

## **A SINGLE-SERVER MARKOVIAN QUEUING SYSTEM WITH DISCOURAGED ARRIVALS AND RETENTION OF RENEGED CUSTOMERS**

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**Abstract:** Customer impatience has a very negative impact on the queuing system under investigation. If we talk from business point of view, the firms lose their potential customers due to customer impatience, which affects their business as a whole. If the firms employ certain customer retention strategies, then there are chances that a certain fraction of impatient customers can be retained in the queuing system. A reneged customer may be convinced to stay in the queuing system for his further service with some probability, say  $q$  and he may abandon the queue without receiving the service with a probability  $p(=1-q)$ . A finite waiting space Markovian single-server queuing model with discouraged arrivals, renegeing and retention of reneged customers is studied. The steady state solution of the model is derived iteratively. The measures of effectiveness of the queuing model are also obtained. Some important queuing models are derived as special cases of this model.

**Keyword:** Probability of Customer Retention, Reneging, Discouraged arrivals, Cost-Profit Analysis.

**MSC:** 60K25-68M20-90B22.

## 1. INTRODUCTION

Queuing theory plays an important role in modeling real life problems involving congestions in wide areas of applied sciences. Applications of queuing with impatience can be seen in traffic modeling, business and industries, computer-communication, health sectors and medical sciences etc.

Queues with discouraged arrivals have applications in computers with batch job processing where job submissions are discouraged when the system is used frequently and arrivals are modeled as a Poisson process with state dependent arrival rate. The discouragement affects the arrival rate of the queuing system. Morse [11] considers discouragement in which the arrival rate falls according to a negative exponential law. We consider a single-server queuing system in which the customers arrive in a Poisson fashion with rate depending on the number of customers present in the system at that time i.e.  $\frac{\lambda}{(n+1)}$ .

Queuing with customer impatience has vast applications in computer-communications, bio- medical modeling, service systems etc. It is important to note that the prevalence of the phenomenon of customer impatience has a very negative impact on the queuing system under investigation. If we talk from business point of view, the firms lose their potential customers due to customer impatience, which affects the business of firms as a whole. If firms employ certain customer retention strategies, then there are chances that a certain fraction of impatient customers can be retained in the queuing system. An impatient customer (due to reneging) may be convinced to stay in the service system for his service by utilizing certain convincing mechanisms. Such customers are termed as retained customers. When a customer gets impatient (due to reneging), he may leave the queue with some probability, say  $p$  and may remain in the queue for service with the probability  $p(=1-q)$ .

Taking these concepts into consideration, a single-server finite capacity Markovian queuing model with discouraged arrivals, reneging and retention of reneged customers is developed. The steady-state solution of the model is derived.

Rest of the paper is structured as follows: In section 2, the literature review is presented. In section 3, queuing model is formulated. The differential-difference equations of the model are derived and solved iteratively in section 4. Measures of effectiveness are derived in section 5. Some queuing models are derived as special cases of this model in section 6. The conclusions are presented in section 7.

## 2. LITERATURE REVIEW

Customer impatience has become the burning problem of private as well as government sector enterprises. Queuing with reneging is firstly studied by Haight [6]. He studies the problem like how to make rational decision while waiting in the queue, the probable effect of this decision etc. Ancker and Gafarian [1] study  $M/M/1/N$  queuing system with balking and reneging, and perform its steady state analysis. Ancker and Gafarian [2] also obtain results for a pure balking system (no reneging) by setting the reneging parameter equal to zero. Multi-server queuing systems with customer

impatience find their applications in many real life situations such as in hospitals, computer-communication, retail stores etc. Xiong and Altiook [16] study multi-server queues with deterministic reneging times with reference to the timeout mechanism used in managing application servers in transaction processing environments. Wang et al. [15] present an extensive review on queuing systems with impatient customers.

Kapodistria [7] studies a single server Markovian queue with impatient customers and considers the situations where customers abandon the system simultaneously. He considers two abandonment scenarios. In the first one, all present customers become impatient and perform synchronized abandonments; while in the second scenario, the customer in service is excluded from the abandonment procedure. He extends this analysis to the M/M/c queue under the second abandonment scenario also. Kumar [8] investigates a correlated queuing problem with catastrophic and restorative effects with impatient customers which have special applications in agile broadband communication networks. Kumar and Sharma [9] apply M/M/1/N queuing model for modeling supply chain situations facing customer impatience. Queuing models where potential customers are discouraged by queue length are studied by many researchers in their research work. Natvig [12] studies the single server birth-death

