Yugoslav Journal of Operations Research
24 (2014) Number 1, 127-143
DOI: 10.2298/YJOR120627014C

GEOM/GEOM[A]/1/ QUEUE WITH LATE ARRIVAL SYSTEM WITH DELAYED ACCESS AND DELAYED MULTIPLE WORKING VACATIONS

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Received: June 2012 / Accepted: April 2013

Abstract: This paper considers a discrete-time bulk-service queue with infinite buffer space and delay multiple working vacations. Considering a late arrival system with delayed access (LAS-AD), it is assumed that the inter-arrival times, service times, vacation times are all geometrically distributed. The server does not take a vacation immediately at service complete epoch but keeps idle period. According to a bulk-service rule, at least one customer is needed to start a service with a maximum serving capacity 'a'. Using probability analysis method and displacement operator method, the queue length and the probability generating function of waiting time at pre-arrival epochs are

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[1] This work is supported by the National Natural Science Foundation of China (No.71171138) and the Talent Introduction Foundation of Sichuan University of Science & Engineering (2012RC23).
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obtained. Furthermore, the outside observer's observation epoch queue length distributions are given. Finally, computational examples with numerical results in the form of graphs and tables are discussed.

Keywords: Discrete time, bulk-service, working vacations, queue, waiting time distribution.

MSC: 60J05.

1. INTRODUCTION

Discrete-time queues with server's vacation have been studied extensively and applied in manufacturing system, telecommunications network and switching systems, etc. In the past, several discrete time queueing models with server vacation (single or multiple) have been investigated by many researchers, and a considerable amount of work has been done. The work related to $Geom/G/1$ queue (including batch arrivals) with various vacation policies can be found in the book by Takagi (1993). Analysis of the $Geom/G/1$ queue with multiple adaptive vacations and the $GI/Geo/1$ queue with multiple vacations are carried out by Zhang and Tian (2001, 2002). The $Geo^X/G/1$ queue with multiple vacations is studied by Fiems and Bruneel (2002). Using the matrix-analytic method, Alfa (2003) analyzed a class of discrete-time vacation models in which distributions of inter-arrival times, service times, vacation times and operational times are of phase type. The $D-MAP/PH/1$ queue with vacations and exhaustive time-limited service has been studied by Alfa (1995). All the aforementioned studies have been carried out by assuming infinite buffer capacities. Simultaneously, some researches on the finite buffer $Geo/G/1/N$ vacation queues can be found in Takagi (1993).

Servi and Finn (2002) studied an $M/M/1$ queuing model with a new type of vacation policy called a working vacation policy. That is, the server does not completely stop serving the customers during a vacation period but it serves customers with a lower rate than in a normal busy period. Wu and Takagi (2006) extended this work to $M/G/1/WV$ model with generally distributed service times as well as vacation durations. Baba (2005) considered the $GI/M/1/WV$ system with the distribution of the vacation duration having an exponential distribution. And, the finite buffer model $GI/M/1/N/WV$ is presented by Banik et al. (2007) with multiple working vacations policy.

Similarly, in the discrete-time counterpart of the $M/M/1/WV$ case, by using quasi-birth-death process and matrix-geometric solution method, Tian et al. (2007) analyzed the $Geom/Geom/1/WV$ queue with geometrically distributed vacation. Subsequently, Li et al.(2007) investigated the $GI/Geo/1$ queue with multiple working vacations in which the vacation time follows geometric distribution. They obtained some stationary distributions and stochastic decomposition properties.

Though the working vacation queues have received wide attention with the rule that the server serves customers singly, many a time there is also a need for bulk-service rules. Yu et al. (2009) considered a finite capacity and bulk-arrival and bulk-service continuous-time queuing system with server working vacations. Vijaya Laxmi (2011) studied a renewal input infinite buffer batch service queue with single exponential working vacation and accessibility to batches. Goswami (2011) investigated a discrete-time batch service renewal input queue with multiple working vacations.

In papers [13-15], the authors assume that the server takes a vacation immediately at a service completion epoch or at a vacation completion epoch. Assuming that the server takes vacation immediately at a service completion epoch, in a late arrival system with delayed access where customers are served depart the service completion epoch in (n, n^{+}), some new customer may arrive in $((n+1)^{-}, n+1)$ due to the very short interval, may happen that the server had hardly left the system when the customers arrived. In this case, degree of satisfaction of customers for the system may decrease and even lead to loss of profit. Similarly, in continuous time queue such as [16], the author assumes that the server takes a vacation immediately at service completion, which will cause a loss to the system, too.

This paper studies a discrete-time bulk-service LAS-DA queuing system with server working vacations. Assume that the server remains dormant between the service completion epoch in (n, n^{+}) and the next arrival epoch in $((n+1)^{-}, n+1)$. If some customers arrive in $((n+1)^{-}, n+1)$, the dormant period will last until the beginning of the epoch of service in $(n+1, (n+1)^{+})$. Otherwise, the server takes a vacation at time $n+1$ immediately. The start and the completion of the vacation happens at time n. On the completion of vacation, if no customers are waiting for service in the system, the server takes another vacation immediately. Application of a probability analysis method is carried out to analyze the queue length and the probability generating function of waiting time at pre-arrival epoch. Furthermore, the queue length distributions of outside observer's observation epoch are given. Finally, computational examples with a variety of numerical results in the form of graphs and tables are discussed.

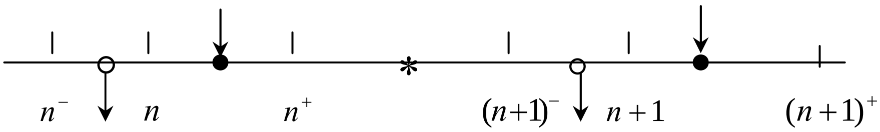
The rest of the paper is arranged as follows. In the next section, the model of the considered queuing system is described. In section 3, the stationary distribution of queue length at pre-arrival epoch is discussed. In section 4, we study the waiting time distribution. In section 5, we discuss the queue length distributions of outside observer's observation epoch. In section 6, some numerical results and the sensitivity analysis of this system are given.

2. SYSTEM DESCRIPTION

We consider a discrete-time bulk-service infinite buffer space queuing system with server delayed multiple working vacations according to the rule of LAS-DA. Assume that the time axis is slotted into intervals of equal length with the length of a slot being unity, marked as 0, 1, 2, …, n, … . A potential arrival occurs in the interval (n^{-}, n) and potential batch-departures occur in (n, n^{+}). The inter-arrival times T of customers are independent and geometrically distributed with probability mass function (p.m.f.) $P\{T = k\} = p\bar{p}^{k-1}, k \geq 1, \bar{p} = 1 - p$.

