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INVENTORY MODELS WITH RAMP-TYPE DEMAND FOR DETERIORATING ITEMS WITH PARTIAL BACKLOGGING AND TIME-VARING HOLDING COST

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Abstract: In this paper, an inventory model with general ramp-type demand rate, partial backlogging of unsatisfied demand and time-varying holding cost is considered; where the "Time-varying holding cost" means that the holding cost is a function of time, i.e. it is time dependent. The model is studied under the following different replenishment policies: (1) starting with no shortages (2) starting with shortages. Two component demand rate has been used. The backlogging rate is any non-increasing function of the waiting time up to the next replenishment. The optimal replenishment policy is derived for both the above mentioned policies.

Keywords: Inventory, Ramp-type demand, Time-varying holding cost, Shortages.

MSC: 90B05.

1. INTRODUCTION

Inventory is an important part of our manufacturing, distribution and retail infrastructure where demand plays an important role in choosing the best inventory policy. Researchers were engaged to develop the inventory models assuming the demand of the items to be constant, linearly increasing or decreasing, or exponential increasing or decreasing with time, stock-dependent etc. Later, it has been realized that the above demand patterns do not precisely depict the demand of certain items such as newly launched fashion items, garments, cosmetics, automobiles etc, for which the demand increases with time as they are launched into the market and after some time, it becomes constant. In order to consider demand of such types, the concept of ramp-type demand is introduced. Ramp-type demand depicts a demand which increases up to a certain time after which it stabilizes and becomes constant.

Maintenance of inventories of deteriorating items is a problem of major concern in the supply chain of almost any business organizations. Many of the physical goods undergo decay or deterioration over time. The inventory lot-size problem for deteriorating items is prominent due to its important connection with commonly used items in daily life. Fruits, vegetables, meat, photographic films, etc are the examples of deteriorating products. Deteriorating items are often classified in terms of their lifetime or utility as a function of time while in stock. A model with exponentially decaying inventory was initially proposed by Ghare and Schrader [4]. Covert and Phillip [2] developed an EOQ model with Weibull distributed deterioration rate. Thereafter, a great deal of research efforts have been devoted to inventory models of deteriorating items, and the details are discussed in the review article by Raafat [15] and in the review article of inventory models considering deterioration with shortages by Karmakar and Dutta Choudhury [10].

Gupta and Vrat [7] developed an inventory model where demand rate is replenishment size (initial stock) dependent. They analyzed the model through cost minimization. Pal et al.[14], Datta and Pal [3] have focused on the analysis of the inventory system which describes the demand rate as a power function, dependent on the level of the on hand inventory; Samanta and Roy [17] developed a production control inventory model for deteriorating items with shortages when demand and production rates are constant, and they studied the inventory model for a number of structural properties of the inventory system with the exponential deterioration. Holding cost is taken as a constant in all of these models.

In most of the models, holding cost is known and constant. But holding cost may not be constant always as there is a regular change in time value of money and change in price index. In this era of globalization, with tough competition in the economic world, holding cost may not remain constant over time. In generalization of EOQ models, various functions describing holding cost were considered by several researchers like Naddor [13], Van deer Veen [23], Muhlemann and Valtis Spanopoulos [12], Weiss [24] and Goh [6]. Giri and Chaudhuri [5] treated the holding cost as a non-linear function of the length of time for which the item is held in stock and as a functional form of the amount of the on-hand inventory. Roy [16] developed an EOQ model for deteriorating items where deterioration rate and holding cost are expressed as linearly increasing functions of time, demand rate is a function of selling price, with shortages allowed and completely backlogged. Tripathi [22] developed an inventory model for non deteriorating items under permissible delay in payments in which holding cost is a function of time.

Again, in most of the above referred papers, complete backlogging of unsatisfied demand is assumed. In practice, there are customers who are willing to wait and receive their orders at the end of the shortage period, while others are not. Teng et al.[21] extended the work done by Chang and Dye's [1] and Skouri and Papachristos [19], assuming backlogging rate to be any decreasing function of the waiting time up to the next replenishment.

The work of the researchers who used ramp-type demand as demand function and various form of deterioration with shortages (allowed/ not allowed), for developing the economic order quantity (EOQ) models are summarized below:

Reference	Objective	Constraints	Contributions	Limitations	Remarks
Mandal & Pal [11],1998	Finding EOQ	Ramp-type demand, constant rate of deterioration, shortages not allowed	An approximate solution for EOQ is obtained	Constant rate of deterioration and shortages not allowed	Holding cost is constant
Wu & Ouyang [26], 2000	Finding EOQ	Ramp-type demand, constant rate of deterioration, shortages allowed	An exact solution for EOQ is obtained	Constant rate of deterioration	Holding cost is constant
Jalan, Giri & Chaudhuri [9], 2001	Finding EOQ	Ramp-type demand, Weibull rate of deterioration, shortages allowed	EOQ is given by Numerical Technique	EOQ cannot be obtained analytically	Holding cost is constant
Wu [25], 2001	Finding EOQ	Ramp-type demand, Weibull rate of deterioration, shortages allowed partial backlogging	EOQ obtained for 3 numerical examples	Method explained by numerical example	Holding cost is constant
Skouri & Konstantaras [18], 2009	Finding EOQ	Ramp-type demand, Weibull rate of deterioration, with shortages and without shortages.	An exact solution for EOQ is obtained	Holding cost is constant	Holding cost is constant
Skouri, Konstantaras, Papachristos & Ganas[20], 2009	Finding EOQ	Ramp-type demand, Weibull rate of deterioration, shortages are allowed, Partial backlogging	An exact solution for EOQ is obtained	Holding cost is constant	Holding cost is constant
Jain & Kumar [8], 2010	Finding EOQ	Ramp-type demand, three -parameter Weibull rate of deterioration, shortages are allowed	An exact solution for EOQ is obtained	Model start with shortages	Holding cost is constant

The above table shows that all the researchers developed EOQ models by taking ramp-type demand, deterioration (constant/Weibull distribution) and shortages (allowed/ not allowed), and holding cost as constant. But holding cost may not be constant over time, as there is a change in time value of money and change in the price index.

