1(t) - u^1(t))^T \left\{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right\} \right. \\
& \quad \left. - (v^1(t) - y^1(t))^T \left\{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right\} \right] dt
\end{aligned}$$

$$\begin{aligned}
 &= \int_I \left[(x^1)^T \{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \} \right. \\
 &\quad - (u^1(t))^T \{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \} \\
 &\quad - (v^1(t))^T \{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \} \\
 &\quad \left. + (y^1(t))^T \{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \} \right] dt
 \end{aligned}$$

Multiplying this by λ^i and summing over i , $i = 1, 2, \dots, p$, we get

$$\begin{aligned}
 &\sum_{i=1}^p \lambda_i \int_I f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) dt - \sum_{i=1}^p \lambda_i \int_I f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) dt \\
 &\geq \int_I \left[(x^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \} \right. \\
 &\quad - (u^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \} \\
 &\quad - (v^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \} \\
 &\quad \left. + (y^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \} \right] dt \\
 &= \int_I \left[(x^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \} \right. \\
 &\quad - (u^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \} \\
 &\quad - \sum_{i=1}^p \lambda_i x^1(t) \omega_i^1(t) + \sum_{i=1}^p \lambda_i u(t) \omega_i^1(t) \\
 &\quad - (v^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \} \\
 &\quad + (y^1(t))^T \sum_{i=1}^p \lambda_i \{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \} \\
 &\quad \left. - \sum_{i=1}^p \lambda_i v^1(t) z_i^1(t) + \sum_{i=1}^p \lambda_i y^1(t) z_i^1(t) \right] dt
 \end{aligned}$$

Using (3), (6), (11) and (14), we get

$$\begin{aligned}
& \sum_{i=1}^p \lambda_i \int_I \left[f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - (y^1)^T \sum_{i=1}^p \lambda_i \left\{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right\} \right. \\
& \left. + \sum_{i=1}^p \lambda_i (x^1(t) \omega_i^1(t)) - \sum_{i=1}^p \lambda_i (y^1(t) z_i^1(t)) \right] \\
& \geq \sum_{i=1}^p \lambda_i \int_I \left[f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - (u^1(t))^T \sum_{i=1}^p \lambda_i \left\{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right\} \right. \\
& \left. + u^1(t) \omega_i^1(t) - y^1(t) z_i^1(t) \right] dt
\end{aligned}$$

In view of

$$s(x^1(t) | C_i^1) \geq (x^1(t))^T \omega_i^1, \quad i = 1, \dots, p$$

and

$$s(v^1(t) | K_i^1) \geq (v^1(t))^T z_i^1, \quad i = 1, \dots, p, \text{ this yields}$$

$$\begin{aligned}
& \sum_{i=1}^p \lambda_i \int_I \left[f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + s(x^1(t) | C_i^1) \right. \\
& \left. - (y^1(t))^T \sum_{i=1}^p \lambda_i \left\{ f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right\} - (y^1(t))^T z_i^1(t) \right] dt \\
& \geq \sum_{i=1}^p \lambda_i \int_I \left[f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - s(v^1(t) | K_i^1) \right. \\
& \left. - (u^1)^T \sum_{i=1}^p \lambda_i \left\{ f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right\} + u^1(t) \omega_i^1(t) \right] dt \quad (19)
\end{aligned}$$

From (12) together with (6) and (13), we have

$$\int_I \left[(x^2(t) - u^2(t))^T \sum_{i=1}^p \lambda_i (g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - \omega_i^2(t) - Dg_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2)) \right] dt \geq 0$$

Which integrated by parts implies

$$\begin{aligned}
& \int_I \left[(x^2(t) - u^2(t))^T \sum_{i=1}^p \lambda_i (g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - \omega_i^2(t)) \right. \\
& \left. + (\dot{x}^2(t) - \dot{u}^2(t))^T \sum_{i=1}^p \lambda_i g_{u^1}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] dt \\
& - (x^2(t) - u^2(t))^T \sum_{i=1}^p \lambda_i g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \Big|_{t=a}^{t=b} \geq 0
\end{aligned}$$

Using (2) and (10), we have

$$\int_I \left[(x^2(t) - u^2(t))^T \sum_{i=1}^p \lambda_i (g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - \omega_i^2(t)) + (\dot{x}^2(t) - \dot{u}^2(t))^T \sum_{i=1}^p \lambda_i g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] dt \geq 0$$

By pseudo-convexity of $\sum_{i=1}^p \lambda_i \int_I (g_{u^2}^i(t, \dots, y^2, \dot{y}^2) + (\cdot)^T \omega_i^2(t)) dt$, this yields

$$\begin{aligned} & \sum_{i=1}^p \lambda_i \int_I (g_{u^2}^i(t, x^2, \dot{x}^2, v^2, \dot{v}^2) + x^2(t)^T \omega_i^2(t)) dt \\ & \geq \sum_{i=1}^p \lambda_i \int_I (g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) + u^2(t)^T \omega_i^2(t)) dt \end{aligned} \tag{20}$$

From (4) together with (14) and (5), we have

$$\begin{aligned} & \int_I \left[(v^2(t) - y^2(t))^T \sum_{i=1}^p \lambda_i (g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t) - Dg_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2)) \right] dt \leq 0 \\ \Rightarrow & \int_I \left[(v^2(t) - y^2(t))^T \sum_{i=1}^p \lambda_i (g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t)) + (\dot{v}^2(t) - \dot{y}^2(t))^T \sum_{i=1}^p \lambda_i g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right] dt \\ & - (\dot{v}^2(t) - \dot{y}^2(t))^T \sum_{i=1}^p \lambda_i g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \Big|_{t=a}^{t=b} \geq 0 \end{aligned}$$

Using (2) and (10), we have

$$\int_I \left[(v^2(t) - y^2(t))^T \sum_{i=1}^p \lambda_i (g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t)) + (\dot{v}^2(t) - \dot{y}^2(t))^T \sum_{i=1}^p \lambda_i g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right] dt \leq 0$$