The customers are served in batches of variable capacity, the maximum service capacity for the server being $a (a \geq 1)$. Service times S_b during normal busy period and service times S_v during a working vacation are assumed to be independent and geometrically distributed with p.m.f. $P\{S_b = k\} = \mu_b \bar{\mu}_b^{k-1}, k \geq 1, \bar{\mu}_b = 1 - \mu_b$ and p.m.f.

$P\{S_v = k\} = \mu_v \bar{\mu}_v^{k-1}, k \geq 1, \bar{\mu}_v = 1 - \mu_v$, respectively. Assume that the server remains dormant between the service completion epoch in (n, n^+) and the next arrival epoch in $((n+1)^-, n+1)$. If some customers arrive in $((n+1)^-, n+1)$, the dormant period will last until the beginning of the epoch of service in $(n+1, (n+1)^+)$. Otherwise, the server takes a vacation at time $n+1$ immediately. The start and completion of the vacation happen at time n. The working vacation time V follows a geometric distribution with parameter $\theta (0 < \theta < 1)$ and its p.m.f. is $P\{V = k\} = \theta \bar{\theta}^{k-1}, k \geq 1, \bar{\theta} = 1 - \theta$. On the completion of vacation, if no customers are waiting for service in the system, the server takes another vacation immediately. If there are some customers being served after the server finishes a vacation, the service interrupted at the end of a vacation is lost, and it is restarted with service rate μ_b at the beginning of the following service period, which means that the normal busy period starts. The various time epochs at which events occur are depicted in Figure 1.



○ : Potential arrival epoch; ● : Potential batch-departure epoch; * : Outside observer's epoch;

$(n^+, (n+1)^-)$: Outside observer's interval; n^- : Epoch before a potential arrival;

n^+ : Epoch after a potential batch-departure;

Figure 1. various time epochs in LAS-DA

3. THE QUEUE LENGTH AT PRE-ARRIVAL EPOCH

When the system becomes empty, let $Q_{0,0}(n^-)$ denote the probability that the server is on vacation and no customers are waiting in the system at time n^-. Let $Q_{0,1_0}(n^-)$ denote the probability that the server is idle and no customers are waiting in the system at time n^-. During a working vacation, let $Q_{k,0_1}(n^-)$ be the probability that the server is on vacation and $k(k \geq 0)$ customers are waiting in the queue (excluding the one in service). Further, let $Q_{k,1}(n^-)$ be the probability that the server is on normal busy period and $k(k \geq 0)$ customers are waiting in queue (excluding the one in service). Define the steady-state probability as follows:

$\tilde{\pi}_{0,0} = \lim_{n^- \to \infty} Q_{0,0}(n^-)$; $\tilde{\pi}_{0,1_0} = \lim_{n^- \to \infty} Q_{0,1_0}(n^-)$; $\tilde{\pi}_{k,0_1} = \lim_{n^- \to \infty} Q_{k,0_1}(n^-)$, $k \ge 0$; $\tilde{\pi}_{k,1} = \lim_{n^- \to \infty} Q_{k,1}(n^-)$, $k \ge 0$. We have

Theorem 1: If $\rho_0 = p / a\mu_b < 1$ *,* $\rho_1 = p / a\mu_v < 1$ *, we get*

1) $\tilde{\pi}_{0,0} = \bar{\mu}_v \omega_0 \tilde{\pi}_{0,0_1}$, $\tilde{\pi}_{k,0_1} = \tilde{\pi}_{0,0_1}\xi^k$, $\tilde{\pi}_{0,1_0} = \dfrac{\mu_v}{\gamma\beta}(\beta\omega_0 - \gamma)\tilde{\pi}_{0,0_1}$,

2) $\tilde{\pi}_{0,1} = \dfrac{\mu_v}{\gamma\beta\bar{p}\mu_b}(\beta\omega_0 - \gamma)\tilde{\pi}_{0,0_1}$, $\tilde{\pi}_{k,1} = c_0' r^k + c''\xi^{k-1} (k \ge 1)$,

where

$$c_0' = \frac{1}{p}\{\frac{(1-\bar{p}\bar{\mu}_b - p\bar{p}\mu_b)\mu_v[\beta\omega_0 - \gamma]}{\gamma\beta\bar{p}\mu_b} - \frac{\mu_b(\beta-1)(\bar{p}\xi + p)(p + \bar{p}r - p\xi^a - \bar{p}r^{a+1})}{\beta(\xi - \omega_1)(1-r)}$$
$$-\frac{(\beta-1)[p\bar{\mu}_v\omega_0 + \bar{p}]}{\beta}\}\tilde{\pi}_{0,0_1} , \beta = \frac{1}{\bar{\theta}} , \gamma = \frac{\bar{p}\mu_v}{p\bar{\mu}_v} , \ \omega_0 = \gamma + \frac{\beta - 1}{p\bar{\mu}_v(1-\xi)} ,$$

$$\omega_1 = p\bar{\mu}_b + \bar{p}\bar{\mu}_b\xi + p\mu_b\xi^a + \bar{p}\mu_b\xi^{a+1} , c'' = \frac{(\beta-1)(\bar{p}\xi + p)\xi\tilde{\pi}_{0,0_1}}{\beta(\xi - \omega_1)} ,$$

$$\tilde{\pi}_{0,0_1} = \{\frac{r}{p(1-r)}\{\frac{(1-\bar{p}\bar{\mu}_b - p\bar{p}\mu_b)\mu_v(\beta\omega_0 - \gamma)}{\gamma\beta\bar{p}\mu_b} - \frac{\mu_b(\beta-1)(\bar{p}\xi + p)(p + \bar{p}r - p\xi^a - \bar{p}r^{a+1})}{\beta(\xi - \omega_1)(1-r)}$$

$$-\frac{(\beta-1)(p\bar{\mu}_v\omega_0 + \bar{p})}{\beta}\} + \bar{\mu}_v\omega_0 + \frac{1}{1-\xi} + \frac{\mu_v(1+\bar{p}\mu_b)(\beta\omega_0 - \gamma)}{\gamma\beta\bar{p}\mu_b} + \frac{(\beta-1)(\bar{p}\xi + p)\xi}{\beta(\xi - \omega_1)(1-\xi)}\}^{-1} .$$

ξ is the root of the equation $\bar{p}\mu_v\bar{\theta}z^{a+1} + p\mu_v\bar{\theta}z^a - (1-\bar{p}\bar{\mu}_v\bar{\theta})z + p\bar{\mu}_v\bar{\theta} = 0$, which is less than 1 and greater than 0. r is the root of the equation $\bar{p}\mu_b z^{a+1} + p\mu_b z^a + p\bar{\mu}_b - (1-\bar{p}\bar{\mu}_b)z = 0$, which is less than 1 and greater than 0.