The motivation behind developing an inventory model in the present article is to prepare a more general inventory model, which includes:

1. holding cost as a linearly increasing function of time
2. deterioration rate as constant

The model has been studied under the following two different replenishment policies (i) starting without shortages (ii) starting with shortages. The optimal replenishment policy is derived for both of the policies and compared with the constant holding cost.

The paper has been arranged according to the following order.

The notations and assumptions used are given in section 2. The model starting without shortages is studied in section 3, and the corresponding one starting with shortages is studied in section 5. For each model, the optimal policy is obtained in section 4 and 6. A numerical example highlighting the obtained results and a sensitivity analysis are given in section 8 and 9, respectively. The paper closes with concluding remarks in section 10.

2. ASSUMPTIONS AND NOTATIONS

The inventory model is developed under the following assumptions:

1. The system operates for a prescribed period T units of time and the replenishment rate is infinite.
2. The ordering quantity brings the inventory level up to the order level Q.
3. Holding cost $C_1(t)$ per unit time is time dependent and is assumed to be as
4. $C_1(t) = h + bt$, where $h > 0$ and $b > 0$.
5. Shortages are backlogged at a rate β(x) which is a non increasing function of x with 0< β(x) ≤ 1, β(0)=1 and x is the waiting time up to the next replenishment. Moreover, it is assumed that β(x) satisfies the relation β(x)+T β/(x) ≥0, where β/(x) is the derivative of β(x). The case with β(x) =1 corresponds to a complete backlogging model.
6. The demand rate D(t) is a ramp-type function of time and is as follows:

$$D(t) = \begin{cases} f(t), t < \mu, \\ f(\mu), t \geq \mu, \end{cases}$$

$f(t)$ is positive, continuous for $t \in$[0,T]

In addition, the following notations are used in developing the proposed inventory model:

1. T the constant scheduling period.
2. t_1 time at which the inventory reaches zero.
3. C_1 holding cost per unit per time unit.
4. C_2 shortage cost per unit per time unit.
5. C_3 cost incurred from the deterioration of one unit.
6. C_4 per unit opportunity cost due to lost sales.
7. θ the constant deterioration rate, where $0 < \theta < 1$
8. μ the parameter of the ramp-type demand function (time point)
9. $I(t)$ the inventory level at time $t \in [0, T]$
10. $H(t)$ be the total holding cost.

3. MATHEMATICAL FORMULATION OF THE MODEL STARTING WITHOUT SHORTAGES

In this section, the inventory model starting with no shortages is studied. The replenishment at the beginning of the cycle brings the inventory level up to Q. Due to deterioration and demand, the inventory level gradually depletes during the period [0, t_1] and falls to zero at $t = t_1$. Thereafter, shortages occur during [t_1,T], which is partially backlogged. The backlogged demand is satisfied at the next replenishment.

The inventory level, (t) , $0 \leq t \leq T$ satisfies the following differential equations.

$$\frac{dI(t)}{dt} + \theta I(t) = -D(t) \, , 0 \leq t \leq t_1 \, , I(t_1) = 0 \qquad (1)$$

$$\frac{dI(t)}{dt} = -D(t)\beta(T-t), t_1 \leq t \leq T \, , \, I(t_1) = 0 \qquad (2)$$

The solutions of these differential equations are affected by the relation between t_1 and μ through the demand rate function. To continue, the two cases:

(i) $t_1 < \mu$

(ii) $t_1 \geq \mu$ must be considered.

3.1 Case I ($t_1 < \mu$)

The realization of the inventory level is depicted in Figure 1

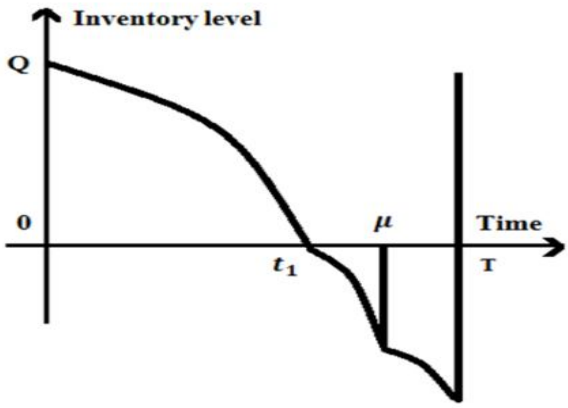

Figure 1: Inventory level for the model starting without shortages over the cycle ($t_1 < \mu$)

In this case, equation (1) becomes

$$\frac{dI(t)}{dt} + \theta I(t) = -f(t) \;\; 0 \leq t \leq t_1 \, , I(t_1) = 0 \qquad (3.1.1)$$

Equation (2) leads to the following two:

$$\frac{dI(t)}{dt} = -f(t)\beta(T-t), \; t_1 \leq t \leq \mu \, , I(t_1) = 0 \qquad (3.1.2)$$

$$\frac{dI(t)}{dt} = -f(\mu)\beta(T-t) \, , \; \mu \leq t \leq T \, , I(\mu_-) = I(\mu_+) \qquad (3.1.3)$$

The solutions of (3.1.1)-(3.1.3) are respectively

$$I(t) = e^{-t\theta} \int_t^{t_1} f(x)e^{\theta x} dx \;,\; 0 \le t \le t_1 \tag{3.1.4}$$

$$I(t) = -\int_{t_1}^{t} f(x)\beta(T-x)dx \,,\, t_1 \le t \le \mu \tag{3.1.5}$$

$$I(t) = -f(\mu)\int_{\mu}^{t} \beta(T-x)dx - \int_{t_1}^{\mu} f(x)\beta(T-x)dx \,,\, \mu \le t \le T \tag{3.1.6}$$

