AN INVENTORY MODEL OF PURCHASE QUANTITY FOR FULLY-LOADED VEHICLES WITH MAXIMUM TRIPS IN CONSECUTIVE TRANSPORT TIME

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Abstract: Products made overseas but sold in Taiwan are very common. Regarding the cross-border or interregional production and marketing of goods, inventory decision-makers often have to think about how to determine the amount of purchases per cycle, the number of transport vehicles, the working hours of each transport vehicle, and the delivery by ground or air transport to sales offices in order to minimize the total cost of the inventory in unit time. This model assumes that the amount of purchases for each order cycle should allow all rented vehicles to be fully loaded and the transport times to reach the upper limit within the time period. The main research findings of this study included the search for the optimal solution of the integer planning of the model and the results of sensitivity analysis.

1. INTRODUCTION

This paper introduces the modified traditional EOQ model of fixed demand rate and unallowed stock-out. There are three key points of modification: 1) different decision variables for inventory decision makers: in addition to determining how many freight vehicles to hire simultaneously, the decision maker has to calculate the times for the back and forth transportation of each vehicle in the upper limit of the given hire time of each

vehicle, in order to determine the purchase amount per cycle; 2) the decision variable of the number of hired freight vehicles is an integer; 3) the purchase amount of each cycle (opening stock level) has to fully load all hired freight vehicles, while satisfying the upper limit for the back and forth transport time.

The transport costs of this model can be divided into two parts: 1) the fixed transport costs: including freight vehicle rental, parking fees, driver salaries and performance bonuses; 2) the variable transport costs: these are mainly subject to the transportation distance and increase proportionally to the distance, such as freight vehicle fuel costs, road tolls, and the wear and tear of freight vehicles. The variable transport costs and the amount of purchase are correlated and are subject to the number of hired freight vehicles as well as the working hours of each vehicle. Regarding previous studies on transportation costs, Weng [14], Serel et al. [13] proposed transport capacity and frequency affect transportation cost. Gupta [5] established the discontinuous transport cost model. Jain and Saksena [9] discuss time minimize transportation problems. Hoque and Goyal [8], Norden and Velde [12] considered the economic bulk model of transport cost.

In recent developments of the EOQ inventory model, the inventory issue has involved an increasing number of factors, and the applications have expanded to more fields. It seems to be a development trend of the inventory theory. The inventory model incorporating transport cost can be applied in a wide area, with numerous application values. Regarding relevant studies integrating transportation and inventory issues, McCann [10], Ahn et al. [1] studied the inventory issue of minimizing transport costs. Yang and Chu[16], Ertogral et al. [4], Çapar et al. [2]discussed supply chain management of transportation costs. Kim and Kim [9] studied load distribution issue of the inventory model. Hill [6], [7], Cetinkaya and Lee [3] developed the distribution transport inventory model for retailers. Mendoza and Ventura [11] studied the inventory model integrating quantity discounts and transportation costs.

Although these EOQ promotion models enhance the range of application of inventory management, and expand the application of inventory theory, they do not take into consideration that inventory management decision-makers are also responsible for transport. Although batch delivery of an order can often reduce cost, cross-border or cross-regional transportation may be restricted by shipping schedules and the periodicity of vessels. This may result in increased transportation costs as part of total expenses, as the goods in each order require back and forth transportation. In this case, batch delivery of an order may not reduce total inventory cost in unit time. Therefore, from the perspective of inventory practice, taking into consideration the transportation cost in the total inventory cost and regarding the continuous transportation of freight vehicles as a constraint of the inventory model are indeed necessary.

2. SYMBOLS

2.1 Parameters

The parameters of the model are as follows:
k:Fixed purchase ordering cost per cycle.
s: Purchase price per unit.

μ:The upper limit of working hours for the continuous back and forth transportation of each freight vehicle from the location of delivery to location of purchase.

t: The length of the delivery time for each freight vehicle (the round trip of each vehicle from the location of delivery to the location of purchase);where $t \leq \mu$.

p: The upper limit of the load of each freight vehicle (the amount of transportable goods).

b:The goods demand rate (goods demand or consumption amount in unit time).

c: The transportation time cost of each trip (increasing with the working hours of each trip of transportation). Each trip (regardless of the load) generates a cost. In general, if the location of delivery and the location of purchase are farther away, the working hours for each trip of the freight vehicles will be longer and the value will be higher.

f :The rental for each vehicle in the time range of $[0, \mu]$. For example, if the transportation costs for renting a freight vehicle for two consecutive trips are $f + 2c$, the transportation costs for renting two freight vehicles for one trip each will be $2f + c$.

h:The inventory cost of unit goods in unit time.