Proof. In order to obtain the steady-state probability, we first construct the difference equations by relating the states of the system at two consecutive prior to potential arrival epochs n^- *and* $(n+1)^-$ *. Using the probabilistic argument, we obtain*

$$Q_{0,0}((n+1)^-) = \bar{p}Q_{0,0}(n^-) + \bar{p}\mu_v\bar{\theta}Q_{0,0_1}(n^-) + \bar{p}Q_{0,1_0}(n^-) , \tag{1}$$

$$Q_{0,0_1}((n+1)^-) = p\bar{\theta}Q_{0,0}(n^-) + \bar{p}\bar{\mu}_v\bar{\theta}Q_{0,0_1}(n^-) + \bar{p}\mu_v\bar{\theta}\sum_{i=1}^{a} Q_{i,0_1}(n^-) + p\mu_v\bar{\theta}\sum_{i=1}^{a} Q_{i-1,0_1}(n^-) \tag{2}$$

$$Q_{k,0_1}((n+1)^-) = \bar{p}\bar{\mu}_v\bar{\theta}Q_{k,0_1}(n^-) + p\bar{\mu}_v\bar{\theta}Q_{k-1,0_1}(n^-) + \bar{p}\mu_v\bar{\theta}Q_{k+a,0_1}(n^-) + p\mu_v\bar{\theta}Q_{k+a-1,0_1}(n^-), (k \ge 1), \tag{3}$$

$$Q_{0,1}((n+1)^-) = p\theta Q_{0,0}(n^-) + \bar{p}\theta Q_{0,0_1}(n^-) + \bar{p}\bar{\mu}_b Q_{0,1}(n^-) + \bar{p}\mu_b\sum_{i=1}^{a} Q_{i,1}(n^-) + p\mu_b\sum_{i=1}^{a} Q_{i-1,1}(n^-) + pQ_{0,1_0}(n^-), \tag{4}$$

$$Q_{k,1}((n+1)^-) = \bar{p}\theta Q_{k,0_1}(n^-) + p\theta Q_{k-1,0_1}(n^-) + \bar{p}\bar{\mu}_b Q_{k,1}(n^-) + p\bar{\mu}_b Q_{k-1,1}(n^-) + \bar{p}\mu_b Q_{k+a,1}(n^-) + p\mu_b Q_{k+a-1,1}(n^-), (k \ge 1), \tag{5}$$

$$Q_{0,1_0}((n+1)^-) = \bar{p}\mu_b Q_{0,1}(n^-) \tag{6}$$

In the steady state, the above Eqs. (1)- (6) reduce to

$$\tilde{\pi}_{0,0} = \bar{p}\tilde{\pi}_{0,0} + \bar{p}\mu_v\bar{\theta}\tilde{\pi}_{0,0_1} + \bar{p}\pi_{0,1_0}, \tag{7}$$

$$\tilde{\pi}_{0,0_1} = p\bar{\theta}\tilde{\pi}_{0,0} + \bar{p}\bar{\mu}_v\bar{\theta}\tilde{\pi}_{0,0_1} + \bar{p}\mu_v\bar{\theta}\sum_{i=1}^{a}\tilde{\pi}_{i,0_1} + p\mu_v\bar{\theta}\sum_{i=1}^{a}\tilde{\pi}_{i-1,0_1}, \tag{8}$$

$$\tilde{\pi}_{k,0_1} = \bar{p}\bar{\mu}_v\bar{\theta}\tilde{\pi}_{k,0_1} + p\bar{\mu}_v\bar{\theta}\tilde{\pi}_{k-1,0_1} + \bar{p}\mu_v\bar{\theta}\tilde{\pi}_{k+a,0_1} + p\mu_v\bar{\theta}\tilde{\pi}_{k+a-1,0_1}, \tag{9}$$

$$\tilde{\pi}_{0,1} = p\theta\tilde{\pi}_{0,0} + \bar{p}\theta\tilde{\pi}_{0,0_1} + \bar{p}\bar{\mu}_b\tilde{\pi}_{0,1} + \bar{p}\mu_b\sum_{i=1}^{a}\tilde{\pi}_{i,1} + p\mu_b\sum_{i=1}^{a}\tilde{\pi}_{i-1,1} + p\tilde{\pi}_{0,1_0}, \tag{10}$$

$$\tilde{\pi}_{k,1} = \bar{p}\theta\tilde{\pi}_{k,0_1} + p\theta\tilde{\pi}_{k-1,0_1} + \bar{p}\bar{\mu}_b\tilde{\pi}_{k,1} + p\bar{\mu}_b\tilde{\pi}_{k-1,1} + \bar{p}\mu_b\tilde{\pi}_{k+a,1} + p\mu_b\tilde{\pi}_{k+a-1,1}, \tag{11}$$

$$\tilde{\pi}_{0,1_0} = \bar{p}\mu_b\tilde{\pi}_{0,1}. \tag{12}$$

According to the characteristic of differential equations let $\tilde{\pi}_{k+j,0_1} = E^j\tilde{\pi}_{k,0_1}$ Spiegel (1971), $j \in Z, k = 0,1,2,\cdots$, where E denote difference operator. Substituting it into (9), we obtain

$$p\bar{\mu}_v\bar{\theta}E^{-1}\tilde{\pi}_{k,0_1} + \bar{p}\mu_v\bar{\theta}E^a\tilde{\pi}_{k,0_1} + p\mu_v\bar{\theta}E^{a-1}\tilde{\pi}_{k,0_1} - (1-\bar{p}\bar{\mu}_v\bar{\theta})\tilde{\pi}_{k,0_1} = 0.$$

The characteristic equation associated with the above equation is given by

$$\frac{\bar{p}\mu_v\bar{\theta}}{1-\bar{p}\bar{\mu}_v\bar{\theta}}z^{a+1} + \frac{p\mu_v\bar{\theta}}{1-\bar{p}\bar{\mu}_v\bar{\theta}}z^a + \frac{p\bar{\mu}_v\bar{\theta}}{1-\bar{p}\bar{\mu}_v\bar{\theta}} - z = 0. \tag{13}$$

Let $f(z) = \frac{\bar{p}\mu_v\bar{\theta}}{1-\bar{p}\bar{\mu}_v\bar{\theta}}z^{a+1} + \frac{p\mu_v\bar{\theta}}{1-\bar{p}\bar{\mu}_v\bar{\theta}}z^a + \frac{p\bar{\mu}_v\bar{\theta}}{1-\bar{p}\bar{\mu}_v\bar{\theta}}$ and $g(z) = z$.