By pseudo concavity of $\int_I \sum_{i=1}^p \lambda_i (g_{y^2}^i(t, x^2, \dot{x}^2, \dots) - (\cdot)^T z_i^2(t)) dt$, we have

$$\begin{aligned} & \sum_{i=1}^p \lambda_i \int_I (g_{y^2}^i(t, x^2, \dot{x}^2, v^2, \dot{v}^2) - v^2(t)^T z_i^2(t)) dt \\ & \leq \sum_{i=1}^p \lambda_i \int_I (g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - y^2(t)^T z_i^2(t)) dt \end{aligned}$$

$$\begin{aligned}
& -\sum_{i=1}^p \lambda_i \int_I \left(g_{y^2}^i(t, x^2, \dot{x}^2, v^2, \dot{v}^2) - v^2(t)^T z_i^2(t) \right) dt \\
& \geq -\sum_{i=1}^p \lambda_i \int_I \left(g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - y^2(t)^T z_i^2(t) \right) dt
\end{aligned} \tag{21}$$

From the relations (20) and (21)

$$\begin{aligned}
& -\sum_{i=1}^p \lambda_i \int_I \left((x^2(t))^T \omega_i^2(t) + v^2(t)^T z_i^2(t) \right) dt \\
& \geq -\sum_{i=1}^p \lambda_i \int_I \left(g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - y^2(t)^T z_i^2(t) \right) dt \\
& \quad + \sum_{i=1}^p \lambda_i \int_I \left(g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - u^2(t)^T w_i^2(t) \right) dt \\
& \quad + \sum_{i=1}^p \lambda_i \int_I \left(g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) + s(x^2(t)|C_i^2) - y^2(t)^T z_i^2(t) \right) dt \\
& \geq \sum_{i=1}^p \lambda_i \int_I \left(g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - s(v(t)|K_i^2) + u^2(t)^T w_i^2(t) \right) dt.
\end{aligned} \tag{22}$$

Combining (19) and (22), we get

$$\begin{aligned}
& \sum_{i=1}^p \lambda_i \int_I \left[f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) + s(x^1(t)|C_i^1) + s(x^2(t)|C_i^2) \right. \\
& \quad \left. + (y^1(t))^T \sum_{i=1}^p \lambda_i \int_I \left(f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right) \right. \\
& \quad \left. - y^1(t)^T z_i^1(t) - y^2(t)^T z_i^2(t) \right] dt \\
& \geq \sum_{i=1}^p \lambda_i \int_I \left[f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + g^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - s(v^1(t)|K_i^1) + s(v^2(t)|K_i^2) \right. \\
& \quad \left. - (u^1(t))^T \sum_{i=1}^p \lambda_i \int_I \left(f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right) \right. \\
& \quad \left. + (u^1(t))^T \omega_i^1(t) + (u^2(t))^T \omega_i^2(t) \right] dt.
\end{aligned}$$

That is,

$$\sum_{i=1}^p \lambda_i \int_I H^i dt \geq \sum_{i=1}^p \lambda_i \int_I G^i dt.$$

This yields

$$\int_I H dt \not\leq \int_I G dt$$

Theorem 2: (Strong Duality): Let $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t), \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ be an efficient solution of (Mix SP). Let $\lambda = \bar{\lambda}$ be fixed in (Mix SD). Furthermore assume that,
(C₁):

$$\int_I \left\{ (\phi^1(t))^T \left(\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - D\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) - D \left[(\phi^1(t))^T \left(-D\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] + D^2 \left[(\phi^1(t))^T \left(-\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \right\} (\phi^1(t))^T dt \geq 0$$

and

$$\int_I \left\{ (\phi^2(t))^T \left(\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - D\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) - D\phi^2(t)^T \left(-D\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) + D^2 \phi^2(t)^T \left(-\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right\} \phi^2(t) dt \geq 0,$$

(C₂):

$$\int_I \left\{ (\phi^1(t))^T \sum_{i=1}^p \left(\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - D\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) - D \left[(\phi^1(t))^T \sum_{i=1}^p \left(-D\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] + D^2 \left[(\phi^1(t))^T \sum_{i=1}^p \left(-\bar{\lambda}^T f_{y^1 y^1} (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \right\} (\phi^1(t))^T dt = 0, t \in I \Rightarrow \phi^1(t) = 0, t \in I$$

and

$$\int_I \left\{ (\phi^2(t))^T \left(\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - D\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) - D\phi^2(t)^T \left(-D\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) + D^2 \phi^2(t)^T \left(-\bar{\lambda}^T g_{y^2 y^2} (t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right\} \phi^2(t) dt = 0, t \in I \Rightarrow \phi^2(t) = 0, t \in I$$

$$(C_3): \quad g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + \bar{\omega}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2), \quad t \in I, i = 1, \dots, p$$

are linearly independent.

Then $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{\omega}_1^1(t), \bar{\omega}_2^1(t), \dots, \bar{\omega}_p^1(t); \bar{\omega}_1^2(t), \bar{\omega}_2^2(t), \dots, \bar{\omega}_p^2(t), \bar{\lambda})$ is feasible for (Mix SD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 1 holds, then there exist $\bar{\omega}_1^1(t), \bar{\omega}_2^1(t), \dots, \bar{\omega}_p^1(t); \bar{\omega}_1^2(t), \bar{\omega}_2^2(t), \dots, \bar{\omega}_p^2(t)$ such that $(u^1(t), u^2(t), v^1(t), v^2(t), \omega_1^1(t), \omega_2^1(t), \dots, \omega_p^1(t); \omega_1^2(t), \omega_2^2(t), \dots, \omega_p^2(t), \bar{\lambda})$ is an efficient solution of dual (Mix SD).

Proof: Since $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t), \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ is efficient, it is weak minimum, there exist $\tau \in R^p, \eta \in R^p, \gamma \in R, \theta^1(t) : I \rightarrow R^{k_1}, \theta^2(t) : I \rightarrow R^{k_2}$ and $\alpha(t) \in R^m, \beta(t) \in R^m$ such that the following Fritz-John optimality conditions [9] are satisfied

$$\begin{aligned} & \sum_{i=1}^p \tau^i \left(f_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + \bar{\omega}_i^1(t) - Df_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\ & + \left(\theta^1(t) - \tau e \bar{y}^1(t) \right)^T \sum_{i=1}^p \lambda_i \left(f_{y^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - Df_{y^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\ & - D \left[\left(\theta^1(t) - \tau e \bar{y}^1(t) \right)^T \sum_{i=1}^p \lambda_i \left(f_{y^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - Df_{y^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right. \\ & \left. - f_{y^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right] \quad (23) \\ & + D^2 \left[\left(\theta^1(t) - \tau e \bar{y}^1(t) \right)^T \sum_{i=1}^p \lambda_i \left(-f_{y^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] - \alpha(t) = 0, \quad t \in I \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^p \tau^i \left(g_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + \bar{\omega}_i^2(t) - Dg_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \\ & + \left(\theta^2(t) - \gamma \bar{y}^2(t) \right)^T \sum_{i=1}^p \bar{\lambda}_i \left(g_{y^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - Dg_{y^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \\ & - D \left[\left(\theta^2(t) - \gamma \bar{y}^2(t) \right)^T \sum_{i=1}^p \bar{\lambda}_i \left(g_{y^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - Dg_{y^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right. \\ & \left. - g_{y^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right] \quad (24) \\ & + D^2 \left[\left(\theta^2(t) - \gamma \bar{y}^2(t) \right)^T \sum_{i=1}^p \bar{\lambda}_i \left(-g_{y^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] - \beta(t) = 0, \quad t \in I. \end{aligned}$$

