

## IMPROVING THE PUBLIC TRANSIT SYSTEM FOR ROUTES WITH SCHEDULED HEADWAYS

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Received: February 2013 / Accepted: April 2013

**Abstract:** This research contributes to the improvement of the optimal headway solution for the transit performance functions (e. g., minimize total cost; maximize social welfare) derived from the traffic model proposed by Hendrickson. The purpose of this paper is threefold. First, we prove that that model has a unique solution for headway. Second, we offer a formulated approximation for headway. Third, numerical examples illustrate that our formulated approximation performs more accurately than the Hendrickson's.

**Keywords:** Analytical approach, headway of bus, stop-spacing, public transportation.

**MSC:** 90B20.

### 1. INTRODUCTION

Researchers developed analytical traffic models to provide a simplified version for the real but too complicated real world situations. The formulated solution for analytical modes is a useful indicator to reveal relations among parameters and decision variables. From the explicit expression, researchers noticed which parameter has significant impact on the optimal solution; so, they could operate a comprehensive examination of the important parameters to obtain more representative mean and variance of the parameters. For examples, Golob et al. [6] examined an analysis of consumer preferences for a public transportation system to improve the quality of information about potential public transportation users, their needs and preferences.

Renault et al. [16] studied discounted and finitely repeated minority games with public signals to extend their previously undiscounted game in Renault et al. [15] to a discounted version and a finitely repeated version of the game. Otsubo and Rapoport [13] built a discrete version of Vickrey's model of traffic congestion to present an algorithm for numerically computing a symmetric mixed-strategy equilibrium solution. Hill et al. [8] obtained a competitive game, that the maximal Nash-equilibrium payoff required quantum resources to attain its optimal alternative to illustrate that quantum entanglement can provide improved solutions. Pop and Sitar [14] examined a new efficient transformation to generalize vehicle routing problem into the classical vehicle routing problem so presenting a new integer programming formulation of the problem. For the green house gas emissions and cost, Traut et al. [17] developed optimal design and allocation of electrified vehicles, and dedicated charging infrastructure to maintain the life cycle with minimum cost. Coffelt and Hendrickson [4] examined a case study of occupant costs in roof management to construct occupant cost model to study the relation between maintenance and replacement costs. Jain and Saksena [10] studied a time minimizing transportation problem with fractional bottleneck objective function to derive an algorithm to find an initial efficient basic solution. An and Zhang [1] constructed a congestion traffic model with heterogeneous commuters. They proved the existence and uniqueness of a nontrivial Nash equilibrium to study the allocation of commuters between public transportation and private vehicles at the equilibrium under gasoline tax affects. However, none of them has provided a further study for Hendrickson [7]. We studied the analytical traffic model of Hendrickson [7] and found its contributions in public transportation operation and management; nevertheless, we also believed that some of his results required further investigation based on our following research. His paper analyzed performance functions with variables in riding and waiting times, transportation fare, frequency and service structure. He considered typical managerial decisions with respect to fare and frequency of service, and discussed the variation in user cost (especially wait cost and in-vehicle cost) resulting from the changes of supply. Various managerial strategies were explored such as maintaining service standards or constant load factors and maximizing service, profit, or net social benefits. An example of a peak-hour, radial transit route was used extensively to illustrate the impact of such decisions. However, only a degenerated model was explained with formulated solution for headway in Hendrickson [7] and it cannot be applied to deal with the general problem. The aim of this paper is to make a contribution in this area by presenting a more adequate solving method for the performance function and developing a proper solution to improve the accuracy of headway. In the analysis and evaluation of bus system operation performance, analytical optimization models are developed to optimize several related decision variables including route length, stop spacing, service headway. Previous studies of Chang and Schonfeld [3, 4] and Chien and Schonfeld [6] discussed the relationship between the aforementioned prevailing variables, and developed closed-form analytic solution. The studies mentioned above revealed that the accurate solution of headway is critical for model performance. Consequently, accurate solution of headway is significant for the performance function. From our previous review, no comprehensive treatment of this topic seems to exist. Moreover, simple results of Hendrickson [7] about travel time and volume relationships are often made erroneously and without rigorous examination. In this paper, we prove that the performance function of total costs have unique solution; also, we provide a formulated approximated solution for headway. From