N:The set of all positive integers.

$x^{(-)}$:The maximum integer smaller than or equal to x, namely, $x^{(-)} = max \ \{n|n\epsilon N, \ and \ n \leq x\}$.

$x^{(+)}$: The maximum integer larger than or equal to x, namely, $x^{(+)} = max \ \{n|n\epsilon N, \ and \ n \geq x\}$.

w : $w = \left(\mu/t\right)^{(-)}$ The upper limit of the trips of each vehicle for back and forth transportation.

2.2 Decision variables

M:The number of rented vehicles for the transportation of goods, in which $M \in N$; namely, Mw is the number of transportation trips and pMw is the opening stock level (purchase amount per cycle).

v:The optimal purchase quantity per cycle without limit integer; it can be shown as $v = \sqrt{2bk/hp^2w^2}$ (c.f. Figure 1).

2.3 Objective function

L: The total inventory cost in unit time; namely, the total inventory costs in a cycle divided by the time length of a cycle, $\frac{pMw}{b}$; namely,

$$L(M) = \frac{\text{Total inventory cost per cycle}}{\text{The length of time per cycle}}$$

$$= \frac{1}{pMw/b}\left[k + s \cdot pMw + c \cdot Mw + fM + h \cdot \frac{pMw}{2} \cdot \frac{pMw}{b}\right]$$

3 MATHEMATICAL MODEL AND OPTIMAL SOLUTION

3.1 Model development

$$min_{M \in N} \ L(M) = \frac{bk}{pMw} + bs + \frac{bc}{p} + \frac{bf}{pw} + \frac{h \cdot pMw}{2} \tag{1}$$

3.2 Find the optimal solution to Model (1)

Deduction 1: If M^* is the optimal solution to (1), then the relationships between M^* and parameters will be $M^* = \left(\sqrt{\frac{2bk}{hp^2w^2}} \right)^{(-)}$ or $M^* = \left(\sqrt{\frac{2bk}{hp^2w^2}} \right)^{(+)}$; namely, $L(M^*) =$ min $\left\{ L\left(\sqrt{\frac{2bk}{hp^2w^2}}^{(-)} \right), L\left(\sqrt{\frac{2bk}{hp^2w^2}}^{(+)} \right) \right\}$.

Proof. Consider M as a real number, and the define function $F(M)$ as:

$$F(M) = \frac{bk}{pMw} + bs + \frac{bc}{p} + \frac{bf}{p} + \frac{h \cdot pMw}{2}, \ M \text{ is a real number}. \tag{2}$$

Function $F(M)$ and function $L(M)$ differ in two ways. The domain of definition of $F(M)$'s M is a real number, and the domain of definition of $L(M)$'s M is a positive integer. When M is a positive integer, the values of $F(M)$ and $L(M)$ are the same.

By (2),

$$\frac{d}{dM}F(M) = \frac{-bk}{pM^2w} + \frac{hpw}{2}. \tag{3}$$

hence,

the necessary and sufficient condition for equation $\frac{d}{dM}F(M) \leq 0$ is $(Mw)^2 \leq \frac{2bk}{hp^2}$ (namely, $M \leq \sqrt{\frac{2bk}{hp^2w^2}}$) $\tag{4}$

By (3),

$$\frac{d^2F(M)}{dM^2} = \frac{2bk}{pM^3w^2} > 0 \ . \tag{5}$$

By (4) and (5),

$$v, v = \sqrt{\frac{2bk}{hp^2w^2}}, \text{ which is the minimum point of function } F(M) \tag{6}$$
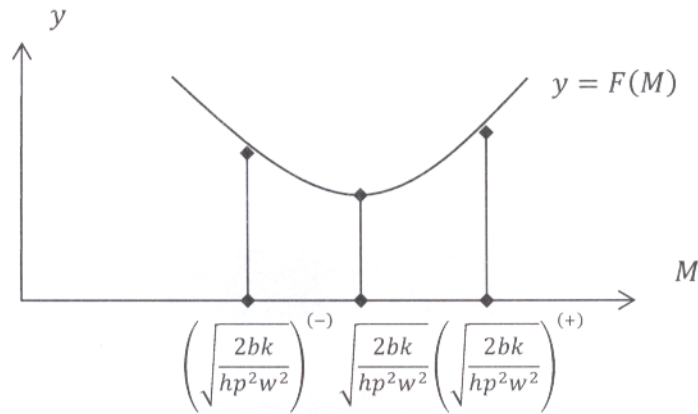
Figure1.$y = F(M)$, M is the real number diagram.