Using Rouché's theorem, it can be shown that there is only one real zero root that falls in the unit circle (Note: the root must be the real root, otherwise there are at least two roots that fall in the unit circle. This is because the imaginary roots of an equation appear in pairs.). We denote this root by $\xi(0<\xi<1)$ and the other a roots by ξ_i , $|\xi_i|\ge 1$ $(i=1,2,3,\cdots,a)$. So ξ satisfies $f(\xi)-g(\xi)=0$. Therefore, the solution of (13) can be written as

$$\tilde{\pi}_{k,0_1}=c_0\xi^k+\sum_{i=1}^{a}c_i\xi_i^k\ ,k\ge 0\ .$$

Since $c_i(i=1,2,3,\cdots,a)=0$ (Otherwise, the probability $\tilde{\pi}_{k,0_1}$ tends to ∞ when k tends to ∞),
we get $\tilde{\pi}_{k,0_1}=c_0\xi^k$. Let $k=0$, we get $c_0=\tilde{\pi}_{0,0_1}$, then

$$\tilde{\pi}_{k,0_1}=\tilde{\pi}_{0,0_1}\xi^k\ . \tag{14}$$

Substituting (14) into (8), we obtain

$$\tilde{\pi}_{0,0}=\bar{\mu}_v\omega_0\tilde{\pi}_{0,0_1}\ , \tag{15}$$

where $\beta=\dfrac{1}{\theta}$, $\gamma=\dfrac{\bar{p}\mu_v}{p\bar{\mu}_v}$.
Substituting (15) into (7), we have

$$\tilde{\pi}_{0,1_0}=\frac{\mu_v}{\gamma\beta}(\beta\omega_0-\gamma)\tilde{\pi}_{0,0_1} \tag{16}$$

Where $\omega_0=\gamma+\dfrac{\beta-1}{p\bar{\mu}_v(1-\xi)}$.

Substituting (15) and (16) into (12), we obtain

$$\tilde{\pi}_{0,1}=\frac{\mu_v}{\gamma\beta\bar{p}\mu_b}(\beta\omega_0-\gamma)\tilde{\pi}_{0,0_1} \tag{17}$$

Now let us solve the equation (11), substituting (14) into (11):

$$\tilde{\pi}_{k,1}=\bar{p}\bar{\mu}_b\tilde{\pi}_{k,1}+p\bar{\mu}_b\tilde{\pi}_{k-1,1}+\bar{p}\mu_b\tilde{\pi}_{k+a,1}+p\mu_b\tilde{\pi}_{k+a-1,1}+\omega_1\theta\tilde{\pi}_{0,0_1}\xi^k\ ,$$

where $\omega_1=p\bar{\mu}_b+\bar{p}\bar{\mu}_b\xi+p\mu_b\xi^a+\bar{p}\mu_b\xi^{a+1}$.

Using $\tilde{\pi}_{k+j,1}=E^j\tilde{\pi}_{k,1}$, $j\in Z,k=1,2,\cdots$, the auxiliary equation of equation (11) such that

$$\bar{p}\mu_b z^{a+1}+p\mu_b z^a-(1-\bar{p}\bar{\mu}_b)z+p\bar{\mu}_b=0 \tag{18}$$

Let $G(z)=\bar{p}\mu_b z^{a+1}+p\mu_b z^a+\bar{p}\bar{\mu}_b z+p\bar{\mu}_b$, obviously $G(1)=1$, $G'(1)=(a+1)\bar{p}\mu_b+ap\mu_b+\bar{p}\bar{\mu}_b=a\mu_b+1-p$ .Since $\rho_0=\dfrac{p}{a\mu_b}<1$, i.e. $p<a\mu_b$, we can see that

$G'(1) > 1$. According to Hunter (1983), the equation $z = G(z)$ has the unique real root in the unit circle, which can be denoted by r, the other a roots can be denoted by r_i, $|r_i| \geq 1$ $(i = 1, 2, \cdots, a)$. The solution of (18) can be written as

$$z^* = c_0' r^k + \sum_{i=1}^{a} c_i' r_i^k, k \geq 1.$$

Hence, the solution of (11) can be written as

$$\pi_{k,1} = c_0' r^k + \sum_{i=1}^{a} c_i' r_i^k + c'' \xi^k.$$

As mentioned above, we have

$$\pi_{k,1} = c_0' r^k + c'' \xi^k. \tag{19}$$

Substituting (19) into (11) and associating with $\bar{p}\mu_b r^{a+1} + p\mu_b r^a + p\bar{\mu}_b - (1 - \bar{p}\bar{\mu}_b) r = 0$, we obtain

$$c'' = \frac{(\beta - 1)(\bar{p}\xi + p)\xi\tilde{\pi}_{0,0_1}}{\beta(\xi - \omega_1)} \tag{20}$$

Substituting (14)-(17), (19), (20) into (10), we obtain c_0'. According to the normalizing condition $\tilde{\pi}_{0,0} + \sum_{i=0}^{\infty} \tilde{\pi}_{i,0_1} + \sum_{i=0}^{\infty} \tilde{\pi}_{i,1} + \tilde{\pi}_{0,1_0} = 1$, we get $\tilde{\pi}_{0,0_1}$.