The total amount of deteriorated items during $[0,t_1]$

$$D = \int_0^{t_1} f(t)e^{\theta t}dt - \int_0^{t_1} f(t)\,dt \tag{3.1.7}$$

The total holding cost in the interval $[0,t_1]$ is given as follows using (3.1.4)

$$H(t) = \int_0^{t_1} C_1(t)I(t)dt = \int_0^{t_1} \left\{(h+bt)e^{-t\theta}\int_t^{t_1} f(x)e^{\theta x}dx\right\}dt$$

$$= h\int_0^{t_1} e^{-\theta t}\int_t^{t_1} f(x)e^{\theta x}dx\,dt \;+ b\int_0^{t_1} te^{-\theta t}\int_t^{t_1} f(x)e^{\theta x}dx\,dt \tag{3.1.8}$$

Time weighted backorders due to shortages during $[t_1,T]$ is

$$I_2(t) = \int_{t_1}^{T}[-I(t)]dt = \int_{t_1}^{\mu}[-I(t)]dt + \int_{\mu}^{T}[-I(t)]dt$$

$$= \int_{t_1}^{\mu}\int_{t_1}^{t} f(x)\beta(T-x)dx\,dt + \int_{\mu}^{T}\left[f(\mu)\int_{\mu}^{t}\beta(T-x)dx + \int_{t_1}^{\mu} f(x)\beta(T-x)dx\right]dt \tag{3.1.9}$$

The amount of lost sales during $[t_1,T]$ is

$$L = \int_{t_1}^{\mu}[1-\beta(T-t)]f(t)dt + f(\mu)\int_{\mu}^{T}[1-\beta(T-t)]dt$$

The order quantity is

$$Q = \int_0^{t_1} f(x)e^{\theta x}dx + \int_{t_1}^{\mu} f(x)\beta(T-x)dx + f(\mu)\int_{\mu}^{T}\beta(T-x)dx \tag{3.1.10}$$

The total cost in the interval $[0,T]$ is the sum of holding, shortage, deterioration and opportunity costs, and is given by

$$TC_1(t_1) = H(t) + C_2(I_2) + C_3D + C_4L$$

$$= h\int_0^{t_1} e^{-\theta t}\int_t^{t_1} f(x)e^{\theta x}dx\,dt \;+ b\int_0^{t_1} te^{-\theta t}\int_t^{t_1} f(x)e^{\theta x}dx\,dt \;+$$

$$C_2\Big\{\int_{t_1}^{\mu}\int_{t_1}^{t} f(x)\beta(T-x)dx\,dt +$$

$$\int_{\mu}^{T}\left[f(\mu)\int_{\mu}^{t}\beta(T-x)dx + \int_{t_1}^{\mu} f(x)\beta(T-x)dx\right]dt\Big\} +$$

$$C_3\,\Big\{\int_0^{t_1} f(t)e^{\theta t}dt - \int_0^{t_1} f(t)\,dt\Big\} +$$

$$C_4\,\Big\{\int_{t_1}^{\mu}[1-\beta(T-t)]f(t)dt + f(\mu)\int_{\mu}^{T}[1-\beta(T-t)]\,dt\Big\} \tag{3.1.11}$$

3.2 Case $\overline{\underline{\text{II}}}$ ($t_1 \geq \mu$)

The realization of the inventory level is depicted in Figure 2

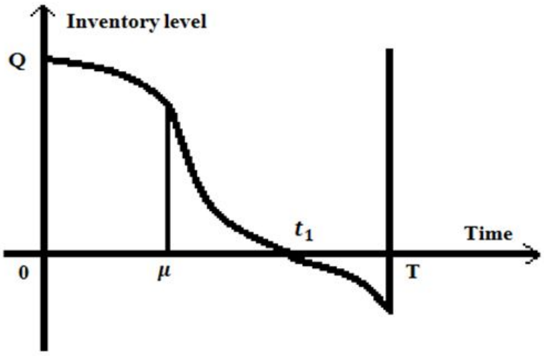

Figure 2: Inventory level for the model starting without shortages over the cycle ($t_1 \geq \mu$)

In this case, equation (1) reduces to the following two:

$$\frac{dI(t)}{dt} + \theta I(t) = -f(t)\,,\ 0 \leq t \leq \mu\,,\ I(\mu_-) = I(\mu_+) \tag{3.2.1}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -f(\mu)\,,\ \mu \leq t \leq t_1,\ I(t_1)=0, \tag{3.2.2}$$

Equation (2) becomes

$$\frac{dI(t)}{dt} = -f(\mu)\beta(T-t)\,,\ t_1 \leq t \leq T\,,\,,\ I(t_1)=0, \tag{3.2.3}$$

Their solutions are respectively,

$$I(t) = e^{-t\theta}\left\{\int_t^{\mu} e^{\theta x} f(x)dx + f(\mu)\int_{\mu}^{t_1} e^{\theta x}dx\right\},\ 0 \leq t \leq \mu \tag{3.2.4}$$

$$I(t) = e^{-t\theta} f(\mu)\int_t^{t_1} e^{\theta x}dx\,,\ \mu \leq t \leq t_1, \tag{3.2.5}$$

$$I(t) = f(\mu)\int_{t_1}^{t} \beta(T-x)dx\,,\ t_1 \leq t \leq T \tag{3.2.6}$$

The total amount of deteriorated items during $[0, t_1]$

$$D = \int_0^{\mu} f(t)e^{\theta t}dt + f(\mu)\int_{\mu}^{t_1} e^{\theta t}dt - \int_{\mu}^{t_1} f(\mu)\,dt \tag{3.2.7}$$