In (1) and (2), when M is a positive integer, $F(M) = L(M)$; hence, it can be easily learnt from Figure 1 that the minimal point M^* of function $L(M)$, $M \in N$ of M^* satisfies the following:

$$L(M^*) = \min\left\{L\left(\left(\sqrt{\frac{2bk}{hp^2w^2}}\right)^{(-)}\right), L\left(\left(\sqrt{\frac{2bk}{hp^2w^2}}\right)^{(+)}\right)\right\}$$

$$= \min\left\{L\left(\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1\right), L\left(\sqrt{\frac{2bk}{hp^2w^2}} + 1 - \theta_1\right)\right\}$$

Where $\theta_1 \in [0,1)$ satisfies: $\left(\sqrt{\frac{2bk}{hp^2w^2}}\right)^{(-)} + \theta_1 = \sqrt{\frac{2bk}{hp^2w^2}}$ $\qquad(7)$

By (1), (2) and (7): when $\theta_1 \in (0, 1)$,

$$L\left(\left(\sqrt{\frac{2bk}{hp^2w^2}}\right)^{(+)}\right) - L\left(\left(\sqrt{\frac{2bk}{hp^2w^2}}\right)^{(-)}\right)$$

$$= L\left(\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1 + 1\right) - L\left(\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1\right)$$

$$= \frac{bk}{pw}\left[\frac{1}{\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1 + 1} - \frac{1}{\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1}\right] + \frac{hpw}{2}$$

$$= \frac{-bk}{pw} \cdot \frac{1}{\left[\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1 + 1\right] \cdot \left[\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1\right]} + \frac{hpw}{2}$$

hence, when $\theta_1 \in (0,1)$,

$$L\left(\sqrt{\frac{2bk}{hp^2w^2}}^{(+)}\right) - L\left(\sqrt{\frac{2bk}{hp^2w^2}}^{(-)}\right) \geq 0. \tag{8}$$

$$\Leftrightarrow \left[\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1 + 1\right] \cdot \left[\sqrt{\frac{2bk}{hp^2w^2}} - \theta_1\right] \geq \frac{2bk}{hp^2w^2}$$

$$\Leftrightarrow (1 - 2\theta_1)\sqrt{\frac{2bk}{hp^2w^2}} + (1 - \theta_1)\theta_1 \geq 0 \tag{9}$$

Since, when $\theta_1 = 0$, equations (8) and (9) are true, it can be learnt from (8) and (9) that when $\theta_1 \in [0,1)$,

$$L\left(\sqrt{\frac{2bk}{hp^2w^2}}^{(+)}\right) - L\left(\sqrt{\frac{2bk}{hp^2w^2}}^{(-)}\right) \geq 0$$

$$\Leftrightarrow (1 - 2\theta_1)\sqrt{\frac{2bk}{hp^2w^2}} + (1 - \theta_1)\theta_1 \geq 0 \tag{10}$$

By (10),

when $\theta_1 \in [0, 1/2]$, equation (10) is always true; and hence, by (7), $M^* = \sqrt{\frac{2bk}{hp^2w^2}}^{(-)}$ \hfill (11)

In the given parameters b, k, h, p, w, we define function $g(\theta)$ as follows:

$$g(\theta) = (1 - 2\theta)v + (1 - \theta)\theta, \theta \in [1/2, 1], \text{ where } v = \sqrt{\frac{2bk}{hp^2w^2}} \tag{12}$$

By (12),

$g\left(\frac{1}{2}\right) = \frac{1}{4}$, $g(1) = -v < 0$, $g'(\theta) = -2v + 1 - 2\theta < 0$, $g''(\theta) = -2 < 0$, $\forall \theta \in [1/2, 1]$; hence $g(\theta)$ is a strictly decreasing concave function of $\theta, \theta \in [1/2, 1]$, with only the solution of $\theta_2, \theta_2 \in \left(\frac{1}{2}, 1\right)$ to satisfy $g(\theta_2) = 0$. (see Figure 2) \hfill (13)
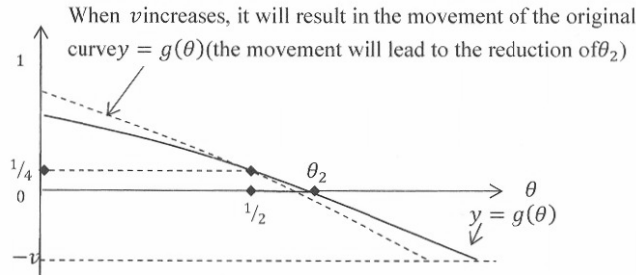
Figure 2. For the given parameters $b, k, h, p, w,$ the diagram of function $g(\theta)$.