$$\begin{aligned}
 & \sum_{i=1}^p (\tau^i - \tau^T e \bar{\lambda}_i) \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\
 & + (\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\
 & - D \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \\
 & + D^2 \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] = 0, t \in I
 \end{aligned} \tag{25}$$

$$\begin{aligned}
 & \sum_{i=1}^p (\theta^2(t) - \gamma \bar{y}^2(t)) \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \\
 & + (\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \\
 & - D \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \\
 & + D^2 \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] = 0, t \in I
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 & (\theta^1(t) - \tau e y^1(t))^T \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\
 & + (\theta^2(t) - \gamma \bar{y}^2(t))^T \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) - \eta = 0, t \in I
 \end{aligned} \tag{27}$$

$$\theta^1(t)^T \sum_{i=1}^p \lambda_i \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) = 0, t \in I \tag{28}$$

$$\theta^2(t)^T \sum_{i=1}^p \lambda_i \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) = 0, t \in I \tag{29}$$

$$\gamma y^2(t)^T \sum_{i=1}^p \lambda_i \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) = 0, t \in I \tag{30}$$

$$\eta^T \lambda = 0 \tag{31}$$

$$\alpha(t)^T \bar{x}^1(t) = 0, t \in I \tag{32}$$

$$\beta(t)^T \bar{x}^2(t) = 0, t \in I \tag{33}$$

$$\bar{y}^1(t) \in N_{K_1^1}(\bar{z}_i^1(t)), t \in I \quad (34)$$

$$-\tau_i \bar{y}^2(t) - (\theta^2(t) - \gamma \bar{y}^2(t)) \in N_{K_2^1}(\bar{z}_i^2(t)), t \in I \quad (35)$$

$$\bar{\omega}^1(t) \in C_1, (\bar{\omega}^1(t))^T \bar{x}^1(t) = s(\bar{x}^1(t)|C_1), t \in I \quad (36)$$

$$\bar{\omega}^2(t) \in C_2, (\bar{\omega}^2(t))^T \bar{x}^2(t) = s(\bar{x}^2(t)|C_2), t \in I \quad (37)$$

$$(\tau, \theta^1(t), \theta^2(t), \gamma, \alpha(t), \beta(t), \eta) \geq 0, t \in I \quad (38)$$

$$(\tau, \theta^1(t), \theta^2(t), \gamma, \alpha(t), \beta(t), \eta) \neq 0, t \in I \quad (39)$$

Since $\lambda > 0$, (31) implies $\eta = 0$. Consequently, (27) reduces to

$$\begin{aligned} & (\theta^1(t) - (\tau^T e) \bar{y}^1(t)) \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\ & + (\theta^2(t) - \gamma \bar{y}^2(t)) \begin{pmatrix} g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) \\ -Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \end{pmatrix} = 0, t \in I \end{aligned} \quad (40)$$

Postmultiplying (25) by $(\theta^1(t) - \tau^T e \bar{y}^1(t))$, (26) by $(\theta^2(t) - \gamma \bar{y}^2(t))$ and then adding, we have

$$\begin{aligned} & \left\{ \sum_{i=1}^p (\tau^i - \tau^T e \bar{\lambda}_i) \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right. \\ & + (\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(f_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - Df_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\ & - D \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Df_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \\ & \left. + D^2 \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-f_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \right\} (\theta^1(t) - \tau^T e \bar{y}^1(t)) \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \sum_{i=1}^p (\tau^i - \gamma \bar{\lambda}_i) \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right. \\
 & + (\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(g_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - Dg_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \\
 & \left. - D \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Dg_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \right. \\
 & \left. + D^2 \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-g_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \right\} (\theta^2(t) - \gamma \bar{y}^2(t)) = 0
 \end{aligned} \tag{41}$$

$t \in I$.

Multiplying (40) λ^i and then using (29) and (30), we obtain

$$\begin{aligned}
 & (\theta^1(t) - \tau e \bar{y}^1(t))^T \sum_{i=1}^p \lambda_i \left(\begin{matrix} f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) \\ -Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \end{matrix} \right) = 0, \quad t \in I \\
 & (\theta^1(t) - \tau e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(\begin{matrix} f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) \\ -Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \end{matrix} \right) \tau^T e = 0, \quad t \in I \\
 & \int_I \left\{ (\theta^1(t) - \tau e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(\begin{matrix} f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \\ (t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \end{matrix} \right) \tau^T e \right\} dt = 0
 \end{aligned} \tag{42}$$

Multiplying (40) by τ^i and summing over i , we have

$$\begin{aligned}
 & \int_I (\theta^1(t) - \tau e \bar{y}^1(t)) \sum_{i=1}^p \tau^i \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \\
 & + (\theta^2(t) - \gamma \bar{y}^2(t)) \sum_{i=1}^p \tau^i \left(\begin{matrix} g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) \\ -Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \end{matrix} \right) dt = 0,
 \end{aligned} \tag{43}$$

By subtraction of (42) and (43) and then using (29) and (30), we have

$$\begin{aligned}
 & \int_I \left\{ (\theta^1(t) - \tau e \bar{y}^1(t))^T \sum_{i=1}^p \tau^i \left(f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) (\tau^1 - (\tau^T e) \lambda_i) \right. \\
 & \left. + (\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p (\tau^i - \bar{\lambda}^i \gamma) \left(\begin{matrix} g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \\ -\bar{z}_i^2 - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \end{matrix} \right) \right\} dt = 0
 \end{aligned} \tag{44}$$

From (41) and (44), we obtain

$$\begin{aligned}
& \int_I \left\{ (\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(f_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - Df_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right. \\
& - D \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Df_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \\
& + D^2 \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-f_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \left. \right\} (\theta^1(t) - \tau^T e \bar{y}^1(t)) dt \\
& + \int_I \left\{ (\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(g_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - Dg_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right. \\
& - D \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Dg_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \\
& + D^2 \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-g_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \left. \right\} (\theta^2(t) - \gamma \bar{y}^2(t)) dt = 0
\end{aligned}$$