The total holding cost in the interval $[0, t_1]$ is given as follows using (3.2.1) & (3.2.2)

$$H(t) = \int_0^{t_1} C_1(t)I(t)dt = \int_0^{\mu}(h+bt)I(t)dt + \int_{\mu}^{t_1}(h+bt)I(t)dt$$

$$=h\left[\int_0^{\mu} e^{-t\theta}\left\{\int_t^{\mu} e^{\theta x} f(x)dx + f(\mu)\int_{\mu}^{t_1} e^{\theta x}dx\right\}dt + \int_{\mu}^{t_1} e^{-t\theta} f(\mu)\int_t^{t_1} e^{\theta x}dx\,dt\right] +$$

$$b\left[\int_0^{\mu} te^{-t\theta}\left\{\int_t^{\mu} e^{\theta x}f(x)dx+\ f(\mu)\int_{\mu}^{t_1} e^{\theta x}dx\right\}dt+\int_{\mu}^{t_1} te^{-t\theta}f(\mu)\int_t^{t_1} e^{\theta x}dx\,dt\right] \quad (3.2.8)$$

Time weighted backorders due to shortages during $[t_1,\text{T}]$ are

$$I_2(t)=\int_{t_1}^{T}[-I(t)]dt$$

$$=\int_{t_1}^{T}\int_{t_1}^{t} f(\mu)\beta(T-x)dx\,dt \quad (3.2.9)$$

The amount of lost sales during $[t_1,\text{T}]$ is

$$L=f(\mu)\int_{t_1}^{T}[1-\beta(T-t)]dt \quad (3.2.10)$$

The order quantity is

$$\text{Q}=\int_0^{\mu} f(x)e^{\theta x}dx+f(\mu)\int_{\mu}^{t_1} dx+f(\mu)\int_{t_1}^{T}\beta(T-x)dx \quad (3.2.11)$$

The total cost in the interval [0,T]

$$TC_2(t_1)=H(t)+C_2(I_2)+C_3D+C_4L$$

$$=$$

$$h\left[\int_0^{\mu} e^{-t\theta}\left\{\int_t^{\mu} e^{\theta x}f(x)dx+\ f(\mu)\int_{\mu}^{t_1} e^{\theta x}dx\right\}dt+\int_{\mu}^{t_1} e^{-t\theta}f(\mu)\int_t^{t_1} e^{\theta x}dx\,dt\right]+$$

$$b\left[\int_0^{\mu} te^{-t\theta}\left\{\int_t^{\mu} e^{\theta x}f(x)dx+\ f(\mu)\int_{\mu}^{t_1} e^{\theta x}dx\right\}dt+\int_{\mu}^{t_1} te^{-t\theta}f(\mu)\int_t^{t_1} e^{\theta x}dx\,dt\right]$$

$$+\,C_2\left\{\int_{t_1}^{T}\int_{t_1}^{t} f(\mu)\beta(T-x)dx\,dt\right\}$$

$$+\,C_3\left\{\int_0^{\mu} f(t)e^{t\theta}dt+f(\mu)\int_{\mu}^{t_1} e^{t\theta}dt-\int_{\mu}^{t_1} f(\mu)\,dt\right\}$$

$$+\,C_4\left\{f(\mu)\int_{t_1}^{T}[1-\beta(T-t)]dt\right\} \quad (3.2.12)$$

Summarizing the total cost of the system over [0,T]

$$TC(t_1)=\begin{cases}TC_1(t_1),t_1<\mu\\ TC_2(t_1),t_1\geq\mu\end{cases}$$

4. THE OPTIMAL REPLENISHMENT POLICY OF THE MODEL STARTING WITHOUT SHORTAGES

The existence of uniqueness t_1, say $t_1{}^*$, which minimizes the total cost function for the model starting without shortages. Although the argument t_1 of the functions $TC_1(t_1)$, $TC_2(t_1)$ is constrained, we shall search for their unconstrained minimum. The first and second order derivatives of $TC_1(t_1)$ are, respectively

$$\frac{dTC_1(t_1)}{dt_1} = f(t_1)g(t_1), \tag{4.1}$$

$$\frac{d^2TC_1(t_1)}{dt_1{}^2} = \frac{df(t_1)}{dt_1}g(t_1) + f(t_1)\frac{dg(t_1)}{dt_1}$$

Where

$$g(t_1) = he^{\theta t_1}\int_0^{t_1} e^{-t\theta}dt + be^{\theta t_1}\int_0^{t_1} te^{-t\theta}\,dt - C_2(T-t_1)\beta(T-t_1) + C_3\left(e^{\theta t_1}-1\right) - C_4(1-\beta(T-t_1)) \tag{4.2}$$

It can be easily verified that, when $0 \le \beta(x) \le 1$, $g(0) < 0$, $g(T) > 0$and further

$$\frac{dg(t_1)}{dt_1} = h + h\theta e^{\theta t_1}\int_0^{t_1} e^{-t\theta}dt + bt_1 + b\theta e^{\theta t_1}\int_0^{t_1} te^{-t\theta}\,dt$$
$$+C_2\left[\beta(T-t_1) + (T-t_1)\beta^{/}(T-t_1)\right] + C_3\theta e^{\theta t_1} - C_4\beta^{/}(T-t_1) > 0 \tag{4.3}$$