queuing process with state dependent parameters  $\lambda_n = \frac{1}{n+1}\lambda, n \geq 0$  and  $\mu_n = \mu, n \geq 1$ . He

reviews state dependent queuing models of different kind and compares his results with M/M/1, M/D/1 and D/M/1 and the single server birth-and-death queuing model with parameters  $\lambda_n = \lambda, n \geq 0$  and  $\mu_n = n\mu, n \geq 1$  numerically. Raynolds [13] presents multi-server queuing model with discouragement. He obtains equilibrium distribution of queue length and derives other performance measures from it. Cuortois and Georges [4] study finite capacity M/G/1 queuing model where the arrival and the service rates are arbitrary functions of the current number of customers in the system. They obtain results for expected value of time needed to complete a service including waiting time distribution and limited probability distribution of the congestion. Hadidi [5] carries out analysis of busy period processes for M/M<sub>n</sub>/1 and M<sub>n</sub>/M/1 queuing models with state dependent service and arrival rates. He also obtains results for busy period and transient state probabilities. Von Doorn [14] obtains exact expressions for transient state probabilities of

the birth death process with parameters  $\lambda_n = \frac{1}{n+1}\lambda, n \geq 0$  and  $\mu_n = \mu, n \geq 1$ . Ammar et al.

[3] study single server finite capacity Markovian queue with discouraged arrivals and reneging using matrix method.

The above-mentioned queuing systems deal with customer impatience and discouragement only. Different extensions of customer impatience in single-server and multi-server queues are carried out here. Furthermore, customer impatience has highly negative impact on the business of any firm as it leads to loss of potential customers. Keeping into mind the negative impact of customer impatience, the novel concept of the retention of renege customers with discouraged arrivals is studied in this paper.

### 3. QUEUING MODEL FORMULATION

In this section, we formulate the queuing model. The Markovian queuing model investigated in this paper is based on the following assumptions:

1. We consider a single-server queuing system in which the customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time i.e.  $\frac{\lambda}{(n+1)}$ .
2. The service times are independently, identically and exponentially distributed with parameter  $\mu$ .
3. The customers are served in order of their arrival.
4. The capacity of the system is finite (say,  $N$ ).
5. Each customer upon joining the queue will wait a certain length of time for his service to begin. If it does not begin by then, he will get impatient (reneged) and may leave the queue without getting service with probability  $p$ , and may remain in the queue for his service with probability  $q(=1-p)$ . The renegeing times follow exponential distribution with parameter  $\xi$ .

### 4. DIFFERENTIAL DIFFERENCE EQUATIONS AND SOLUTION OF THE QUEUING MODEL

Let  $P_n(t)$  be the probability that there are  $n$  customers in the system at time  $t$ .

The differential-difference equations are derived by using the general birth-death arguments. These equations are solved iteratively in steady-state in order to obtain the steady state solution.

The differential-difference equations of the model are:

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t) + \mu P_1(t) \quad (1)$$

$$\begin{aligned} \frac{d}{dt} P_n(t) = & - \left[ \left( \frac{\lambda}{n+1} \right) + \mu + (n-1)\xi p \right] P_n(t) \\ & + (\mu + n\xi p) P_{n+1}(t) + \left( \frac{\lambda}{n} \right) P_{n-1}(t); \quad n = 1, 2, 3, \dots, N-1 \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d}{dt} P_N(t) = & - [\mu + (N-1)\xi p] P_N(t) \\ & + \left( \frac{\lambda}{N} \right) P_{N-1}(t) \end{aligned} \quad (3)$$

In steady state,  $\lim_{t \rightarrow \infty} P_n(t) = P_n$  and therefore,  $\frac{dP_n(t)}{dt} = 0$  as  $t \rightarrow \infty$  and hence, the solution of equations (1) to (3) gives the difference equations

$$0 = -\lambda P_0 + \mu P_1 \quad (4)$$

$$0 = -\left[ \left( \frac{\lambda}{n+1} \right) + \mu + (n-1)\xi p \right] P_n + (\mu + n\xi p) P_{n+1} + \left( \frac{\lambda}{n} \right) P_{n-1}; \quad n = 1, 2, 3, \dots, N-1 \quad (5)$$

$$0 = -(\mu + (N-1)\xi p) P_N + \left( \frac{\lambda}{N} \right) P_{N-1} \quad (6)$$

Solving iteratively equations (4) – (6), we get

$$P_n = \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0; \quad 1 \leq n \leq N \quad (7)$$

Using the normalization condition,  $\sum_{n=0}^N P_n = 1$ , we get

$$P_0 = \frac{1}{1 + \sum_{n=1}^N \left( \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right)} \quad (8)$$

Hence, the steady-state probabilities of the system size are derived explicitly.