In view of the hypothesis (C₁), we have

$$\begin{aligned}
& \int_I \left\{ (\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(f_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - Df_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right. \\
& - D \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Df_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \\
& + D^2 \left[(\theta^1(t) - \tau^T e \bar{y}^1(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-f_{y^1 y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \left. \right\} (\theta^1(t) - \tau^T e \bar{y}^1(t)) dt = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \int_I \left\{ (\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(g_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - Dg_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right. \\
& - D \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-Dg_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \\
& + D^2 \left[(\theta^2(t) - \gamma \bar{y}^2(t))^T \sum_{i=1}^p \bar{\lambda}_i \left(-g_{y^2 y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \left. \right\} (\theta^2(t) - \gamma \bar{y}^2(t)) dt = 0
\end{aligned}$$

From this, in view of the hypothesis (C₂), we have

$$\phi^1(t) = \theta^1(t) - \tau^T e \bar{y}^1(t) = 0, \quad t \in I \quad (45)$$

$$\phi^2(t) = \theta^2(t) - \gamma \bar{y}^2(t) = 0, \quad t \in I \quad (46)$$

From (46) and (26)

$$\sum_{i=1}^p (\tau^i - \gamma \bar{\lambda}_i)^T \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) = 0$$

In view of hypothesis (C₃), we have

$$\tau^i = \gamma \bar{\lambda}_i \quad , \quad i = 1, \dots, p \tag{47}$$

If possible, let, $\gamma = 0$. From (47), we have $\tau = 0$, and therefore, from (45) and (46), we have

$$\phi^1(t) = 0, \quad \phi^2(t) = 0, \quad t \in I$$

Also from (23) and (24), we have $\alpha(t) = 0 = \beta(t)$, $t \in I$.

Hence $(\tau, \theta^1(t), \theta^2(t), \gamma, \alpha(t), \beta(t)) = 0$, contradicting Fritz John condition (39). Hence $\gamma > 0$ and, consequently $\tau > 0$. From (23) and (24) along with (47), we obtain

$$\sum_{i=1}^p \bar{\lambda}^i \left(f_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + \bar{\omega}_i^1(t) - Df_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) - \alpha(t) = 0, \quad t \in I \tag{48}$$

and

$$\sum_{i=1}^p \bar{\lambda}^i \left(g_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + \bar{\omega}_i^2(t) - Dg_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) - \beta(t) = 0, \quad t \in I \tag{49}$$

In view of (38) this, yields

$$\sum_{i=1}^p \bar{\lambda}^i \left(f_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + \bar{\omega}_i^1(t) - Df_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \geq 0, \quad t \in I \tag{50}$$

$$\sum_{i=1}^p \bar{\lambda}^i \left(g_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + \bar{\omega}_i^2(t) - Dg_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \geq 0, \quad t \in I \tag{51}$$

and, in view of (32) and (33) together with (6) it gives,

$$\bar{x}^1(t)^T \sum_{i=1}^p \bar{\lambda}^i \left(f_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + \bar{\omega}_i^1(t) - Df_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) = 0, \quad t \in I \tag{52}$$

$$\bar{x}^2(t)^T \sum_{i=1}^p \bar{\lambda}^i \left(g_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + \bar{\omega}_i^2(t) - Dg_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) = 0, \quad t \in I \tag{53}$$

The relation (53) can be written as

$$\int_I \bar{x}^2(t)^T \sum_{i=1}^p \bar{\lambda}^i \left(g_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + \bar{\omega}_i^2(t) - Dg_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) dt = 0 \tag{54}$$

From (45) and (46), we have

$$\begin{aligned} & (\bar{y}^1(t), \bar{y}^2(t)) \geq 0 \quad t \in I \\ & \bar{y}^1(t)^T \sum_{i=1}^p \bar{\lambda}^i \left[f_{y^i}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^i}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right] = 0 \end{aligned} \quad (55)$$

Also from (34) and (35), we have

$$\bar{y}^1(t)^T \bar{z}_i^1(t) = s(\bar{y}^1(t) | K_i^1), \quad t \in I, i = 1, 2, \dots, p \quad (56)$$

$$\bar{y}^2(t)^T \bar{z}_i^2(t) = s(\bar{y}^2(t) | K_i^2), \quad t \in I, i = 1, 2, \dots, p \quad (57)$$

Consequently, from (51), (54), (55) and (56), the feasibility of $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ for (Mix SD) follows.

Consider,

$$\begin{aligned} H^i &= f^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + g^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + s(\bar{x}^1(t) | C_i^1) + s(\bar{x}^2(t) | C_i^2) \\ &\quad - \bar{y}^1(t)^T \sum_{i=1}^p \bar{\lambda}^i \left[f_{y^i}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^i}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right] \\ &\quad - \bar{y}^1(t)^T \bar{z}_i^1(t) - \bar{y}^2(t)^T \bar{z}_i^2(t) \end{aligned}$$

Using (35), (36), (54), (56), (57) and (58) in proper sequence, we obtain

$$\begin{aligned} H^i &= f^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + g^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \\ &\quad - s(\bar{y}^1(t) | K_i^1) - s(\bar{y}^2(t) | K_i^2) + \bar{x}^1(t)^T \bar{\omega}_i^1(t) + \bar{x}^2(t)^T \bar{\omega}_i^2(t) \\ &\quad - \bar{x}^1(t)^T \sum_{i=1}^p \bar{\lambda}^i \left[f_{x^i}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{\omega}_i^1(t) - Df_{x^i}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right] \\ &= G^i \text{ for } i = 1, 2, \dots, p \end{aligned}$$

Therefore,

$$\int_I (H^1, H^2, \dots, H^i, \dots, H^p) dt = \int_I (G^1, G^2, \dots, G^i, \dots, G^p) dt$$

This proves the efficiency of $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{\omega}_1^1(t), \bar{\omega}_2^1(t), \dots, \bar{\omega}_p^1(t); \bar{\omega}_1^2(t), \bar{\omega}_2^2(t), \dots, \bar{\omega}_p^2(t), \bar{\lambda})$ of the dual problem (Mix SD) by an application of Theorem 1.