The assumption (4.1) made for $\beta(x)$, ($\beta(x)$ non-increasing so $\beta^{/}(x)$< 0 and $\beta(x) + T\beta^{/}(x) \ge 0$) implies that $g(t_1)$ is strictly increasing. By assumption $f(t_1) > 0$ and so the derivative $\frac{dTC_1(t_1)}{dt_1}$ vanishes at $t_1{}^*$, with 0< $t_1{}^*$<T, which is the unique root of

$$g(t_1) = 0 \tag{4.4}$$

For this $t_1{}^*$ we have

$$\frac{d^2TC_1(t_1)}{dt_1{}^2} = \frac{df(t_1)}{dt_1}g(t_1) + f(t_1)\frac{dg(t_1)}{dt_1} > 0$$

So that $t_1{}^*$ corresponds to unconstrained global minimum.
For the branch $TC_2(t_1)$, the first and second order derivatives are

$$\frac{dTC_2(t_1)}{dt_1} = f(\mu)k(t_1),$$

$$\frac{d^2TC_2(t_1)}{dt_1{}^2} = f(\mu)\frac{dk(t_1)}{dt_1} > 0 \tag{4.5}$$

Where the function $k(t_1)$ is given by

$$k(t_1) = he^{\theta t_1}\left\{\int_0^{\mu} e^{-t\theta}dt + \int_{\mu}^{t_1} e^{-t\theta}dt\right\} + be^{\theta t_1}\left\{\int_0^{\mu} te^{-t\theta}dt + \int_{\mu}^{t_1} te^{-t\theta}dt\right\} - C_2(T-t_1)\beta(T-t_1) + C_3\left(e^{\theta t_1}-1\right) - C_4(1-\beta(T-t_1)) \tag{4.6}$$

The inequality (4.5) follows from (4.3) and ensures the strict convexity of $TC_2(t_1)$. When $0 \le \beta(x) \le 1$ and based on the properties of $k(t_1)$, we conclude that $\frac{dTC_2(t_1)}{dt_1}$ vanishes at $t_1{}^*$, with $0 < t_1{}^* < T$, which is the unique root of

$$k(t_1) = 0 \tag{4.7}$$

Now, an algorithm is proposed that leads to the optimal policy

Step1. compute $t_1{}^*$ from equation (4.4) or (4.7)

Step2. compare $t_1{}^*$ to μ

2.1 If ${t_1}^* \leq \mu$, then the total cost function and the optimal order quantity are given by (3.1.11) and (3.1.10)

2.2 If ${t_1}^* > \mu$, then the total cost function and the optimal order quantity are given by (3.2.12) and (3.2.11)

5. MATHEMATICAL FORMULATION OF THE MODEL STARTING WITH SHORTAGES

The cycle now starts with shortages that occur during the period $[0, t_1]$ and are partially backlogged. After time t_1, a replenishment brings the inventory level up to Q. Demand and deterioration of the items depletes the inventory level during the period $[t_1$,T] until this falls to zero at t = T. Again, the two cases

(i) $t_1 < \mu$

(ii) $t_1 \geq \mu$ must be examined.

5.1 Case I ($t_1 < \mu$)

The realization of the inventory level is depicted in Figure 3

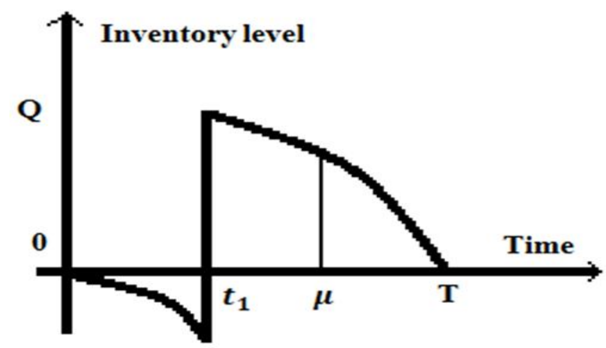

Figure 3: Inventory level for the model starting with shortages over the cycle ($t_1 < \mu$)

The inventory level $I(t)$, 0≤ t ≤ T satisfies the following differential equations:

$$\frac{dI(t)}{dt} = -f(t)\beta(t_1 - t)\,,\ 0 \leq t \leq t_1\,,\ I(0) = 0, \tag{5.1.1}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -f(t)\,,\ t_1 \leq t \leq \mu\,,\ I(\mu_-) = I(\mu_+)\,, \tag{5.1.2}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -f(\mu)\,,\ \leq t \leq T\,,\ I(T) = 0, \tag{5.1.3}$$

The solutions of (5.1.1) – (5.1.3) are respectively

$$I(t) = -\int_0^t f(x)\beta(t_1 - x)dx,\ 0 \leq t \leq t_1 \tag{5.1.4}$$

$$I(t) = e^{-t\theta}\left\{\int_t^\mu f(x)e^{\theta x}dx + f(\mu)\int_\mu^T e^{\theta x}dx\right\},\ t_1 \leq t \leq \mu \qquad 5.1.5)$$

$$I(t) = e^{-t\theta} f(\mu) \int_t^T e^{\theta x} dx \,, \mu \le t \le T \quad (5.1.6)$$

The total amount of deterioration during $[t_1, T]$ is

$$D = e^{-\theta t_1} \left\{ \int_{t_1}^{\mu} f(x) e^{\theta x} dx + f(\mu) \int_{\mu}^{T} e^{\theta x} dx \right\} - \int_{t_1}^{\mu} f(x) dx - (T - \mu) f(\mu) \quad (5.1.7)$$

The total inventory carried during the interval $[t_1, T]$ is

$$H(t) = h\left[\int_{t_1}^{\mu} e^{-t\theta} \left\{ \int_t^{\mu} f(x) e^{\theta x} dx + f(\mu) \int_{\mu}^{T} e^{\theta x} dx \right\} dt + \int_{\mu}^{T} e^{-t\theta} f(\mu) \int_t^T e^{\theta x} dx dt \right] +$$

$$b\left[\int_{t_1}^{\mu} t e^{-t\theta} \left\{ \int_t^{\mu} f(x) e^{\theta x} dx + f(\mu) \int_{\mu}^{T} e^{\theta x} dx \right\} dt + \int_{\mu}^{T} t e^{-t\theta} f(\mu) \int_t^T e^{\theta x} dx dt \right] \quad (5.1.8)$$