Remark: *If $\beta \to 1$ and $a = 1$, this queuing system is equivalent to $Geom / Geom / 1$ queuing system where the server serves customers singly. We have*

$$\tilde{\pi}_{0,1_0} = 0, \tilde{\pi}_{0,1} = 0, \tilde{\pi}_{k,1} = 0, \tilde{\pi}_{0,0_1} = \frac{1-\xi}{1+\gamma-\gamma\xi},$$

$$\tilde{\pi}_{k,0_1} = \frac{1-\xi}{1+\gamma-\gamma\xi}\xi^k, \tilde{\pi}_{0,0} = \frac{(1-\xi)\gamma}{1+\gamma-\gamma\xi}.$$

where $\gamma = \frac{\bar{p}\mu_v}{p\bar{\mu}_v}$, since $\rho_1 = p / a\mu_v < 1$ .Hence, when $a = 1$ we have $p / \mu_v < 1$, i.e., $p - p\mu_v < \mu_v - p\mu_v$, and further, we obtain $\gamma = \frac{\bar{p}\mu_v}{p\bar{\mu}_v} > 1$. Since $\xi = \frac{1}{\gamma} = \frac{p\bar{\mu}_v}{\bar{p}\mu_v}$ we obtain

$$\tilde{\pi}_{0,0_1} = \xi(1-\xi), \quad \tilde{\pi}_{k,0_1} = (1-\xi)\xi^{k+1}, \quad \tilde{\pi}_{0,0} = (1-\xi).$$

Therefore, $P\{\tilde{L}_n = k\} = \begin{cases} \tilde{\pi}_{0,0} = 1-\xi, k=0 \\ \tilde{\pi}_{k-1,0_1} = (1-\xi)\xi^k, k \ge 1 \end{cases}$, which are matched with the results given by Tian et al.(2007), where $\tilde{L}$ denote the steady-state queue length at slot point n^- (including the customers in service).

Corollary: The steady state probability of each state of the system can be written as

$$P\{J=0\} = \tilde{\pi}_{0,0}\ , P\{J=0_1\} = \frac{1}{1-\xi}\tilde{\pi}_{0,0_1}\ ,$$

$$P\{J=1_0\} = \frac{\mu_v[\beta\omega_0-\gamma]\tilde{\pi}_{0,0_1}}{\gamma\beta}\ ,$$

$$P\{J=1\} = \frac{\mu_v}{\gamma\beta\bar{p}\mu_b}[\beta\omega_0-\gamma]\tilde{\pi}_{0,0_1} + c_0'\frac{r}{1-r} + c''\frac{1}{1-\xi}$$

Theorem 2 *If* $|z| \le 1$*, the probability generating function (PG.F) of steady state queue length is given by*

$$L(z) = [1+\bar{\mu}_v\omega_0 + \frac{1}{1-\xi} + \frac{(\bar{p}\mu_b\mu_v+\mu_v)(\beta\omega_0-\gamma)}{\gamma\beta\bar{p}\mu_b}]\tilde{\pi}_{0,0_1} + \frac{c_0'rz}{1-rz} + \frac{zc''+\tilde{\pi}_{0,0_1}\xi z}{1-\xi z} \tag{21}$$

And the average queue length is

$$E(L) = \frac{\xi\pi_{0,0_1}+c''}{(1-\xi)^2} + \frac{rc_0'}{(1-r)^2} \tag{22}$$

Proof. In the steady state the queue length L *(excluding the customers in service) at time* n^- *has the following marginal distribution:*

$$P\{L=0\}$$
$$= \tilde{\pi}_{0,0} + \tilde{\pi}_{0,1_0} + \tilde{\pi}_{0,1} + \tilde{\pi}_{0,0_1}$$
$$= [1+\bar{\mu}_v\omega_0 + \frac{1}{1-\xi} + \frac{(\bar{p}\mu_b\mu_v+\mu_v)[\beta\omega_0-\gamma]}{\gamma\beta\bar{p}\mu_b}\tilde{\pi}_{0,0_1}\ ,$$
$$P\{L=k\} = \tilde{\pi}_{k,0_1} + \tilde{\pi}_{k,1} = c_0'r^k + c''\xi^{k-1} + \tilde{\pi}_{0,0_1}\xi^k, (k \ge 1).$$

Using $L(z) = P\{L=0\} + \sum_{k=1}^{\infty} P\{L=k\}z^k$, we can obtain (21) easily. Furthermore, taking derivation to $L(z)$ and letting $z=1$, we can get (22).

4. THE WAITING TIME DISTRIBUTION

Let the random variable T_q be the total waiting time of the arriving customer in the queue. Assume that if the arriving customer sees i customers waiting for service, the distribution law that he waits for k slots is object to $w_i(k) = P\{T_q = k\}$, $i = 0,1,2,\cdots, k = 0,1,2,\cdots$, and the PGF is $W_i(z) = \sum_{k=0}^{\infty} w_i(k) z^k$. In the steady state the PGF of waiting time is $w_q(z)$ and $w_q(z) = \sum_{i=0}^{\infty} \tilde{\pi}_{il} W_i(z), l = 0_1, 1$

Theorem 3 *In the steady state the PGF of waiting time of the arriving customer is given by*

$$w_q(z) = \frac{(\beta-1)(\bar{p}\xi + p)(\xi - \xi^a)}{\beta(\xi - \omega_1)(1-\xi)} \tilde{\pi}_{0,0_1} q(z) + \tilde{\pi}_{0,1} q(z) + c' \frac{r - r^a}{1-r} q(z)$$
$$+ c' \frac{(r^a - r^{2a-1}) q^2(z)}{(1-r)(1 - r^a q(z))} + \frac{(\beta-1)(1-\xi^a)\xi^a \tilde{q}(\frac{z}{\beta}) q(z)}{(1-\xi)(1-\bar{\mu}_v \frac{z}{\beta})[1 - q(z)\xi^a][1 - \tilde{q}(\frac{z}{\beta})\xi^a]} \tilde{\pi}_{0.0_1}, \quad (23)$$
$$+ \frac{(\beta-1)(\bar{p}\xi + p)(\xi^a - \xi^{2a-1}) \tilde{\pi}_{0,0_1} q^2(z)}{\beta(\xi - \omega_1)(1-\xi)(1 - \xi^a q(z))} + \frac{\beta \tilde{q}(\frac{1}{\beta} z)(1-\xi^a)}{(1-\xi)[1 - \tilde{q}(\frac{1}{\beta} z)\xi^a]} \tilde{\pi}_{0,0_1}$$