The proof of the following converse duality theorem follows automatically by symmetry of the formulation:

Theorem 3: (Converse Duality): Let $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\omega}_1^1(t), \bar{\omega}_2^1(t), \dots, \bar{\omega}_p^1(t); \bar{\omega}_1^2(t), \bar{\omega}_2^2(t), \dots, \bar{\omega}_p^2(t), \bar{\lambda})$ be an efficient solution of (Mix SP). Let $\lambda = \bar{\lambda}$ be fixed in (Mix SD). Furthermore assume that
(A₁):

$$\int_I \left\{ (\psi^1(t))^T \left(\bar{\lambda}^T f_{x^1 x^1}(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - D\bar{\lambda}^T f_{x^1 x^1}(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) - D \left[(\psi^1(t))^T \left(-D\bar{\lambda}^T f_{x^1 x^1}(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] + D^2 \left[(\psi^1(t))^T \left(-\bar{\lambda}^T f_{x^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \right\} (\psi^1(t))^T dt \geq 0$$

and

$$\int_I \left\{ (\psi^2(t))^T \left(\bar{\lambda}^T g_{x^2 x^2}(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - D\bar{\lambda}^T g_{x^2 x^2}(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) - D \left[(\psi^2(t))^T \left(-D\bar{\lambda}^T g_{x^2 x^2}(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] + D^2 \left[(\psi^2(t))^T \left(-\bar{\lambda}^T g_{x^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \right\} (\psi^2(t))^T dt \geq 0$$

(A₂):

$$\int_I \left\{ (\psi^1(t))^T \left(\bar{\lambda}^T f_{x^1 x^1}(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - D\bar{\lambda}^T f_{x^1 x^1}(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) - D \left[(\psi^1(t))^T \left(-D\bar{\lambda}^T f_{x^1 x^1}(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] + D^2 \left[(\psi^1(t))^T \left(-\bar{\lambda}^T f_{x^1 x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right) \right] \right\} (\psi^1(t))^T dt = 0, t \in I \Rightarrow \psi^1(t) = 0, t \in I$$

and

$$\int_I \left\{ (\psi^2(t))^T \left(\bar{\lambda}^T g_{x^2 x^2}(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - D\bar{\lambda}^T g_{x^2 x^2}(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) - D \left[(\psi^2(t))^T \left(-D\bar{\lambda}^T g_{x^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] + D^2 \left[(\psi^2(t))^T \left(-\bar{\lambda}^T g_{x^2 x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \right\} (\psi^2(t))^T dt = 0, t \in I \Rightarrow \psi^2(t) = 0, t \in I$$

and

(A₃): $g_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + \bar{\omega}_i^2(t) - Dg_{x^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2)$, $i = 1, \dots, p$ are linearly independent.

Then $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t), \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \lambda)$ is feasible for (Mix SD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 1 hold, then there exist $\bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t)$ such that

$$\begin{aligned} & (u^1(t), u^2(t), v^1(t), v^2(t), z_1^1(t), z_2^1(t), \dots, z_p^1(t); z_1^2(t), z_2^2(t), \dots, z_p^2(t), \lambda) \\ & = (\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda}) \end{aligned}$$

is an efficient solution of dual (Mix SD).

5. SELF DUALITY

A mathematical problem is said to be self-dual if it is formally identical with its dual. In general, the problems (Mix SP) and (Mix SD) cannot be formally identical if the kernel function does not owe any special characteristics. Hence, skew symmetric of f^i and g^i is assumed in order to validate the following self-duality theorem for the pair of problems treated in the preceding section.

Theorem 4. (Self Duality) Let f^i and g^i , $i = 1, 2, \dots, p$, be skew symmetric and $C_i^1 = K_i^1$ and $C_i^2 = K_i^2$ with $\omega_i^1(t) = z_i^1(t)$, $\omega_i^2(t) = z_i^2(t)$, $t \in I$. Then the problem (Mix SP) is self dual. If the problems (Mix SP) and (Mix SD) are dual problems, and $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ is a joint optimal solution of (MixSP) and (MixSD), then so is $(\bar{y}^1(t), \bar{y}^2(t), \bar{x}^1(t), \bar{x}^2(t), \bar{\omega}_1^1(t), \bar{\omega}_2^1(t), \dots, \bar{\omega}_p^1(t); \bar{\omega}_1^2(t), \bar{\omega}_2^2(t), \dots, \bar{\omega}_p^2(t), \bar{\lambda})$, and the common functional value is zero, i.e.

$$\text{Minimum (Mix SP)} = \int_I \left\{ f^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + g^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right\} dt = 0$$

Proof: By skew symmetric of f^i and g^i , we have

$$f_{x^1}^i(t, \bar{x}^1(t), \dot{\bar{x}}^1(t), \bar{y}^1(t), \dot{\bar{y}}^1(t)) = -f_{y^1}^i(t, \bar{y}^1(t), \dot{\bar{y}}^1(t), \bar{x}^1(t), \dot{\bar{x}}^1(t))$$

$$g_{x^2}^i(t, \bar{x}^2(t), \dot{\bar{x}}^2(t), \bar{y}^2(t), \dot{\bar{y}}^2(t)) = -g_{y^2}^i(t, \bar{y}^2(t), \dot{\bar{y}}^2(t), \bar{x}^2(t), \dot{\bar{x}}^2(t))$$

$$f_{y^1}^i(t, \bar{x}^1(t), \dot{\bar{x}}^1(t), \bar{y}^1(t), \dot{\bar{y}}^1(t)) = -f_{x^1}^i(t, \bar{y}^1(t), \dot{\bar{y}}^1(t), \bar{x}^1(t), \dot{\bar{x}}^1(t))$$

$$g_{y^2}^i(t, \bar{x}^2(t), \dot{\bar{x}}^2(t), \bar{y}^2(t), \dot{\bar{y}}^2(t)) = -g_{x^2}^i(t, \bar{y}^2(t), \dot{\bar{y}}^2(t), \bar{x}^2(t), \dot{\bar{x}}^2(t))$$

$$f_{x^1}^i(t, \bar{x}^1(t), \dot{\bar{x}}^1(t), \bar{y}^1(t), \dot{\bar{y}}^1(t)) = -f_{y^1}^i(t, \bar{y}^1(t), \dot{\bar{y}}^1(t), \bar{x}^1(t), \dot{\bar{x}}^1(t))$$

$$g_{x^2}^i(t, \bar{x}^2(t), \dot{\bar{x}}^2(t), \bar{y}^2(t), \dot{\bar{y}}^2(t)) = -g_{y^2}^i(t, \bar{y}^2(t), \dot{\bar{y}}^2(t), \bar{x}^2(t), \dot{\bar{x}}^2(t))$$

$$f_{y^1}^i(t, \bar{x}^1(t), \dot{\bar{x}}^1(t), \bar{y}^1(t), \dot{\bar{y}}^1(t)) = -f_{x^1}^i(t, \bar{y}^1(t), \dot{\bar{y}}^1(t), \bar{x}^1(t), \dot{\bar{x}}^1(t))$$

$$g_{y^2}^i(t, \bar{x}^2(t), \dot{\bar{x}}^2(t), \bar{y}^2(t), \dot{\bar{y}}^2(t)) = -g_{x^2}^i(t, \bar{y}^2(t), \dot{\bar{y}}^2(t), \bar{x}^2(t), \dot{\bar{x}}^2(t))$$