Time weighted backorder during the time interval $[0, t_1]$ is

$$I_2 = \int_0^{t_1} (t_1 - t) f(t) \beta(t_1 - t) dt \quad (5.1.9)$$

The amount of lost sales during the time interval $[0, t_1]$ is

$$L = \int_0^{t_1} \big(1 - \beta(t_1 - t)\big) f(t) dt \quad (5.1.10)$$

The order quantity is

$$Q' = \int_0^{t_1} f(t) \beta(t_1 - t) dt + e^{-\theta t_1} \left\{ \int_{t_1}^{\mu} f(x) e^{\theta x} dx + f(\mu) \int_{\mu}^{T} e^{\theta x} dx \right\} \quad (5.1.11)$$

Using similar argument as in the previous model, the total cost of this model is

$$TC_1'(t_1) = h\left[\int_{t_1}^{\mu} e^{-t\theta} \left\{ \int_t^{\mu} f(x) e^{\theta x} dx + f(\mu) \int_{\mu}^{T} e^{\theta x} dx \right\} dt + \int_{\mu}^{T} e^{-t\theta} f(\mu) \int_t^T e^{\theta x} dx dt \right]$$
$$+ b\left[\int_{t_1}^{\mu} t e^{-t\theta} \left\{ \int_t^{\mu} f(x) e^{\theta x} dx + f(\mu) \int_{\mu}^{T} e^{\theta x} dx \right\} dt + \int_{\mu}^{T} t e^{-t\theta} f(\mu) \int_t^T e^{\theta x} dx dt \right]$$
$$+ C_2 \left\{ \int_0^{t_1} (t_1 - t) f(t) \beta(t_1 - t) dt \right\} + C_3 \left\{ e^{-\theta t_1} \left\{ \int_{t_1}^{\mu} f(x) e^{\theta x} dx + f(\mu) \int_{\mu}^{T} e^{\theta x} dx \right\} - \int_{t_1}^{\mu} f(x) dx - (T - \mu) f(\mu) \right\}$$
$$+ C_4 \left\{ \int_0^{t_1} \big(1 - \beta(t_1 - t)\big) f(t) dt \right\} \quad (5.1.12)$$

5.2 Case <u>II</u> ($t_1 \ge \mu$)

The realization of the inventory level is depicted in Figure 3

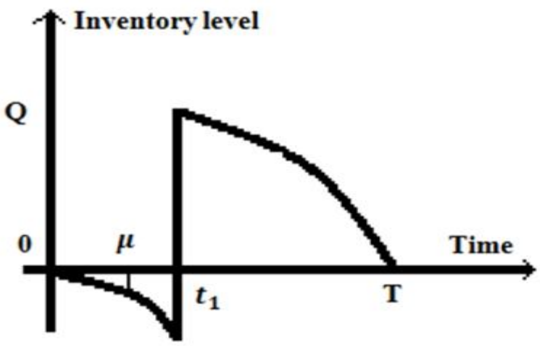

Figure 2: Inventory level for the model starting without shortages over the cycle ($t_1 \geq \mu$)

The inventory level $I(t)$, $0 \leq t \leq T$ satisfies the following differential equations:

$$\frac{dI(t)}{dt} = -f(t)\beta(t_1 - t)\ , 0 \leq t \leq \mu\ , I(0) = 0, \tag{5.2.1}$$

$$\frac{dI(t)}{dt} = -f(\mu)\beta(t_1 - t)\ , \leq t \leq t_1\ , I(\mu_-) = I(\mu_+), \tag{5.2.2}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -f(\mu),\ t_1 \leq t \leq T\ , I(T) = 0, \tag{5.2.3}$$

The solutions are respectively,

$$I(t) = -\int_0^t f(x)\,\beta(t_1 - x)dx\ , 0 \leq t \leq \mu, \tag{5.2.4}$$

$$I(t) = -\int_0^\mu f(x)\,\beta(t_1 - x)dx - f(\mu)\int_\mu^t \beta(t_1 - x)dx, \mu \leq t \leq t_1, \tag{5.2.5}$$

$$I(t) = e^{-t\theta} f(\mu)\int_t^T e^{\theta x}dx\ , t_1 \leq t \leq T\ , \tag{5.2.6}$$

Total amount of deteriorated during [t_1,T] is

$$D = e^{-\theta t_1} f(\mu)\int_{t_1}^T e^{\theta x}dx - (T - t_1)f(\mu)\ , \tag{5.2.7}$$

The total inventory carried during the interval [t_1,T] is

$$H(t) = h\left[\int_{t_1}^T e^{-t\theta} f(\mu)\int_t^T e^{\theta x}dx\,dt\right] + b\left[\int_{t_1}^T te^{-t\theta} f(\mu)\int_t^T e^{\theta x}dx\,dt\right] \tag{5.2.8}$$

Time weighted backorder during the time interval [0, t_1] is

$$I_2 = \int_0^\mu \left[\int_0^t f(x)\beta(t_1 - x)dx\right]dt + \int_\mu^{t_1}\left[\int_0^\mu f(x)\,\beta(t_1 - x)dx + f(\mu)\int_\mu^t \beta(t_1 - x)dx\right]dt \tag{5.2.9}$$

The amount of lost sales during the time interval [0, t_1] is

$$L = \int_0^\mu f(x)\big(1 - \beta(t_1 - x)\big)dx + f(\mu)\int_\mu^{t_1}\big(1 - \beta(t_1 - x)\big)dx \tag{5.2.10}$$