the same numerical examples, our formulated approximated solution for headway gives more accurate results than the Hendrickson's. Instead, we find the closed form of total costs and headway relationships, and we propose analytic functions to approximate the optimal headway. There are three papers published in Yugoslav Journal of Operations Research having similar analytical approach as ours. Wu et al. [18] investigated the Newton method for determining the optimal replenishment policy for EPQ model with present value, and their findings are more efficient than the bisection method. Lin et al. [11] constructed inventory models from ramp type demand to a generalized setting such that the optimal solution for replenishment policy is independent of demand type. Hung [9] developed continuous review inventory models with the present value of money and crashable lead time; he also obtained several lemmas and one theorem to estimate optimal solutions.

## 2. REVIEW OF HENDRICKSON'S MODEL

To be compatible with Hendrickson [7], we used the same assumptions and notation:

$d$	route length
$h$	scheduled inter-vehicle headway
$k$	constant parameter
$n$	number of potential stops on a route
$q$	patron arrival rate along a route per unit time
$Q$	expected volume carried by a single vehicle ( $Q = hq$ )
$r$	expected riding time
$s$	expected number of stops as a function of potential stops and volume
$v$	average vehicle cruising velocity (apart from patron stops)
$w$	expected waiting time
$\sigma$	standard derivation of inter-vehicle headways at a stop
$t_n(h)$	expected vehicle travel time over a route with headway $h$
$t_p$	average patron boarding and unloading time
$t_s$	average extra time required to decelerate and accelerate for a patron stop
$C_f$	fixed cost per vehicle dispatch on a route (including mileage-related costs)
$C_h$	cost per unit time of operating a vehicle
$C_b$	cost of a vehicle run on a route ( $C_b = C_f + C_h t_n(h)$ )
$C_r$	average value of patron's riding time per unit time
$C_w$	average value of patron's waiting time per unit time

The total system operating costs may be expressed as a fixed charge per vehicle dispatch plus an hourly charge. In this case, the total system operating costs per patron are:

$$C(h) = C_r r + C_w w + \frac{C_b}{hq} \quad (1)$$

with  $r = \frac{t_n(h)}{2}$ ,  $t_n(h) = \frac{d}{v} + t_s s + t_p qh$ ,  $s = n \left(1 - e^{-\frac{qh}{n}}\right)$  if not all stops are made, or

$s = n$  if all stops are made,  $w = \frac{h}{2}$  for random patron arrivals, or  $w = \frac{h}{2} \left(1 + \frac{\sigma^2}{h^2}\right)$  with

some variation (Osuna and Newell [12]), and  $\frac{C_b}{hq} = \frac{C_f}{hq} + \frac{C_h}{hq} t_n(h)$ .

Therefore, we face the following minimizing problem:

$$C(h) = \left(\frac{C_r}{2} + \frac{C_h}{hq}\right) \left(\frac{d}{v} + t_s n \left(1 - e^{-\frac{qh}{n}}\right) + t_p qh\right) + C_w \left(\frac{h}{2} + \frac{\sigma^2}{2h}\right) + \frac{C_f}{hq} \quad (2)$$

In Hendrickson [7], by conviction or for analytical convenience, he only considered the special case with  $s = n$  and  $w = hk$ , then

$$C(h) = \left(\frac{C_r}{2} + \frac{C_h}{hq}\right) \left(\frac{d}{v} + t_s n + t_p qh\right) + C_w kh + \frac{C_f}{hq} \quad (3)$$

For simplicity, we assume that  $a_0 = \frac{C_r}{2} \left(\frac{d}{v} + t_s n\right) + C_h t_p$ ,  $a_1 = \frac{q}{2} C_r t_p + C_w k$  and

$a_2 = \frac{C_h}{q} \left(\frac{d}{v} + t_s n\right) + \frac{C_f}{q}$ , then we can rewrite Eq. (3) as

$$C(h) = a_0 + a_1 h + \frac{a_2}{h} \quad (4)$$

Hence, it is not surprising that for this special case Hendrickson derived that the

minimum value occurs at  $h^* = \left(\frac{a_2}{a_1}\right)^{\frac{1}{2}} = \left(\frac{C_f + C_h \left(\frac{d}{v} + t_s n\right)}{(0.5) C_r t_p q^2 + C_w k q}\right)^{\frac{1}{2}}$ . However, he did not

examine the general case. In this paper, we prove that the generalized total costs, Eq. (2) still has one critical point and that this point is the minimum solution.