By (10), (11) and (12), in the given parameters $b, k, h, p, w,$ the necessary and sufficient equation of $M^* = \upsilon^{(-)} = \sqrt{\dfrac{2bk}{hp^2w^2}}^{(-)}$ is $\theta_1 \le \theta_2$ where θ_1 is defined as in (7); and θ_2 is defined as in (13), namely,

$$0 = g(\theta_2) = (1 - 2\theta_2)\upsilon + (1 - \theta_2)\theta_2, \theta\_2 \in (1/2, 1) \tag{14}$$

4. SENSITIVITY ANALYSIS OF PARAMETER   CHANGE

In Deduction 1, when parameter b or k increases while h or p decreases, it will result in the increase of $\upsilon = \sqrt{\dfrac{2bk}{hp^2w^2}}$; however, M^* should be a positive integer (M^* is one of $\upsilon^{(-)}$ and $\upsilon^{(+)}$). After increasing or reducing to certain level of the parameters b, k, h, p, M^* may change. Assume parameter υ changes and record (7)'s θ_1 as $\theta_1(\upsilon)$; the θ_2 of Figure 2 will be recorded as $\theta_2(\upsilon)$.

When υ changes in the range of $[n, n+1)$, as $\upsilon = n + \theta_1(\upsilon)$, thus

$$\theta_1'(\upsilon) = \frac{d}{d\upsilon}\upsilon = 1 > 0, \theta_1''(\upsilon) = 0 \tag{14}$$

therefore, function $\theta_1(\upsilon)$ is a linear function of slope at 1 (see Figure 3).

Consider (14)'s θ_2 as $\theta_2(\upsilon)$, differentiating $\theta_2(\upsilon)$ with respect to υ, it yields

$$0 = -2\theta_2'(\upsilon)\upsilon + (1 - 2\theta_2(\upsilon)) + \theta_2'(\upsilon)(1 - 2\theta_2(\upsilon))$$

and hence,

$$\theta_2'(\upsilon) = \frac{(1 - 2\theta_2(\upsilon))}{2\theta_2(\upsilon) - 1 + 2\upsilon} < 0 \text{ (it can be learnt from (14) that } \theta_2(\upsilon) \in (1/2, 1]) \tag{15}$$

From (1) and (7), we have the following properties

(a) If $v = \sqrt{\dfrac{2bk}{hp^2w^2}} < 1$, then $M^* = \sqrt{\dfrac{2bk}{hp^2w^2}}^{(+)} = 1$ (16)

(b) If $v = \sqrt{\dfrac{2bk}{hp^2w^2}} \geq 1$, then there exists a unique positive integer n that satisfies $v \in [n, n+1)$.   (17)

In (14), assume v as $v = n + \theta_1$, and suppose $\bar{\theta}_n$ as the θ solution to (14)'s $\theta_1 = \theta_2 = \bar{\theta}_n$ (see Figure 3), i.e. $\bar{\theta}_n$ satisfies: $0 = (1 - 2\bar{\theta}_n)(n + \bar{\theta}_n) + (1 - \bar{\theta}_n)\bar{\theta}_n = -3\bar{\theta}_n^2 - 2(n-1)\bar{\theta}_n + n$, namely,

$$\bar{\theta}_n = \frac{-(n-1)+\sqrt{(n-1)^2+3n}}{3} = \frac{-(n-1)^2+[(n-1)^2+3n]}{3[(n-1)+\sqrt{(n-1)^2+3n}]} = \frac{n}{(n-1)+\sqrt{(n-1)^2+3n}} \quad (18)$$
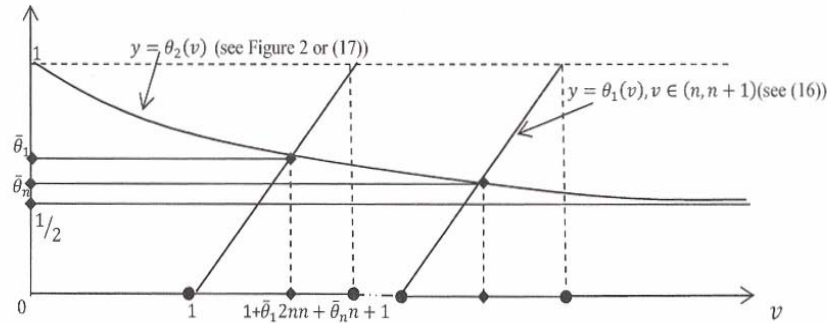