Recasting the dual problem (Mix SD) as a minimization problem and using the above relations, we have

$$\text{Mix (SD1): Minimize } -\int_I (G^1, G^2, \dots, G^p) dt$$

Subject to

$$x^1(a) = 0 = x^1(b) \quad , \quad y^1(a) = 0 = y^1(b)$$

$$x^2(a) = 0 = x^2(b) \quad , \quad y^2(a) = 0 = y^2(b)$$

$$\sum_{i=1}^p \bar{\lambda}^i \left[f_{y^1}^i(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) - \bar{\omega}_i^1(t) - Df_{y^1}^i(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) \right] \leq 0, \quad t \in I$$

$$= \sum_{i=1}^p \bar{\lambda}^i \left[g_{y^2}^i(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) - \bar{\omega}_i^2(t) - Dg_{y^2}^i(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) \right] \leq 0, \quad t \in I$$

$$\int_I \bar{u}^T(t) \left[g_{y^2}^i(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) - \bar{\omega}_i^2(t) - Dg_{y^2}^i(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) \right] dt \geq 0, \quad t \in I$$

$$(\bar{v}^1(t), \bar{v}^2(t)) \geq 0, \quad t \in I$$

$$\bar{\omega}_i^1(t) \in C_i^1 \quad \text{and} \quad \bar{\omega}_i^2(t) \in C_i^2$$

$$\bar{\lambda} > 0, \quad \lambda^T e = 1, \quad e^T = (1, \dots, 1)$$

also,

$$\begin{aligned} -G^i &= -f^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - g^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \\ &\quad - s(\bar{y}^1(t) | K_i^1) - s(\bar{y}^2(t) | K_i^2) - \bar{x}^1(t)^T \bar{\omega}_i^1(t) - \bar{x}^2(t)^T \bar{\omega}_i^2(t) \\ &\quad + \bar{x}^1(t)^T \sum_{i=1}^p \bar{\lambda}^i \left[f_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{\omega}_i^1(t) - Df_{x^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right] \end{aligned}$$

$$\begin{aligned} &= f^i(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) + g^i(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) \\ &\quad - s(\bar{y}^1(t) | K_i^1) - s(\bar{y}^2(t) | K_i^2) - \bar{x}^1(t)^T \bar{\omega}_i^1(t) - \bar{x}^2(t)^T \bar{\omega}_i^2(t) \\ &\quad - \bar{x}^1(t)^T \sum_{i=1}^p \bar{\lambda}^i \left[f_{y^1}^i(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) - \bar{\omega}_i^1(t) - Df_{y^1}^i(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) \right] \\ &= H^i(t, \bar{y}^1, \bar{y}^2, \dot{\bar{y}}^1, \dot{\bar{y}}^2, \bar{x}^1, \bar{x}^2, \dot{\bar{x}}^1, \dot{\bar{x}}^2, \bar{\omega}_i^1, \bar{\omega}_i^2, \bar{\lambda}) \end{aligned}$$

Hence, by using various hypotheses of this theorem, we have

$$\text{(MIX SP-1): } \int_I (H^1, H^2, \dots, H^p) dt$$

Subject to

$$x^1(a) = 0 = x^1(b) \quad , \quad y^1(a) = 0 = y^1(b)$$

$$x^2(a) = 0 = x^2(b) \quad , \quad y^2(a) = 0 = y^2(b)$$

$$\sum_{i=1}^p \bar{\lambda}^i \left[f_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) - \bar{z}_i^1(t) - Df_{y^1}^i(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) \right] \leq 0, \quad t \in I,$$

$$\sum_{i=1}^p \bar{\lambda}^i \left[g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right] \leq 0, \quad t \in I,$$

$$\int_I \bar{y}^2(t)^T \left[\sum_{i=1}^p \bar{\lambda}^i \left(g_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) - \bar{z}_i^2(t) - Dg_{y^2}^i(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right) \right] \geq 0,$$

$$(\bar{y}^1(t), \bar{y}^2(t)) \geq 0, \quad t \in I$$

$$\bar{z}_i^1(t) \in C_i^1 \quad \text{and} \quad \bar{z}_i^2(t) \in C_i^2 \quad , \quad i = 1, 2, \dots, p$$

$$\bar{\lambda} > 0, \quad \lambda^T e = 1 \quad , \quad e^T = (1, \dots, 1)$$

which is just the primal problem (Mix SP). Therefore $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ is an efficient solution of dual problem, which implies that $(\bar{y}^1(t), \bar{y}^2(t), \bar{x}^1(t), \bar{x}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ is an efficient solution of the primal.

Similarly $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ is an efficient solution of (MixSP), which implies that $(\bar{y}^1(t), \bar{y}^2(t), \bar{x}^1(t), \bar{x}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \dots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \dots, \bar{z}_p^2(t), \bar{\lambda})$ is an efficient solution of the dual problem (MixSD).

In view of (36), (37), (54), (56), (57) and (58), we have

$$\text{Minimum (MixSP)} = \int_I \left\{ f^1(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + g^1(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right.$$

$$\left. , \dots, f^p(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + g^p(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right\} dt$$

Corresponding, to the solution $\omega_i^1 \in C_i^1$, we have

$$\text{Minimum (Mix SP)} = \int_I \left\{ f^1(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) + g^1(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) \right.$$

$$\left. , \dots, f^p(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) + g^p(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) \right\} dt$$

By the skew-symmetry of each f^i , we have

$$\begin{aligned} & \int_I \left\{ f^1(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + g^1(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2), \dots, f^p(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + g^p(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right\} dt \\ &= \int_I \left\{ f^1(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) + g^1(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2), \dots, f^p(t, \bar{y}^1, \dot{\bar{y}}^1, \bar{x}^1, \dot{\bar{x}}^1) + g^p(t, \bar{y}^2, \dot{\bar{y}}^2, \bar{x}^2, \dot{\bar{x}}^2) \right\} dt \\ &= - \int_I \left\{ f^1(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1) + g^1(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2), \dots, f^p(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) + g^p(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2) \right\} dt = 0 \end{aligned}$$

5. SPECIAL CASES

If $J_2 = 0$, $L_2 = 0$, the pair of mixed type nondifferentiable symmetric dual problems (Mix SP) and (Mix SD) are reduced to the following Wolfe type nondifferentiable symmetric dual variational problem (WP) and (WD), recently studied by Husain and Rumana [10].