and the average waiting time is

$$E(w_q) = \frac{1}{\mu_b}\{\tilde{\pi}_{0,1} + \frac{(r - r^a)[1 - (1-r)(r^{2a-1} - 2r^{a-1})]c_0'}{(1-r)(1-r^a)^2} + \{(\frac{(\beta-1)(\xi^a - \xi^{2a-1})(2-\xi^a)(\bar{p}\xi + p)}{\beta(\xi - \omega_1)(1-\xi)(1-\xi^a)^2}$$
$$+ \frac{\beta u_v \mu_b (1-\xi^a)}{(1-\xi)(\beta - \bar{\mu}_v - \xi^a u_v)^2} + \frac{(\beta-1)(\xi - \xi^a)(\bar{p}\xi + p)}{\beta(\xi - \omega_1)(1-\xi)} + \frac{\beta(\beta-1) u_v \xi^a [\mu_b(\bar{\mu}_v + 1)(1-\xi^a) + \beta - \bar{\mu}_v]}{(1-\xi)(\beta - \bar{\mu}_v)^2 (1-\xi^a)(\beta - \bar{\mu}_v - u_v \xi^a)} \quad (24)$$
$$+ \frac{(\beta-1)\xi^{2a} u_v^2 \mu_b}{(1-\xi)(\beta - \bar{\mu}_v)(\beta - \bar{\mu}_v - u_v \xi^a)^2}\} \tilde{\pi}_{0.0_1}\}$$

Proof. Firstly, we define $\lfloor x \rfloor$ as the greatest integer function (floor), which returns the greatest integer less than or equal to a real number x. An arriving customer may observe the system in any of the following two cases.

Case 1. *Since the system considered is a late arrival delayed access system, we have*

$$P\{T_q = 0\} = 0 \quad (25)$$

Case 2. *When* $T_q = m, (m \geq 1)$, *there are two cases as follows:*

1) The server is on normal busy period and i customers are waiting for service.

Under this condition, the arriving customer has to wait for $1+\left\lfloor \frac{i}{a} \right\rfloor$ periods of service and each period of service S_{b_i} $(i=1,2,\cdots)$ is independent and geometrically distributed with p.m.f. $P\{S_{b_i}=k\}=\mu_b\bar{\mu}_b^{k-1},\ k\geq 1, \bar{\mu}_b=1-\mu_b$. Its PGF is $\frac{u_b z}{1-\bar{\mu}_b z}$. We have

$$w_i(m)=P\{T_q=m\}$$
$$=P\{s_{b_1}+s_{b_2}+\cdots+s_{b_{1+\left\lfloor \frac{i}{a} \right\rfloor}}=m\}\cdot$$

Hence

$$W_i(z)=\left(\frac{u_b z}{1-\bar{\mu}_b z}\right)^{1+\left\lfloor \frac{i}{a} \right\rfloor}.$$

Let $q(z)=\frac{u_b z}{1-\bar{\mu}_b z}$, the PGF of waiting time can be given by

$$\begin{aligned}\sum_{i=0}^{\infty}\tilde{\pi}_{i,1}W_i(z)&=\tilde{\pi}_{0,1}q(z)+[c'\frac{r-r^a}{1-r}+\frac{(\beta-1)(\bar{p}\xi+p)\xi(\xi-\xi^a)\tilde{\pi}_{0,0_1}}{\beta(\xi-\omega_1)(1-\xi)}]q(z)\\&+\frac{(\beta-1)(\bar{p}\xi+p)(\xi^a-\xi^{2a-1})q^2(z)\xi\tilde{\pi}_{0,0_1}}{\beta(\xi-\omega_1)(1-\xi)(1-\xi^a q(z))}+c'\frac{(r^a-r^{2a-1})q^2(z)}{(1-r)(1-r^a q(z))}\end{aligned} \tag{26}$$

The arriving customer finds that the server is on vacation.

In this case, if the arriving customer finds i customers waiting for service, he has to wait for $1+\left\lfloor \frac{i}{a} \right\rfloor$ periods of service, and each period of service S_{v_i} $(i=1,2,\cdots)$ is independent and geometrically distributed with p.m.f. $P\{S_{v_i}=k\}=\mu_v\bar{\mu}_v^{k-1}, k\geq 1, \bar{\mu}_v=1-\mu_v$, It's PGF is $\frac{u_v z}{1-\bar{\mu}_v z}$ and $W_i(z)=\left(\frac{u_v z}{1-\bar{\mu}_v z}\right)^{1+\left\lfloor \frac{i}{a} \right\rfloor}$. Let $\tilde{q}(z)=\frac{u_v z}{1-\bar{\mu}_v z}$, let s_{v_j} be the j th length of period of service with service rate μ_v and let $s^{(j)}{}_v$ be the sum of lengths of j periods of service with service rate μ_v, where $s^{(0)}{}_v=0$ and $j=1,2,3,\cdots$. There are two cases to consider to be in this condition:

A) The server is on vacation whereas $1+\left\lfloor \frac{i}{a} \right\rfloor$ periods of service ended. We have

$$\begin{aligned}
w_i(m) &= P\{T_q = m; V \ge m\} \\
&= P\{s_{v_1} + s_{v_2} + \cdots + s_{v_{1+\lfloor \frac{i}{a} \rfloor}} = m; V \ge m\} \\
&= P\{s_{v_1} + s_{v_2} + \cdots + s_{v_{1+\lfloor \frac{i}{a} \rfloor}} = m\} P\{V \ge m\} \\
&= \sum_{u=m}^{\infty} P\{V = u\} P\{s_{v_1} + s_{v_2} + \cdots + s_{v_{1+\lfloor \frac{i}{a} \rfloor}} = m\} \\
&= \bar{\theta}^{m-1} P\{s_{v_1} + s_{v_2} + \cdots + s_{v_{1+\lfloor \frac{i}{a} \rfloor}} = m\}
\end{aligned}$$

Hence

$$W_i(z) = \frac{1}{1-\theta} \left(\frac{(1-\theta)\mu_v z}{1-(1-\theta)(1-\mu_v)z} \right)^{1+\lfloor \frac{i}{a} \rfloor}.$$

And the PGF of waiting time can be given by

$$\sum_{i=0}^{\infty} \tilde{\pi}_{i,0_1} W_i(z) = \sum_{i=0}^{\infty} \tilde{\pi}_{i,0_1} \frac{1}{1-\theta} \left(\frac{(1-\theta)\mu_v z}{1-(1-\theta)(1-\mu_v)z} \right)^{1+\lfloor \frac{i}{a} \rfloor} = \frac{\beta \tilde{q}(\frac{1}{\beta} z)(1-\xi^a)}{(1-\xi)[1-\tilde{q}(\frac{1}{\beta} z)\xi^a]} \tilde{\pi}_{0,0_1}. \tag{27}$$