Figure 3. Function $\theta_1(v)$ and function $\theta_2(v)$ diagrams

According to (11) and Figure 3, that

(4-1). For given parameters b, k, h, p, w, and $v = \sqrt{\dfrac{2bk}{hp^2w^2}}$ which is written as $v = n_v + \theta_v$, where $\theta_v \in [0, 1]$ (namely, n_v is integer part of v, and θ_v is decimal part of v); then,

$$\begin{cases} \text{If } \theta_v \leq \dfrac{n_v}{(n_v-1)+\sqrt{(n_v-1)^2+3n_v}}, \text{then the optimal solution } M^*(v) \text{ of (1) is } n_v \\ \quad (\text{i.e. } M^*(v) \text{ is to truncate the decimals of } v \text{ to obtain integer } n_v) \\ \\ \text{If } \theta_v > \dfrac{n_v}{(n_v-1)+\sqrt{(n_v-1)^2+3n_v}}, \text{then the optimal solution } M^*(v) \text{ of (1) is } n_v + 1 \\ \quad (\text{i.e. } M^*(v) \text{ is to carry the decimal of } v \text{ to obtain integer } n_v + 1) \end{cases} \quad (19)$$

(4-2). By (18),

$$\bar{\theta}_1 = \frac{1}{\sqrt{3}} \geq \bar{\theta}_2 = \frac{1}{1+\sqrt{7}} \geq \bar{\theta}_3 = \frac{1}{2+\sqrt{13}} \ldots \geq \lim_{n\to\infty} \bar{\theta}_n = \frac{1}{2} = 0.5 \quad (20)$$

(4-3).When $v = \bar{v}_n$, $M^*(v) = n_v$ or $M^*(v) = n_v + 1$, namely, $L(M^*(v)) = L(n_v) = L(n_v + 1)$.

(4-4).When $v \in [n, n + \bar{\theta}_n]$, in (4-1) the removal of the decimal part of v gets M^*; when $v \in [n + \theta_n, n + 1)$,in (4-2), the addition of the decimal part of v gets M^*. By (18), we know that key point, $\bar{\theta}_n$ decreases with the increase of n , and $\bar{\theta}_n \in (1/2 , 1/\sqrt{3}] \cong (0.5, 0.577], \forall v$.

5. CONCLUSIONS

This study made a number of findings.
(1) This paper developed the purchase amount and EOQ problem of delivering goods to the location of sales by continuous transportation into a model that could be concretely discussed (see Model 1).
(2) When the freight vehicles are required to transport a full load under the condition of a full trip, the optimal purchase amount of each cycle proposed by this model is M^*wp, in which M^*can be obtained by Deduction 1 and (19), (See Figure 1).
(3) In the given parameters b, k, h, p, w, the optimal number of rented freight vehicles is $M^*(v)$, and the relationship of v, $v = \sqrt{\frac{2bk}{hp^2w^2}}$ and $M^*(v)$are as shown in (19).
(4) The key point $\bar{\theta}_v$ for optimal solution $M^*(v)$ after truncation should satisfy the condition of $\bar{\theta}_v \in [1/2 , 1/\sqrt{3}), \forall n$. The key point $\bar{\theta}_v$ which truncates the decimals of v into the optimal solution $M^*(v)$, decreases with the increase of v. This indicates that when the demand rate for goods b increases, purchase cost k, unit inventory cost h decreases, the upper limit for the load of each vehicle p decreases, and the upper limit for back and forth transportation w decreases, the integer part, n of $v = \sqrt{\frac{2bk}{hp^2w^2}}$ (where, $n = v^{(-)}$) increases, causing a decrease of $\overline{\theta}_n$.
(5) In the respect of practical applications, inventory decision makers may summarize the results of (19) into a table (see Table 1) or write them into a software program to be used by freight vehicle dispatchers. After determining the values of various parameters, the corresponding v values, the integer parts $n = v^{(-)}$, and the decimal parts θ_v can be obtained.

Table 1. The relation between n and $\bar{\theta}_n$

$n = v^{(-)}$	1	2	3	4	5	6	7	8	9	~	∞
$\bar{\theta}_n \cong$	0.577	0.549	0.535	0.528	0.523	0.519	0.517	0.515	0.513	~	0.50

Numerical cases

Case 1.If v=8.944, then $n = v^{(-)} = 8$, $\theta_v = 0.944 > \bar{\theta}_8 \cong 0.515$; therefore, $M^* = n + 1 = 8 + 1 = 9$.

Case 2: If v=7.303, then $n = v^{(-)} = 7$, $\theta_v = 0.303 \leq \bar{\theta}_\infty = 0.5$; therefore, $M^* = n = 7$.

Using Table 1, the freight vehicle dispatchers at the local level can quickly find the optimal number of rented vehicles.

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