(SWPo): Minimize: $\int_I (H_0^1, H_0^2, \dots, H_0^p) dt$

Subject to:

$$x^1(a) = 0 = x^1(b) \quad , \quad y^1(a) = 0 = y^1(b)$$

$$\sum_{i=1}^p \lambda^i \left(f_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right) \leq 0, \quad t \in I$$

$$z_i^1(t) \in C_i^1, \quad i = 1, \dots, p, \quad t \in I$$

$$x^1(t) \geq 0, \quad t \in I$$

$$\lambda \in \Lambda^+ = \left\{ \lambda \in R^p \mid \lambda > 0, \lambda^T e = 1, e = (1, 1, \dots, 1)^T \in R^p \right\}$$

where,

1. $H_0^i = f^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + s(x^1(t) | C_i^1)$

$$- y^1(t)^T \sum_{i=1}^p \lambda^i \left(f_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + z_i^1(t) - Df_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right) - y^1(t)^T z_i^1(t)$$

(SWDo): Maximize: $\int_I (G_0^1, G_0^2, \dots, G_0^p) dt$

Subject to:

$$u^1(a) = 0 = u^1(b) \quad , \quad v^1(a) = 0 = v^1(b)$$

$$\sum_{i=1}^p \lambda^i \left(f_u^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_u^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right) \geq 0, \quad t \in I$$

$$\omega_i^1(t) \in K^i, \quad i = 1, \dots, p, \quad t \in I$$

$$v(t) \geq 0, \quad t \in I$$

$$\lambda \in \Lambda^+$$

where,

$$G_0^i = f^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + s(v^1(t)|K_i^1)$$

$$-u^1(t)^T \sum_{i=1}^p \lambda^i \left(f_u^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_u^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right) - x^1(t)^T \omega_i^1(t)$$

If $J_1 = 0$, $L_1 = 0$, the pair of mixed type symmetric dual problem (Mix SP) and (Mix SD) are reduced to the following Mond-Weir type symmetric dual variational problem (M-WP) and (M-WD), recently studied by Husain and Rumana [10].

$$(SM-WPo): \text{Maximize: } \int_I (H_1^1, H_1^2, \dots, H_1^p) dt$$

Subject to:

$$x^2(a) = 0 = x^2(b) \quad , \quad y^2(a) = 0 = y^2(b)$$

$$\sum_{i=1}^p \lambda^i \left[g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t) - Dg_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right] \leq 0, \quad t \in I,$$

$$\int_I y^2(t)^T \left[\sum_{i=1}^p \lambda^i \left(g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t) - Dg_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right) \right] \geq 0,$$

$$x^2(t) \geq 0, \quad t \in I$$

$$z_i^2(t) \in K_i^2$$

$$\lambda > 0$$

Where,

$$H_1^i = g^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) + s(x^2(t)|C_i^2) - z_i^2(t) y^2(t)$$

$$(SM-WDo) : \text{Maximize: } \int_I (G_1^1, G_1^2, \dots, G_1^p) dt$$

Subject to:

$$u^2(a) = 0 = u^2(b) \quad , \quad v^2(a) = 0 = v^2(b) .$$

$$\sum_{i=1}^p \lambda^i \left[g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) + \omega_i^2(t) - Dg_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] \geq 0, \quad t \in I$$

$$\int_I u^2(t)^T \left[g_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) + \omega_i^2(t) - Dg_{u^2}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] \leq 0, \quad t \in I, \quad ,$$

$$v^2(t) \geq 0, \quad t \in I$$

$$\omega_i^2(t) \in C_i^2$$

$$\lambda > 0$$

where,

$$G_i^1 = g^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) + s(v^2(t) | K_i^2) + u_i^2(t) \omega_i^2(t)$$

If $C_i^1 = C_i^2 = \{0\}$ and $K_i^1 = K_i^2 = \{0\}$, $i = 1, 2, \dots, p$, i.e., support functions are suppressed from the formulation of the dual models (Mix SP) and (Mix SD), we have the following pair of reduced dual models, studied by Husain and Rumana [11].

$$\text{(Mix SP*)}: \text{Minimize } \int_I \left\{ f(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right.$$

$$\left. - y^1(t)^T \left(\lambda^T f_{y^1}(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right) \right.$$

$$\left. - D\lambda^T f_{y^1}(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right\} dt$$

Subject to

$$x^1(a) = 0 = x^1(b) \quad , \quad y^1(a) = 0 = y^1(b) \quad ,$$

$$x^2(a) = 0 = x^2(b) \quad , \quad y^2(a) = 0 = y^2(b) \quad ,$$

$$\lambda^T f_{y^1}(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - D\lambda^T f_{y^1}(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \leq 0, \quad t \in I, \quad ,$$

$$\lambda^T g_{y^2}(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - D\lambda^T g_{y^2}(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \leq 0, \quad t \in I, \quad ,$$

$$\int_I y^2(t)^T \left(\lambda^T g_{y^2}(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right.$$

$$\left. - D\lambda^T g_{y^2}(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right) \geq 0, \quad ,$$

$$\lambda \in \Lambda^+ .$$

(Mix SD*) Maximize $\int_I \{f(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + g(t, u^2, \dot{u}^2, v^2, \dot{v}^2)$

$$\begin{aligned} & -u^1(t)^T \left(\lambda^T f_{y^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right. \\ & \left. - D\lambda^T f_{y^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right) e \} dt \end{aligned}$$

Subject to

$$u^1(a) = 0 = u^1(b) \quad , \quad v^1(a) = 0 = v^1(b) ,$$

$$u^2(a) = 0 = u^2(b) \quad , \quad v^2(a) = 0 = v^2(b) ,$$

$$\lambda^T f_{u^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - D\lambda^T f_{u^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \geq 0, \quad t \in I ,$$

$$\lambda^T g_{u^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - D\lambda^T g_{u^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \geq 0, \quad t \in I ,$$

$$\int_I u^2(t)^T \left(\lambda^T g_{u^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right.$$

$$\left. - D\lambda^T g_{u^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right) \geq 0, ,$$

$$\lambda \in \Lambda^+ .$$

where $\Lambda^+ = \{ \lambda \in R^p \mid \lambda > 0, \lambda^T e = 1, e = (1, 1, \dots, 1)^T \in R^p \}$

6. NATURAL BOUNDARY VALUES

The pairs of mixed type symmetric nondifferentiable multiobjective variational problem can be formulated by natural boundary values rather than by fixed end points. The problems with natural boundary conditions are needed to establish well-defined relationship between the pairs of dual continuous programming problems and nonlinear programming problems.