### 3. OUR IMPROVEMENT FOR THE GENERAL MODEL

From Eq. (2), with  $w = hk$ , where  $k = \frac{1}{2} \left( 1 + \frac{\sigma^2}{h^2} \right)$  when  $\frac{\sigma}{h}$  is a constant, we know that

$$\begin{aligned} \frac{d}{dh} C(h) &= \frac{qC_r t_p}{2} + kC_w - \frac{1}{h^2} \left( \frac{C_f}{q} + \frac{dC_h}{qv} + \frac{nC_h t_s}{q} \right) \\ &+ e^{-\frac{q}{n}h} \left( \frac{qC_r t_s}{2} + \frac{C_h t_s}{h} + \frac{nC_h t_s}{qh^2} \right). \end{aligned} \quad (5)$$

Motivated by Eq. (5), we assume  $G(h)$  as  $G(h) = h^2 \frac{dC(h)}{dh}$  then

$$\begin{aligned} G(h) &= \left( \frac{qC_r t_p}{2} + kC_w \right) h^2 - \frac{C_h}{q} \left( \frac{C_f}{C_h} + \frac{d}{v} + nt_s \right) \\ &+ e^{-\frac{q}{n}h} C_h t_s \left( \frac{qC_r}{2C_h} h^2 + h + \frac{n}{q} \right). \end{aligned} \quad (6)$$

We obtain that  $G(0) = - \left( \frac{C_f}{q} + \frac{dC_h}{qv} \right)$  and  $\lim_{h \rightarrow \infty} G(h) = \infty$ . Next, we find the criterion to insure that  $G(h)$  is an increasing function for  $h > 0$ .

We know that

$$\frac{dG(h)}{dh} = \left( qC_r t_p + 2kC_w \right) h + h e^{-\frac{q}{n}h} \left( qC_r t_s - \frac{q^2}{2n} C_r t_s h - \frac{q}{n} C_h t_s \right). \quad (7)$$

Hence, to prove that  $\frac{dG(h)}{dh} > 0$  is equivalent as to show that

$$\left( qC_r t_p + 2kC_w \right) e^{\frac{q}{n}h} + qC_r t_s > \frac{q^2}{2n} C_r t_s h + \frac{q}{n} C_h t_s. \quad (8)$$

Since the parameters in Eq. (8) have their practical meaning, therefore, we quote the data from Hendrickson [7], then  $C_h = 30$ ,  $C_r = 5$ ,  $C_w = 10$ , the value of  $n$  are 10 or 20, the range of  $t_p$  from 4.5 second to 5 second,  $t_s = 12$  second, and the range for  $q$  from 86 to 213 passengers per hour. As a result, we know that  $C_r = 5$  and  $\frac{C_h}{n} = \frac{3}{2}$  or 3, and then it follows

$$qC_r t_s > \frac{q}{n} C_h t_s. \quad (9)$$

By the Taylor's series expansion, we have  $e^{\frac{q}{n}h} > \frac{q}{n}h$  and the definition of  $k$ , we know that  $2k > 1$ , so the following are equivalent:

$$(a) \left( qC_r t_p + 2kC_w \right) \frac{q}{n} h > \frac{q^2}{2n} C_r t_s h,$$

$$(b) qC_r t_p + 2kC_w > \frac{q}{2} C_r t_s,$$

and

$$(c) \frac{2kC_w}{C_r} > 2 > q \left( \frac{t_s}{2} - t_p \right) = q \left( \frac{12}{2} - 5 \text{ or } 4.5 \right) = q \left( 1 \text{ or } \frac{3}{2} \right).$$