The order quantity is

$$Q' = \int_0^{\mu} f(t)\beta(t_1 - t)dt + f(\mu)\int_{\mu}^{t_1} \beta(t_1 - t)dt + e^{-\theta t_1}\left\{f(\mu)\int_{t_1}^{T} e^{\theta t}dt\right\} \quad (5.2.11)$$

Using similar argument as in the previous model, the total cost of this model is

$$TC'_2(t_1) = h\left[\int_{t_1}^{T} e^{-t\theta} f(\mu)\int_{t}^{T} e^{\theta x}dx\, dt\right] + b\left[\int_{t_1}^{T} te^{-t\theta} f(\mu)\int_{t}^{T} e^{\theta x}dx\, dt\right] + C_2\left\{\int_0^{\mu}\left[\int_0^{t} f(x)\beta(t_1 - x)dx\right]dt + \int_{\mu}^{t_1}\left[\int_0^{\mu} f(x)\,\beta(t_1 - x)dx + f(\mu)\int_{\mu}^{t}\beta(t_1 - x)dx\right]dt\right\} + C_3\left\{e^{-\theta t_1} f(\mu)\int_{t_1}^{T} e^{\theta x}dx - (T - t_1)f(\mu)\right\} + C_4\left\{\int_0^{\mu} f(x)\big(1 - \beta(t_1 - x)\big)dx + f(\mu)\int_{\mu}^{t_1}\big(1 - \beta(t_1 - x)\big)dx\right\} \quad (5.2.12)$$

Summarizing the total cost of the system over [0,T]

$$TC'(t_1) = \begin{cases} TC'_1(t_1), t_1 < \mu \\ TC'_2(t_1), t_1 \geq \mu \end{cases}$$

6. THE OPTIMAL REPLENISHMENT POLICY OF THE MODEL STARTING WITH SHORTAGES

We derive the optimal replenishment policy of the model starting with shortages, we calculate the value of t_1, say t^*_1, which minimize the total cost function. Taking the first order derivative of $TC_1{}'(t_1)$ and equating to zero gives

$$-(h + bt_1 + \theta C_3)e^{-\theta t_1}\left[\int_{t_1}^{\mu} f(x)e^{\theta x}dx + f(\mu)\int_{\mu}^{T} e^{\theta x}dx\right] + \int_0^{t_1}\left[C_2\beta(t_1 - t) + C_2(t_1 - t)\beta'(t_1 - t) - C_4\beta'(t_1 - t)\right]f(t)dt = 0 \quad (6.1)$$

If t^*_1 is a root of (6.1), for this root the second order condition for minimum is

$$(h\theta + b(t^*_1\theta - 1) + C_3\theta^2)e^{-\theta t^*_1}\left[\int_{t^*_1}^{\mu} f(x)e^{\theta x}dx + f(\mu)\int_{\mu}^{T} e^{\theta x}dx\right] + (h + bt^*_1 + \theta C_3)e^{-2\theta t^*_1} f(t^*_1) + \int_0^{t^*_1}\left[2C_2\beta'(t^*_1 - t) + C_2(t^*_1 - t)\beta''(t^*_1 - t) - C_4\beta''(t^*_1 - t)\right]f(t)dt + \left(C_2\beta(0) - C_4\beta'(0)\right)f(t^*_1) > 0 \quad (6.2)$$

Equating the first order derivative of $TC'_2(t_1)$ to zero gives

$$-(h + bt_1 + \theta C_3)e^{-\theta t_1} f(\mu)\int_{t_1}^{T} e^{\theta x}dx + \int_0^{\mu}\left[C_2\beta(t_1 - t) + C_2(t_1 - t)\beta'(t_1 - t) - C_4\beta'(t_1 - t)\right]f(t)dt$$

$$+f(\mu)\int_{\mu}^{t_1}[C_2\beta(t_1-t)+C_2(t_1-t)\beta^{/}(t_1-t)-C_4\beta^{/}(t_1-t)]dt=0 \qquad (6.3)$$

If t^*_1 is a root of (6.3) for this root, the second order condition for minimum is

$$\left(h\theta+b(t^*_1\theta-1)\right)+C_3\theta^2e^{-\theta t^*_1}\int_{t^*_1}^{T}e^{\theta x}dx+(h+bt^*_1+\theta C_3)f(\mu)+$$
$$\int_0^{\mu}[2C_2\beta^{/}(t^*_1-t)+C_2(t^*_1-t)\beta^{//}(t^*_1-t)-C_4\beta^{//}(t_1-t)]f(t)dt+$$
$$f(\mu)\int_{\mu}^{t^*_1}[2C_2\beta^{/}(t^*_1-t)+C_2(t^*_1-t)\beta^{//}(t^*_1-t)-C_4\beta^{//}(t_1-t)]dt+$$
$$(C_2\beta(0)-C_4\beta^{/}(0))>0 \qquad (6.4)$$

Now, an algorithm is proposed that leads to the optimal policy:

Step1. Find the global minimum, t^*_1 for $TC_1{}^{/}(t_1)$. This will be one of the following:

a) a root $t_1{}^*$ from equation (6.1) , which satisfies (6.2)
b) $t_1{}^*=\mu$
c) $t_1{}^*=0$

the total cost function and the optimal order quantity are given by (5.1.12) and (5.1.11)

Step2. Find the global minimum, t^*_1 for $TC_2{}^{/}(t_1)$. This will be one of the following:

a) a root of $t_1{}^*$ from equation (6.3) , which satisfies (6.4)
b) $t_1{}^*=\mu$
c) $t_1{}^*=T$

the total cost function and the optimal order quantity are given by (5.2.12) and (5.2.11)

7. NUMERICAL EXAMPLE

An example is given to illustrate the results of the models developed in this study with the following parameters,

$C_2=$ ₹ $15\ per\ unit\ per\ year;\ C_3=$ ₹ $5\ per\ unit;\ C_4=$ ₹ $20\ per\ unit;$

$h=$ ₹$1\ per\ unit;\ b=$ ₹$.01\ per\ unit;\ \theta=.001;\ \mu=.2year;\ T=1year; f(t)=3e^{4.5t}$;

$\beta(x)=e^{-0.2x}$

To find the value of t_1 and hence the total cost, and the optimal order quantity.

