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ESTIMATION OF $P\{X < Y\}$ FOR GAMMA EXPONENTIAL MODEL

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Abstract: In this paper, we estimate probability $P\{X < Y\}$ when X and Y are two independent random variables from gamma and exponential distribution, respectively. We obtain maximum likelihood estimator and its asymptotic distribution. We also perform a simulation study.

Keywords: Reliability; gamma distribution; exponential distribution; maximum likelihood estimator; asymptotic normality.

MSC2010: 62F12, 60E05.

1. INTRODUCTION

Estimation of the probability $R = P\{X < Y\}$ is a very important subject of interest, especially in the field of a system reliability. There are a lot of papers in statistical literature considering derivation of an explicit expression for R. For some well-known distributions, the algebraic form of R has been worked out.

Estimation of R when X and Y are normal has been considered by Downtown [4]. Tong [16] analyzed estimation of R when X and Y are exponential variables. Constantine and Karson [2], Ismail *et al*. [7], and Constantine *et al*. [3] estimated R when X and Y are from gamma distributions with known shape parameters. Reliability for logistic distribution is analyzed in Nadarajah [10], and for Laplace distribution in Nadarajah [11]. Bivariate beta and bivariate gamma distribution are considered in

Nadarajah [12, 13]. Nadarajah and Kotz [14] derived forms of R for generalizations of the exponential, gamma, beta, extreme value, logistic, and the Pareto distributions. Kundu and Gupta [8, 9] estimated $P\{X < Y\}$ for generalized exponential distribution and for Weibull distributions. Rezaei *et al*. [15] considered generalized Pareto distributions.

But for many other distributions (including generalization of distributions or mixed distributions), expression of R has not been derived. The aim of this paper is to estimate R when X follows gamma and Y follows exponential distribution with unknown shape and scale parameters. Note that exponential distribution is a special case of gamma distribution, but we can say that our results are different from those in Constantine and Karson [2], Ismail *et al*. [7] and Constantine *et al*. [3] because we have three parameters that should be estimated.

2. ESTIMATION OF *R*

Let us suppose that random variable X has gamma distribution with parameters α and β, where $\alpha>0$ and $\beta>0$. We denote it with X: $G(\alpha, \beta)$. Its probability density function is given by:

$$f(x;\alpha,\beta)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}},\quad x\geq 0,$$

where α is a shape parameter and β is a scale parameter.

Let us suppose that random variable Y has exponential distribution with parameter λ, where $\lambda>0$. We denote it with Y: $E(\lambda)$. Its probability density function is given by:

$$g(y;\lambda)=\frac{1}{\lambda}e^{-\frac{y}{\lambda}},\quad y\geq 0,$$

where λ is a scale parameter. It is known that $E(\lambda)$ distribution is indeed a $G(1, \lambda)$ distribution.

2.1. Maximum likelihood estimator of R

Let X: $G(\alpha, \beta)$ and Y:$E(\lambda)$, where X and Y are independent random variables. Therefore

$$\begin{aligned}R=P\{X<Y\}&=\int_0^{\infty}dx\int_x^{\infty}f(x;\alpha,\beta)g(y;\lambda)dy\\&=\int_0^{\infty}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}}dx\int_x^{\infty}\frac{1}{\lambda}e^{-\frac{y}{\lambda}}dy\end{aligned}\tag{1}$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1} e^{-x(\frac{1}{\beta}+\frac{1}{\lambda})} dx$$

$$= \left(\frac{\lambda}{\lambda+\beta}\right)^{\alpha}.$$

Joint distribution for (X,Y) is given by $h(x,y;\alpha,\beta,\lambda) = \frac{1}{\lambda\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-(\frac{x}{\beta}+\frac{y}{\lambda})}$, $x \geq 0$, $y \geq 0$. Let $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ be a random sample from that distribution. Therefore, the likelihood function and its ln are given by

$$L(\alpha,\beta,\lambda) = \frac{1}{(\lambda\beta^{\alpha}\Gamma(\alpha))^n}(\prod_{i=1}^{n} x_i)^{\alpha-1} e^{-(\frac{1}{\beta}\sum_{i=1}^{n} x_i + \frac{1}{\lambda}\sum_{i=1}^{n} y_i)}$$
$$\ln L(\alpha,\beta,\lambda) = -n(\ln\lambda + \alpha\ln\beta + \ln\Gamma(\alpha)) + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i - \frac{1}{\lambda}\sum_{i=1}^{n} y_i, \quad (2)$$

Taking partial derivatives of ln$L$ with respect to α,β, and λ, we get

$$\frac{\partial \ln L}{\partial\alpha} = -n(\ln\beta + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}) + \sum_{i=1}^{n}\ln x_i ,$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{-n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i ,$$

$$\frac{\partial \ln L}{\partial\lambda} = \frac{-n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} y_i .$$

Given the above identities to be equal to 0 and solving those equations, we obtain