Hence, we consider  $2$  or  $\frac{4}{3} > q$ , per second, that means,  $7200$  or  $4800 > q$ , per hour. From the range of  $q$  from  $86$  to  $213$ , per hour; therefore, we can say that

$$\left( qC_r t_p + 2kC_w \right) e^{\frac{q}{n}h} > \left( qC_r t_p + 2kC_w \right) \frac{q}{n} h > \frac{q^2}{2n} C_r t_s h. \quad (10)$$

Combining Eq. (9) and (10), we obtain  $\frac{dG(h)}{dh} > 0$  so  $G(h)$  is an increasing function for  $h > 0$ , from  $G(0) < 0$  to  $\lim_{h \rightarrow \infty} G(h) = \infty$ . Hence, there is a unique point, say

$h^*$ , such that  $G(h^*) = 0$  and  $h^*$  is the unique positive solution for  $\frac{dC(h)}{dh} = 0$ .

Therefore,  $h^*$  is the minimum point for the total costs. We summarize our results in the following Theorem.

**Theorem 1.** From the practical point of view, the following two inequalities:  $C_r > \frac{C_h}{n}$

and  $\left( qC_r t_p + 2kC_w \right) e^{\frac{q}{n}h} > \frac{q^2}{2n} C_r t_s h$  are satisfied. Moreover,  $\frac{dC(h)}{dh} = 0$  has a unique positive solution.

Hence, the total costs have a unique minimum solution.

#### 4. THE CONVEXITY PROPERTY OF THE PERFORMANCE FUNCTION

Next, for the convexity property of  $G(h)$ , we consider that

$$\frac{d^2G(h)}{dh^2} = (qC_r t_p + 2kC_w) + qt_s e^{-\frac{q}{n}h} \left( C_r \left( \frac{q^2 h^2}{2n^2} - \frac{2qh}{n} + 1 \right) + \frac{C_h}{n} \left( \frac{qh}{n} - 1 \right) \right). \quad (11)$$

We show that from the practical point of view,  $\frac{d^2G(h)}{dh^2} > 0$ .

First, we observe that the following are equivalent: (a)  $qC_r t_p + 2kC_w > 2qC_r t_s$ , and (b)  $\frac{2kC_w}{C_r} > 2 > q(2t_s - t_p) = q(24 - 5 \text{ or } 4.5) = q(19 \text{ or } 19.5)$ . So, we consider that  $\frac{2}{19}$  or  $\frac{4}{39} > q$ , per second, that means,  $378\frac{18}{19}$  or  $369\frac{3}{13} > q$ , per hour. Hence, from the practical point of view, the range of  $q$  from 86 to 213, we still imply

$$qC_r t_p + 2kC_w > 2qC_r t_s. \quad (12)$$

By Eq. (11), we get that  $\frac{d^2G(h)}{dh^2} > 0$  is equivalent to

$$(qC_r t_p + 2kC_w) e^{\frac{q}{n}h} + qt_s \left( C_r \left( \frac{q^2 h^2}{2n^2} - \frac{2qh}{n} + 1 \right) + \frac{C_h}{n} \left( \frac{qh}{n} - 1 \right) \right) > 0. \quad (13)$$

By the Taylor's series expansion, we have  $e^{\frac{q}{n}h} > 1 + \frac{q}{n}h + \frac{q^2}{2n^2}h^2$ . Hence, from the practical point of view, we prove that

$$\begin{aligned} & (qC_r t_p + 2kC_w) \left( 1 + \frac{qh}{n} + \frac{q^2 h^2}{2n^2} \right) \\ & + qt_s \left( C_r \left( \frac{q^2 h^2}{2n^2} - \frac{2qh}{n} + 1 \right) + \frac{C_h}{n} \left( \frac{qh}{n} - 1 \right) \right) > 0 \end{aligned} \quad (14)$$

We rewrite the left hand side of Eq. (14) as

$$\begin{aligned} & \frac{q^2 h^2}{2n^2} (qC_r (t_p + t_s) + 2kC_w) + \frac{qh}{n} \left( \frac{q}{n} C_r t_s + 2kC_w + qC_r (t_p - 2t_s) \right) \\ & + 2kC_w + qC_r t_p + qt_s \left( C_r - \frac{C_h}{n} \right) \end{aligned} \quad (15)$$

Combining Eq. (9) and (12), we derive that Eq. (15) is positive, hence by Eq. (14), from the practical point of view, we prove that  $\frac{d^2G(h)}{dh^2} > 0$  and  $G(h)$  is a concave up function. We summarize the results in the next Theorem.