B) The vacation is finished and $j(j < 1+\lfloor \frac{i}{a} \rfloor)$ periods of service ended, the service rate is converted to μ_b from μ_v, the normal busy period begins. The waiting time of the arriving customer should be equal to the sum of the server's vacation times and $1+\lfloor \frac{i}{a} \rfloor - j$ periods of service, the service rate of $1+\lfloor \frac{i}{a} \rfloor - j$ periods of service is μ_b. We have

$$\begin{aligned}
w_i(m) &= \sum_{j=0}^{\lfloor \frac{i}{a} \rfloor} P\{T_q = m; s_v^{(j)} \le V < s_v^{(j+1)}\} \\
&= \sum_{j=0}^{\lfloor \frac{i}{a} \rfloor} P\{V + s_{b_1} + s_{b_2} + \cdots + s_{b_{1+\lfloor \frac{i}{a} \rfloor - j}} = m; s_v^{(j)} \le V < s_v^{(j+1)}\} \\
&= \sum_{j=0}^{\lfloor \frac{i}{a} \rfloor} \sum_{u=1}^{m-1-\lfloor \frac{i}{a} \rfloor + j} P\{V = u\} P\{s_{b_1} + s_{b_2} + \cdots + s_{b_{1+\lfloor \frac{i}{a} \rfloor - j}} = m-u\} \\
&\times P\{s_v^{(j)} \le V < s_v^{(j+1)}\}
\end{aligned}$$

The PGF of waiting time can be given by

$$\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}\tilde{\pi}_{i,0_1}\sum_{j=0}^{\lfloor \frac{i}{a}\rfloor}\sum_{u=1}^{m-1-\lfloor \frac{i}{a}\rfloor+j} P\{V=u\}P\{s_{b_1}+s_{b_2}+\cdots+s_{b_{1+\lfloor \frac{i}{a}\rfloor-j}}=m-u\} P\{s_v^{(j)}\le V<s_v^{(j+1)}\}z^m$$

$$=\frac{(\beta-1)(1-\xi^a)\xi^a\tilde{q}(\frac{z}{\beta})q(z)}{(1-\xi)(1-\overline{\mu}_v\frac{z}{\beta})[1-q(z)\xi^a][1-\tilde{q}(\frac{z}{\beta})\xi^a]}\tilde{\pi}_{0.0_1} \tag{28}$$

Adding equations (25)-(28), we can get (23); using $\frac{dw_q(z)}{dz}\Big|_{z=1}$, we can obtain (24).

5. OUTSIDER OBSERVER'S DISTRIBUTIONS

For the late arrival system with delayed access, an outside observer's observation epoch falls in the time interval after a potential departure epoch and before a potential arrival epoch. Let $\hat{\pi}_{0,0}$, $\hat{\pi}_{n,0_1}$, $\hat{\pi}_{n,1}$ and $\hat{\pi}_{0,1_0}$ be the probabilities that the outside observer observes no customers in the system and the server is on vacation, n customers in the system (excluding the servicing customers)and the server is on vacation, n customers in the system(excluding the servicing customers)and the server is in normal busy period and the probability of the server is in idle time, respectively. By observing the relationship between arbitrary time t^- and the observation epoch $(*)$ of the outside observer, we have

$$\hat{\pi}_{0,0}=\overline{p}\tilde{\pi}_{0,0}+\overline{p}\tilde{\pi}_{0,1_0},$$

$$\hat{\pi}_{0,0_1}=\overline{p}\overline{\mu}_v\overline{\theta}\tilde{\pi}_{0,0_1}+p\overline{\theta}\tilde{\pi}_{0,0}+p\mu_v\overline{\theta}\sum_{k=1}^{a}\tilde{\pi}_{k-1,0_1}+\overline{p}\mu_v\overline{\theta}\sum_{k=1}^{a}\tilde{\pi}_{k,0_1},$$

$$\hat{\pi}_{n,0_1}=\overline{p}\overline{\theta}\overline{\mu}_v\tilde{\pi}_{n,0_1}+p\overline{\theta}\overline{\mu}_v\tilde{\pi}_{n-1,0_1}+\overline{p}\overline{\theta}\mu_v\tilde{\pi}_{n+a,0_1}+p\overline{\theta}\mu_v\tilde{\pi}_{n-1+a,0_1}\ (n\ge 1),$$

$$\hat{\pi}_{0,1}=p\theta\tilde{\pi}_{0,0}+p\tilde{\pi}_{0,1_0}+(\overline{p}\overline{\mu}_b+p\mu_b)\tilde{\pi}_{0,1}+\overline{p}\theta\tilde{\pi}_{0,0_1}+(\overline{p}\sum_{k=1}^{a}\tilde{\pi}_{k,1}+p\sum_{k=1}^{a}\tilde{\pi}_{k-1,1})\mu_b$$

$$\hat{\pi}_{n,1}=\overline{p}\overline{\mu}_b\tilde{\pi}_{n,1}+p\overline{\mu}_b\tilde{\pi}_{n-1,1}+\overline{p}\theta\pi_{n,0_1}+p\theta\tilde{\pi}_{n-1,0_1}+\overline{p}\mu_b\tilde{\pi}_{n+a,1}+p\mu_b\tilde{\pi}_{n-1+a,1}$$

$$(n\ge 1),\hat{\pi}_{0,1_0}=\overline{p}\mu_b\tilde{\pi}_{0,1}$$

6. NUMERICAL RESULTS AND THE SENSITIVITY ANALYSIS

In this section, we present some numerical results in tables for queue length distributions at the different states of the system. All numerical results have been obtained using the results derived in this paper. We observe that $\tilde{\pi}_{n,0_1}$, $\tilde{\pi}_{n,1}$ $\hat{\pi}_{n,0_1}$ and $\hat{\pi}_{n,1}$ monotonically decrease as n increases in table 1, table 2, table 3 and table 4. $E(L)$ and $E(w_q)$ monotonically decrease as a increases. In Fig.2 and Fig.3, Let $a=10$, $p=0.3$ and $\mu_b=0.5$, we have plotted the effect of various vacation service rates on the average queue length and the average waiting time, respectively, we observe that the average queue length and the average waiting time decrease as vacation service rate increases. In Fig.4, let $p=0.3$, $\mu_v=0.4$, $\mu_b=0.5$, $\theta=0.3$, we observe that the average queue length and the average waiting time decrease as the batch size a increases; meanwhile, we find that the average queue length is equal to 0.2554 from $a=6$ on, and the average waiting time is equal to 0.8105 from $a=8$ on, they do not change as the batch size increases.