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \frac{1}{n}\sum_{i=1}^{n}\ln x_i - \ln\beta, \quad (3)$$

$$\beta = \frac{1}{n\alpha}\sum_{i=1}^{n} x_i , \quad (4)$$

$$\lambda = \frac{1}{n}\sum_{i=1}^{n} y_i \quad (5)$$

Therefore,

$$\hat{\lambda} = \overline{Y}_n . \quad (6)$$

From (3) and (4), we obtain

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \frac{1}{n}\sum_{i=1}^{n}\ln x_i - \ln(\frac{1}{n}\sum_{i=1}^{n}x_i) + \ln\alpha .$$

Solving this differential equation, we get

$$\ln\Gamma(\alpha) = (\frac{1}{n}\sum_{i=1}^{n}\ln x_i - \ln(\frac{1}{n}\sum_{i=1}^{n}x_i))\alpha + \alpha\ln\alpha - \alpha + C ,$$

where C is some arbitrarily constant. Therefore, $\hat{\alpha}$ can be obtained as a solution of the equation of the form

$$s(\alpha) = \alpha ,$$

where

$$\mathrm{s}(\alpha) = (\frac{1}{n}\sum_{i=1}^{n}\ln x_i - \ln(\frac{1}{n}\sum_{i=1}^{n}x_i))\alpha + \alpha\ln\alpha - \ln\Gamma(\alpha) + C . \tag{7}$$

Since $\hat{\alpha}$ is a fixed point of the function s, it can be obtained by using the iterative procedure:

$$s(\alpha_k) = \alpha_{k+1} , \tag{8}$$

where α_k is the kth iterate of $\hat{\alpha}$. The procedure should be stopped when $|\alpha_{k+1} - \alpha_k|$ is sufficiently small. Once $\hat{\alpha}$ is obtained, from (4) we get

$$\hat{\beta} = \frac{\overline{X}_n}{\hat{\alpha}} . \tag{9}$$

Maximum likelihood estimates have invariance property, so we obtain

$$\hat{R} = \left(\frac{\hat{\lambda}}{\hat{\lambda} + \hat{\beta}}\right)^{\hat{\alpha}} . \tag{10}$$

2.2. Asymptotic distribution

Denote the Fisher information matrix of (α, β, λ) as $I(\alpha, \beta, \lambda)$, where

$$I(\alpha,\beta,\lambda) = -\begin{bmatrix} E(\frac{\partial^2 \ln h}{\partial\alpha^2}) & E(\frac{\partial^2 \ln h}{\partial\alpha\partial\beta}) & E(\frac{\partial^2 \ln h}{\partial\alpha\partial\lambda}) \\ E(\frac{\partial^2 \ln h}{\partial\beta\partial\alpha}) & E(\frac{\partial^2 \ln h}{\partial\beta^2}) & E(\frac{\partial^2 \ln h}{\partial\beta\partial\lambda}) \\ E(\frac{\partial^2 \ln h}{\partial\lambda\partial\alpha}) & E(\frac{\partial^2 \ln h}{\partial\lambda\partial\beta}) & E(\frac{\partial^2 \ln h}{\partial\lambda^2}) \end{bmatrix}. \quad (11)$$

The ln of joint distribution h is given by

$$\ln h(x,y;\alpha,\beta,\lambda) = -(\ln\lambda + \alpha\ln\beta + \ln\Gamma(\alpha)) + (\alpha-1)\ln x - (\frac{x}{\beta} + \frac{y}{\lambda}).$$

The second partial derivatives of ln h are:

$$\frac{\partial^2 \ln h}{\partial\alpha^2} = -\frac{\Gamma''(\alpha)\Gamma(\alpha) - (\Gamma'(\alpha))^2}{\Gamma^2(\alpha)},$$

$$\frac{\partial^2 \ln h}{\partial\alpha\partial\beta} = \frac{\partial^2 \ln h}{\partial\beta\partial\alpha} = -\frac{1}{\beta},$$

$$\frac{\partial^2 \ln h}{\partial\alpha\partial\lambda} = \frac{\partial^2 \ln h}{\partial\lambda\partial\alpha} = 0,$$

$$\frac{\partial^2 \ln h}{\partial\beta^2} = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3},$$

$$\frac{\partial^2 \ln h}{\partial\beta\partial\lambda} = \frac{\partial^2 \ln h}{\partial\lambda\partial\beta} = 0,$$

$$\frac{\partial^2 \ln h}{\partial\lambda^2} = \frac{1}{\lambda^2} - \frac{2y}{\lambda^3}.$$

Using $EX = \alpha\beta$ and $EY = \lambda$, we obtain

$$I(\alpha,\beta,\lambda) = \begin{bmatrix} \frac{\Gamma''(\alpha)\Gamma(\alpha) - (\Gamma'(\alpha))^2}{\Gamma^2(\alpha)} & \frac{1}{\beta} & 0 \\ \frac{1}{\beta} & \frac{\alpha}{\beta^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix}. \quad (12)$$

Theorem 1. As $n \to \infty$ then,

$$(\sqrt{n}(\hat{\alpha}-\alpha), \sqrt{n}(\hat{\beta}-\beta), \sqrt{n}(\hat{\lambda}-\lambda)) \xrightarrow{D} N_3(\mathbf{0}, I^{-1}(\alpha,\beta,\lambda)).$$

Proof: The proof follows from the asymptotic normality of maximum likelihood estimates (see [6]).