**Theorem 2.** From the practical point of view, the following two inequalities:  $C_r > \frac{C_h}{n}$  and  $2kC_w + qC_r(t_p - 2t_s) > 0$  are satisfied. It is legitimate to use the Newton's method to locate the solution for  $G(h) = 0$  that is  $\frac{dC(h)}{dh} = 0$ .

## 5. THE FORMULATED APPROXIMATION FOR HEADWAY

Here, we consider a formulated approximation for  $h^*$ . From Eq. (5) and (6), and the Taylor's series expansion for  $e^{-\frac{q}{n}h}$ , then we have

$G(h) = -\left(\frac{C_f}{q} + \frac{dC_h}{qv}\right) + \left(\frac{qC_r t_p}{2} + kC_w + \frac{qt_s}{2}\left(C_r - \frac{C_h}{n}\right)\right)h^2 +$  those terms with order than  $h^2$ .

Hence, our formulated approximation for  $h^*$  is constructed as

$$h = \left( \frac{C_f + \frac{d}{v}C_h}{(0.5)C_r t_p q^2 + C_w k q + (0.5)\left(C_r - \frac{C_h}{n}\right)t_s q^2} \right)^{\frac{1}{2}}. \quad (16)$$

In the numerical examples, we demonstrate that our formulated approximation is a very good estimation for  $h^*$ .

## 6. NUMERICAL EXAMPLES AND SENSITIVE ANALYSIS

Since  $G(h)$  is a concave up function for  $h > 0$ , so the Newton's method is suitable to locate  $h^*$ . We examine the same numerical example as Hendrickson [7]. The data of parameters are listed below:  $C_h = 30$ ,  $C_r = 5$ ,  $C_w = 10$ , the value of  $n$  are 10 or 20, the range of  $t_p$  from 4.5 second to 5 second,  $t_s = 12$  second, and the range for  $q$

from 86 to 213 passengers per hour. Moreover,  $d = 8$ ,  $v = 32$ ,  $\frac{\sigma}{h} = 0.35$ ,

$k = \frac{1}{2}\left(1 + \frac{\sigma^2}{h^2}\right) = 0.56125$  and  $C_f = 0$ . Our first example uses the data of  $n = 20$ ,

$t_p = \frac{4.5}{3600} = \frac{1}{800}$  and  $q = 86$ . For simplicity, we assume that the solution of

$\frac{d}{dh}C(h)=0$  is  $h^*$ , and then the formulated approximation of Hendrickson [7] is expressed as

$$h_1 = \left( \frac{C_f + C_h \left( \frac{d}{v} + t_s n \right)}{(0.5)C_r t_p q^2 + C_w k q} \right)^{\frac{1}{2}},$$

and our formulated approximation is expressed as

$$h_2 = \left( \frac{C_f + \frac{d}{v} C_h}{(0.5)C_r t_p q^2 + C_w k q + (0.5) \left( C_r - \frac{C_h}{n} \right) t_s q^2} \right)^{\frac{1}{2}}.$$

From the comparison of headway as optimal headway  $h^* = 0.119$ , Hendrickson’s approximated headway  $h_1 = 0.137$ , and our approximated headway  $h_2 = 0.116$ , we can say that our formulated approximation is a better estimation for  $h^*$ . Moreover, the comparison of total costs as optimal total costs  $C(h^*) = 2.240$ , Hendrickson’s approximated total costs  $C(h_1) = 2.255$ , and our approximated total costs  $C(h_2) = 2.240$ , we can say that our formulated approximation is a very good estimation for total costs. Next, we examine the sensitive analysis of our numerical example. In each example, we only change one parameter of  $n = 20$ ,  $t_p = \frac{1}{800}$  and  $q = 86$  by  $n = 10$ ,  $t_p = 5$  or  $q = 213$ . We list them in Table 1 for headway, and Table 2 for total costs. To be more accurate, in Tables 1 and 2, the expression for the results is calculated to the sixth decimal place.