Table 1. queue size distribution with $a=2$, $p=0.3$, $\mu_v=0.4$, $\mu_b=0.5$, $\theta=0.3$.

n	$\tilde{\pi}_{n,0_1}$	$\tilde{\pi}_{n,1}$	$\hat{\pi}_{n,0_1}$	$\hat{\pi}_{n,1}$
1	0.02	0.1416	0.0142	0.1475
2	0.0037	0.0361	0.0026	0.0369
3	6.80E-04	0.0092	4.80E-04	0.0093
4	1.25E-04	0.0023	8.85E-05	0.0023
5	2.31E-05	5.96E-04	1.63E-05	5.92E-04
6	4.25E-06	1.52E-04	3.00E-06	1.50E-04
7	7.83E-07	3.86E-05	5.54E-07	3.81E-05
8	1.44E-07	9.83E-06	1.03E-07	9.67E-06
9	2.66E-08	2.50E-06	2.02E-08	2.46E-06
10	4.89E-09	6.37E-07	4.89E-09	6.25E-07
sum	0.0246	0.19	1.73E-02	1.97E-01

$E(L)=0.2850$, $E(w_q)=$ 0.8514

Table 2. queue size distribution with $a=5$, $p=0.3$, $\mu_v=0.4$, $\mu_b=0.5$, $\theta=0.3$.

n	$\tilde{\pi}_{n,0_1}$	$\tilde{\pi}_{n,1}$	$\hat{\pi}_{n,0_1}$	$\hat{\pi}_{n,1}$
1	0.02	0.1337	0.0141	0.1429
2	0.0036	0.0309	0.0025	0.0325
3	6.38E-04	0.0071	4.50E-04	0.0074
4	1.14E-04	0.0016	8.03E-05	0.0017
5	2.03E-05	3.81E-04	1.43E-05	3.90E-04
6	3.63E-06	8.80E-04	2.56E-06	8.96E-05
7	6.47E-07	2.03E-05	4.58E-07	2.06E-05
8	1.16E-07	4.70E-06	8.27E-08	4.75E-06
9	2.06E-08	1.08E-06	1.56E-08	1.09E-06
10	3.68E-09	2.51E-07	3.68E-09	2.52E-07
sum	0.0244	0.1739	1.72E-02	1.85E-01

$E(L)=0.2557$, $E(w_q)=$ 0.8114

Table 3. queue size distribution with $a = 8$, $p = 0.3$, $\mu_v = 0.4$, $\mu_b = 0.5$, $\theta = 0.3$.

n	$\tilde{\pi}_{n,0_1}$	$\tilde{\pi}_{n,1}$	$\hat{\pi}_{n,0_1}$	$\hat{\pi}_{n,1}$
1	0.02	0.1336	0.0141	0.1428
2	0.0036	0.0308	2.50E-03	0.0325
3	6.37E-04	0.0071	4.50E-04	0.0074
4	1.14E-04	1.60E-03	8.03E-05	1.70E-03
5	2.03E-05	3.79E-04	1.43E-05	3.88E-04
6	3.62E-06	8.75E-05	2.56E-06	8.91E-05
7	6.47E-07	2.02E-05	4.58E-07	2.05E-05
8	1.15E-07	4.66E-06	8.25E-08	4.71E-06
9	2.06E-08	1.07E-06	1.56E-08	1.08E-06
10	3.68E-09	2.48E-07	3.68E-09	2.50E-07
sum	0.0244	0.1736	1.72E-02	1.85E-01

$E(L) = 0.2554$, $E(w_q) = 0.8105$

Table 4. queue size distribution with $a = 15$, $p = 0.3$, $\mu_v = 0.4$, $\mu_b = 0.5$, $\theta = 0.3$.

n	$\tilde{\pi}_{n,0_1}$	$\tilde{\pi}_{n,1}$	$\hat{\pi}_{n,0_1}$	$\hat{\pi}_{n,1}$
1	0.02	0.1336	0.0141	0.1428
2	0.0036	0.0308	2.50E-03	0.0325
3	6.37E-04	0.0071	4.50E-04	0.0074
4	1.14E-04	1.60E-03	8.03E-05	1.70E-03
5	2.03E-05	3.79E-04	1.43E-05	3.88E-04
6	3.62E-06	8.74E-05	2.56E-06	8.91E-05
7	6.47E-07	2.02E-05	4.58E-07	2.05E-05
8	1.15E-07	4.66E-06	8.25E-08	4.71E-06
9	2.06E-08	1.07E-06	1.56E-08	1.08E-06
10	3.68E-09	2.48E-07	3.68E-09	2.50E-07
sum	0.0244	0.1736	1.72E-02	1.85E-01

$E(L) = 0.2554$, $E(w_q) = 0.8105$

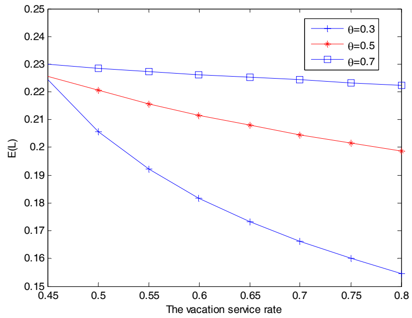


Figure 2. Effect of μ_v on the average queue length.

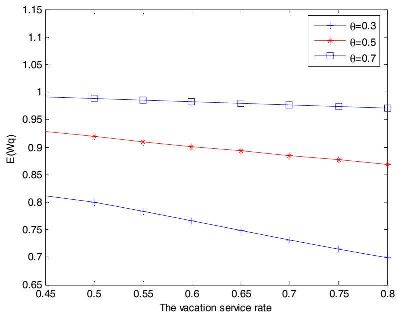


Figure 3. Effect of μ_v on the average waiting time.

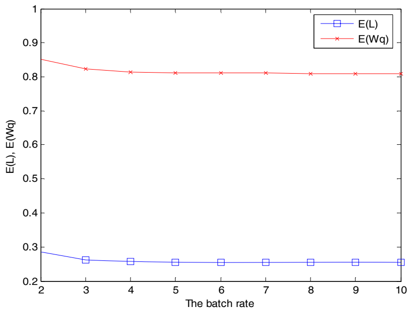


Figure4. Effect of a on the average queue length and the average waiting time.

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