Following is the pair of mixed type symmetric dual problems with natural boundary values:

Primal (Mix SP₀): Maximize $\int_I (H^1, H^2, \dots, H^p) dt$

Subject to

$$\sum_{i=1}^p \lambda^i \left[f_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - z_i^1(t) - Df_{y^i}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \right] \leq 0, \quad t \in I ,$$

$$\sum_{i=1}^p \lambda^i \left[g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t) - Dg_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right] \leq 0, \quad t \in I ,$$

$$\int_I y^2(t)^T \left[\sum_{i=1}^p \lambda^i \left(g_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - z_i^2(t) - Dg_{y^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \right) \right] \geq 0,$$

$$(x^1(t), x^2(t)) \geq 0, t \in I$$

$$z_i^1(t) \in K_i^1 \quad \text{and} \quad z_i^2(t) \in K_i^2, \quad i = 1, 2, \dots, p$$

$$\lambda > 0, \quad \lambda^T e = 1, \quad e^T = (1, \dots, 1)$$

$$f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \Big|_{t=a} = 0, \quad f_{y^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \Big|_{t=b} = 0, \quad i = 1, 2, \dots, p$$

$$\lambda^T g_{y^1}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \Big|_{t=a} = 0, \quad \lambda^T g_{y^1}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \Big|_{t=b} = 0$$

$$\text{Dual (Mix SD}_0\text{): Maximize } \int_I (G^1, G^2, \dots, G^p) dt$$

Subject to:

$$\sum_{i=1}^p \lambda^i \left[f_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) + \omega_i^1(t) - Df_{u^1}^i(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \right] \geq 0, t \in I,$$

$$\sum_{i=1}^p \lambda^i \left[g_{u^1}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) + \omega_i^2(t) - Dg_{u^1}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] \geq 0, t \in I,$$

$$\int_I u^2(t)^T \sum_{i=1}^p \lambda^i \left[g_{u^1}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) + \omega_i^2(t) - Dg_{u^1}^i(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \right] dt \leq 0,$$

$$(v^1(t), v^2(t)) \geq 0, t \in I$$

$$\omega_i^1(t) \in C_i^1 \quad \text{and} \quad \omega_i^2(t) \in C_i^2, \quad i = 1, 2, \dots, p$$

$$\lambda > 0, \quad \lambda^T e = 1, \quad e^T = (1, \dots, 1)$$

$$f_{x^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \Big|_{t=a} = 0, \quad f_{x^1}^i(t, x^1, \dot{x}^1, y^1, \dot{y}^1) \Big|_{t=b} = 0, \quad i = 1, 2, \dots, p$$

$$\lambda^T g_{x^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \Big|_{t=a} = 0, \quad \lambda^T g_{x^2}^i(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \Big|_{t=b} = 0$$

The duality results for each of the above pairs of dual variational problems can be proved easily on the lines of the proofs of the Theorems 1-4, with slight modifications in the arguments, as in Mond and Hanson [17].

7. MULTIOBJECTIVE NONLINEAR PROGRAMMING

If the time dependency is removed from the variational problems (Mix SPo) and (Mix SDo) with natural boundary values and $b-a=1$, we obtain the following pair of static mixed type multiobjective dual problems involving support functions, which are not explicitly reported in the literature with their correct formulations..

Primal (Mix SP₁): Minimize $\hat{H} = (\hat{H}^1, \hat{H}^2, \dots, \hat{H}^p)$

Subject to

$$\sum_{i=1}^p \lambda^i [f_{y^1}^i(x^1, y^1) - z_i^1] \leq 0,$$

$$\sum_{i=1}^p \lambda^i [g_{y^2}^i(x^2, y^2) - z_i^2] \leq 0,$$

$$(y^2)^T \left[\sum_{i=1}^p \lambda^i (g_{y^2}^i(x^2, y^2) - z_i^2) \right] \geq 0,$$

$$z_i^1 \in K_i^1 \quad \text{and} \quad z_i^2 \in K_i^2, \quad i = 1, \dots, p$$

$$\lambda \in \Lambda^+$$

$$\begin{aligned} \hat{H}^i &= f^i(x^1, y^1) + g^i(x^2, y^2) + s(x^1 | C_i^1) + s(x^2 | C_i^2) \\ &- y^1 \sum_{i=1}^p \lambda^i [f_{y^1}^i(x^1, y^1) - z_i^1 - Df_{y^1}^i(x^1, y^1)] - z_i^1 y^1 - z_i^2 y^2 \end{aligned}$$

Dual (Mix SD1): Maximize $\hat{G} = (\hat{G}^1, \hat{G}^2, \dots, \hat{G}^p)$

Subject to:

$$\sum_{i=1}^p \lambda^i [f_{u^1}^i(u^1, v^1) + \omega_i^1] \geq 0,$$

$$\sum_{i=1}^p \lambda^i [g_{u^1}^i(u^2, v^2) + \omega_i^2] \geq 0,$$

$$(u^2)^T \sum_{i=1}^p \lambda^i [g_{u^1}^i(u^2, v^2) + \omega_i^2] \leq 0,$$

$$\omega_i^1 \in C_i^1 \quad \text{and} \quad \omega_i^2 \in C_i^2 \quad i = 1, 2, \dots, p$$

$$\lambda \in \Lambda^+$$

where,

$$\hat{G}^i = f^i(x^1, y^1) + g^i(x^2, y^2) + s(v^1 | K_i^1) + s(v^2 | K_i^2) \\ + u^1 \omega_i^1 + u_i^2 \omega^2 - u^1 \sum_{i=1}^p \lambda^i \left[f_{u^1}^i(x^1, y^1) + \omega_i^1 - Df_{u^1}^i(x^1, y^1) \right]$$

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