Table 1. Sensitive analysis for headway

$n$	$t_p$	$q$	$h^*$	$h_1$	$h_2$	$\frac{h_1 - h^*}{h^* - h_2}$
20	1/800	86	0.119015	0.137050	0.116889	8.48
20	1/800	213	0.072080	0.084286	0.068425	3.34
20	1/720	86	0.118724	0.136703	0.116616	8.53
20	1/720	213	0.071676	0.083794	0.068091	3.38
10	1/800	86	0.121452	0.129636	0.118908	3.21
10	1/800	213	0.075111	0.079727	0.070984	1.12
10	1/720	86	0.121144	0.129308	0.118621	3.24
10	1/720	213	0.074663	0.079261	0.070611	1.14

Table 2. Sensitive analysis for total costs

$n$	$t_p$	$q$	$C(h^*)$	$C(h_1)$	$C(h_2)$	$\frac{C(h_1)-C(h^*)}{C(h_2)-C(h^*)}$
20	1/800	86	2.240247	2.254633	2.240482	61.23
20	1/800	213	1.762656	1.774166	1.763935	9.00
20	1/720	86	2.247964	2.262370	2.248196	61.97
20	1/720	213	1.772139	1.783685	1.773389	9.24
10	1/800	86	2.210908	2.213929	2.211227	9.49
10	1/800	213	1.719236	1.720875	1.720709	1.11
10	1/720	86	2.218697	2.221727	2.219012	9.61
10	1/720	213	1.728941	1.730598	1.730385	1.15

From Table 1, the range  $\frac{h_1-h^*}{h^*-h_2}$  for  $\frac{h_1-h^*}{h^*-h_2}$  (the relative ratio between the approximated errors for headway of Hendrickson divided by ours) is from 8.53 to 1.12 with mean 4.06. As a result, we may conclude that our formulated approximation is better than the Hendrickson's. From Table 2, the range  $\frac{C(h_1)-C(h^*)}{C(h_2)-C(h^*)}$  for

$\frac{C(h_1)-C(h^*)}{C(h_2)-C(h^*)}$  (the relative ratio between the total costs of Hendrickson divided by ours) is from 61.97 to 1.11 with mean 20.35. Therefore, we may imply that our approximated total costs are also superior to the Hendrickson's.

Comparing Hendrickson's headway approximation  $h_1$  and our headway approximation  $h_2$ , we know that in  $h_1$ , the term  $t_s$  is in the numerator and in  $h_2$ , the term  $t_s$  disappears in the formula in the denominator. Also, the term  $(0.5)(C_r - C_h/n)t_p q^2$  is added to the optimal  $h_2$  in the part of denominator. Apparently, the different results reflect implicitly the optimal cost affects. The optimal solution  $h_1$  indicates that  $h_1$  increase with  $t_s$  increases, but it can result in wrong deterministic analysis under real conditions. Namely, when the increment of average extra time required decelerating and accelerating for a patron stop will erroneously enable us to make a large headway decision. The optimal solution  $h_2$  indicates that if  $h_2$  increases, decreases should vary with  $t_p$  increases, that is to say, the increase of the average patron boarding and unloading time will reduce headway for decision.

Comparing Hendrickson's headway approximation  $h_1$  and our headway approximation  $h_2$ , we know that in  $h_1$ , the term  $t_s$  is in the numerator and in  $h_2$ , the term  $t_s$  is in the denominator. From practical sense, if  $t_s$  increases, then the headway  $h$  should be decreased for the operation management. Meanwhile, according to the aforementioned numerical examples, the results demonstrate that our approximation is

more exact than that of Hendrickson. Therefore, our approximation is physically more reasonable than that of Hendrickson. We conclude that the better approximation headway model should have the term  $t_s$ , average extra time for a patron stop, in the denominator and not in the numerator.

## 7. CONCLUDINS

This paper makes a rigorous investigation into how to obtain the optimal headway solution in the analytical model for the transit systems. A new approximation headway solution and its implications are presented. Based on the same numerical example comparison for fixed-route public transit system, the results indicate that the new approximation headway solution are more practical and accurate for cost function so, better than that of Hendrickson. The present paper can be of assistance in improving the solution of performance function.

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