## 5. MEASURES OF EFFECTIVENESS

In this section, some important measures of effectiveness are derived. These can be used to study the performance of the queuing system under consideration.

### The Expected System Size (L<sub>s</sub>)

$$L_s = \sum_{n=1}^N n \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0$$

### The Expected Queue Length (L<sub>q</sub>)

$$L_q = \left[ \sum_{n=1}^N n \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0 - \frac{\lambda}{\mu} \right]$$

**The Expected Waiting Time in the System (Ws)**

$$W_s = \left[ \frac{1}{\lambda} \sum_{n=1}^N n \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] \right] P_0$$

**The Expected Waiting Time in the queue (Wq)**

$$W_q = \left[ \frac{1}{\lambda} \sum_{n=1}^N n \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0 - \frac{1}{\mu} \right]$$

**The Expected Number of customers Served, E(Customer Served)**

$$E(\text{Customer Served}) = \sum_{n=1}^N n \mu P_n$$

$$E(\text{Customer Served}) = \sum_{n=1}^N n \mu \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0$$

**Rate of Abandonment,  $R_{abond}$** 

$$R_{abond} = \lambda \sum_{n=0}^N P_n - E(\text{Customer Served})$$

$$R_{abond} = \lambda - \sum_{n=1}^N n \mu \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0$$

**Expected number of waiting customers, who actually wait,  $E(\text{Customer Waiting})$** 

$$E(\text{Customer Waiting}) = \frac{\sum_{n=2}^N (n-1) P_n}{\sum_{n=2}^N P_n}$$

$$E(\text{Customer Waiting}) = \frac{\sum_{n=2}^N (n-1) \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0}{\sum_{n=2}^N \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0}$$

**Probability distribution of busy period, Prob (Busy period)**

$$\text{Prob}(\text{Busy period}) = \text{Prob. } (n \geq 1)$$

$$\text{Prob}(\text{Busy period}) = \sum_{n=1}^N \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right] P_0$$

Where  $P_0$  has been computed in (8).

## 6. SPECIAL CASES

### When there is no retention of renege customers ( i. e. $q = 0$ ).

The queuing system is reduced to a system with discouraged arrivals and renegeing with

$$P_n = \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi} \right] P_0; 1 \leq n \leq N$$

Using the normalization condition,  $\sum_{n=0}^N P_n = 1$ , we get

$$P_0 = \frac{1}{1 + \sum_{n=1}^N \left[ \frac{1}{n!} \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi} \right]}$$

### When there is no discouragement.

We study two sub-cases:

- (i) The model reduces to an M / M /1 / N queuing system with retention of renege customers as studied by Kumar and Sharma [10] with

$$P_n = \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} P_0; 1 \leq n \leq N-1$$

Also for  $n = N$  we get

$$P_N = \prod_{k=1}^N \frac{\lambda}{\mu + (k-1)\xi p} P_0$$

Using the normalization condition,  $\sum_{n=0}^N P_n = 1$ , we get

$$P_0 = \frac{1}{1 + \sum_{n=1}^N \left[ \prod_{k=1}^n \frac{\lambda}{\mu + (k-1)\xi p} \right]}$$

- (ii) When there is no renegeing (i.e. the customers do not get impatient).

In this case, the probability of renegeing ( $p$ ) is zero, implies that  $\xi = 0$ . As there is no renegeing, there is no question of customer retention. All the customers who enter into the system leave after getting service. Therefore, from equations (7) and (8) it follows that

$$P_n = \left( \frac{\lambda}{\mu} \right)^n P_0; 1 \leq n \leq N$$

and using the normalization condition, we get

$$P_0 = \frac{1}{1 + \sum_{n=0}^N \left(\frac{\lambda}{\mu}\right)^n}$$

It is evident that the model reduces to a simple M/M/1/N queuing model.

## 7. CONCLUSIONS

This paper studies a single server queuing model with discouraged arrivals, reneging and retention of renege customers. We obtain the steady-state solution and different measures of effectiveness are also derived. Some queuing models are derived as special cases of this model.

The model analysis is limited to finite capacity. The infinite capacity case of the model can also be studied. Further, the model can be solved in transient state to get time-dependent results. The cost-profit analysis of the model can also be carried to study its economic analysis. The same idea can be extended to some non-Markovian queuing models.

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