Model starting without shortages:

Using (6.4) or (6.6), the optimal value of t_1 is $t^*_1=.949076>\mu$. The optimal ordering quantity is $Q_2=6.8775$ (from 6.2.22), and the minimum cost is TC_2= ₹ 0.207622.

Since the nature of the cost function is highly non linear, the convexity of the function is shown graphically.

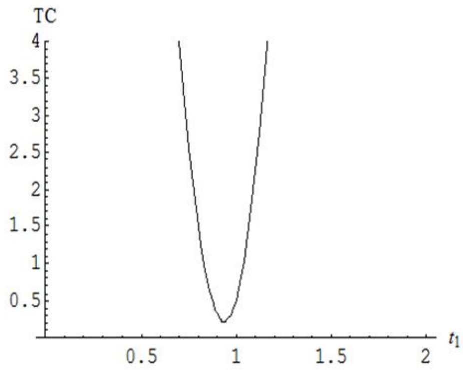

Figure 5. The graphical representation of the total cost function model starting without shortage.

Model starting with shortages:

Using (6.12), the optimal value of $t_1$is $t^*{}_1 = .141333 < \mu$. The optimal ordering quantity is $Q_1 = 6.87141$ (from 6.3.12) and the minimum cost is TC_1^*= ₹ 3.44586.

Since the nature of the cost function is highly non linear, the convexity of the function is shown graphically.

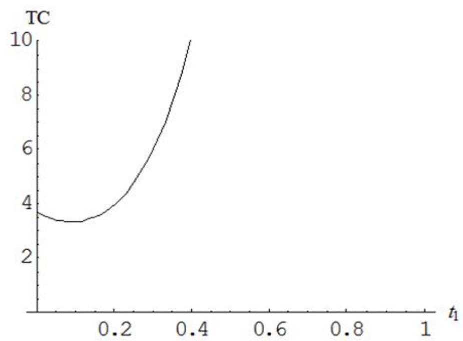

Figure 6. The graphical representation of the total cost function model starting with shortages.

8. SENSIVITY ANALYSIS

Here, we have studied the effects of changing the values of the parameters C_2 , C_3 , C_4, h, b, θ on the optimal cost and on stock level derived by the proposed method.

The sensitivity analysis is performed by changing the value of each of the parameters by –50%, –25%, 25%, and 50%, taking one parameter at a time and keeping the remaining five parameters unchanged. Using the numerical example given in section 8, a sensitivity analysis is performed to explore the sensitiveness of the parameters to the optimal policy.

		Model starting without shortages		Model starting with shortages	
parameter	Percentage change(%)	TC*(₹)	Q*	TC/*(₹)	Q/*
		0.207622	6.8775	3.44586	6.87141
C_2	-50	0.305874	6.87429	3.95661	6.84309
	-25	0.246166	6.87643	3.51139	6.86481
	25	0.180725	6.87811	3.44493	6.87434
	50	0.160903	6.87850	3.45870	6.87591
C_3	-50	0.198554	6.87751	3.43792	6.87144
	-25	0.203088	6.87750	3.44189	6.87142
	25	0.212154	6.87749	3.44982	6.87139
	50	0.216687	6.87749	3.45399	6.87138
C_4	-50	0.226457	6.87700	3.32786	6.87324
	-25	0.216562	6.87727	3.37777	6.87241
	25	0.199502	6.87769	3.53923	6.87017
	50	0.192097	6.87786	3.66855	6.86860
h	-50	0.076866	6.87907	1.79969	6.87678
	-25	0.132428	6.87839	2.63796	6.87678
	25	0.30108	6.87642	4.22508	6.86834
	50	0.411519	6.87642	4.97755	6.86514
b	-50	0.201569	6.87751	3.43972	6.87142
	-25	0.204596	6.8775	3.44279	6.87141
	25	0.210647	6.87749	3.43512	6.87142
	50	0.213671	6.87749	3.45199	6.8714
θ	-50	0.193895	6.87505	3.43322	6.86942
	-25	0.203046	6.87668	3.44164	6.87074
	25	0.212197	6.87831	3.45007	6.87207
	50	0.216773	6.87913	3.45428	6.87274

From the results of the above table, the following observations can be made.

1) For the negative change of the parameter C_2, in both the models starting with shortages and without shortage, TC will increase.
2) Other parameters are less sensitive.

9. CONCLUSION

In this paper, an order level inventory model for deteriorating items with time-varying holding cost has been studied. The model is fairly general as the demand rate is any function of time up to the time-point of its stabilization (general ramp-type demand rate), and the backlogging rate is any non increasing function of waiting time, up to the next replenishment. Moreover, the traditional parameter of holding cost is assumed here to be time varying. As time value of money and price index change, holding cost may not remain constant over time. It is assumed that the holding cost is linearly increasing function of time. The inventory model is studied under two different replenishment policies: (i) starting with no shortages and (ii) starting with shortages.

Again, if holding cost is constant, then the model starting with no shortages, $t_1^* = 0.949305 > \mu$, TC= ₹ 0.195515 and Q= 6.87752 and for the model starting with shortages $t_1^* = 0.141197 < \mu$, TC= ₹ 3.43359 and Q=6.87143.

The total cost with constant holding cost is less than the total cost with time-varying holding cost, which is a realistic situation.

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