□

Let $r(\alpha,\beta,\lambda)$ be a transformation such that the matrix of partial derivatives

$$B=\left[\frac{\partial r}{\partial\alpha}\quad\frac{\partial r}{\partial\beta}\quad\frac{\partial r}{\partial\lambda}\right]$$

has continuous elements and does not vanish in a neighborhood of (α,β,λ). Let $\hat{\eta}=r(\hat{\alpha},\hat{\beta},\hat{\lambda})$. Then $\hat{\eta}$ is the maximum likelihood estimate of η, where $\eta=r(\alpha,\beta,\lambda)$ and

$$\sqrt{n}(\hat{\eta}-\eta)\overset{D}{\rightarrow}N_3(0,BI^{-1}(\alpha,\beta,\lambda)B'). \tag{13}$$

Considering our transformation $R=\left(\frac{\lambda}{\lambda+\beta}\right)^{\alpha}$ and using the previous, we get the next result.

Theorem 2. As $n\rightarrow\infty$ then,

$$\sqrt{n}(\hat{R}-R)\overset{D}{\rightarrow}N(0,V),$$

where

$$V=(\frac{\lambda}{\lambda+\beta})^{2\alpha}\frac{\alpha}{\alpha i_{11}-1}\left[\ln^2\frac{\lambda}{\lambda+\beta}+\frac{2\beta}{\lambda+\beta}\ln\frac{\lambda}{\lambda+\beta}+\frac{\alpha\beta^2}{(\lambda+\beta)^2}(\alpha i_{11}+i_{11}-1)\right] \tag{14}$$

and

$$i_{11}=\frac{\Gamma''(\alpha)\Gamma(\alpha)-(\Gamma'(\alpha))^2}{\Gamma^2(\alpha)}. \tag{15}$$

Proof: We obtain

$$B=(\frac{\lambda}{\lambda+\beta})^{\alpha}\left[\ln\frac{\lambda}{\lambda+\beta}\quad -\frac{\alpha}{\lambda+\beta}\quad \frac{\alpha\beta}{\lambda(\lambda+\beta)}\right]$$

and

$$I^{-1}(\alpha,\beta,\lambda)=\frac{1}{\alpha i_{11}-1}\begin{bmatrix}\alpha & -\beta & 0\\ -\beta & \beta^2 i_{11} & 0\\ 0 & 0 & \lambda^2(\alpha i_{11}-1)\end{bmatrix}.$$

After multiplication, we get the result. □

Theorem 2 can be used to construct asymptotic confidence intervals. To construct them, we only have to estimate variance V. Estimate $\hat{V}$, we obtain by changing α,β and λ in formula (14) with $\hat{\alpha},\hat{\beta}$ and $\hat{\lambda}$.

3. A SIMULATION STUDY

In this section, we present a simulation study to see the performance of an estimator R. We take different values for α,β,λ, actually we use arbitrary parameters settings. The whole approach can be applied on any other parameters setting.

For each parameters setting, we generate samples of size 15, 25, 50 from gamma and exponential distribution. From the sample, we compute the estimate of λ using (6). For estimate of α, we use an iterative process (8). We use initial value to be 1, and the iterative process stops when the difference between two iterative values are less than 10^{-6}. When we estimate α, we compute the estimate of β by using (9). Finally, we compute the MLE of R by using (10). The mean square error (MSE) of $\hat{R}$ over 1000 replications is presented in Table 1.

Table 1. MSE of $\hat{R}$

(α,β,λ)	Sample size	MSE($\hat{R}$)
(1,1,1)	15	0.00956
	25	0.00563
	50	0.00297
(1.5,2,1)	15	0.00796
	25	0.00551
	50	0.00265
(2,1.5,1.5)	15	0.00493
	25	0.00283
	50	0.00160
(0.5,0.5,1)	15	0.01408
	25	0.00875
	50	0.00475
(0.5,1.5,3)	15	0.01735
	25	0.01081
	50	0.00588

From Table 1, we can see that all MSE($\hat{R}$) decrease as the sample size increases. It verifies the consistency property of the MLE estimators of R.

The MLE of R, $\hat{R}$ and its variance are reported in Table 2. We compute the variance of $\hat{R}$ by using (14). We use this variance to construct confidence interval for R. These results are reported in Table 2.

Table 2. Confidence interval for R

(α,β,λ)	Sample size	R	$\hat{R}$	Var($\hat{R}$)	95% Confidence Interval
(1,1,1)	15	0.50	0.46	0.00922	(0.263, 0.657)
	25	0.50	0.48	0.00541	(0.332, 0.628)
	50	0.50	0.49	0.00271	(0.388, 0.592)
(1.5,2,1)	15	0.19	0.28	0.00764	(0.101, 0.459)
	25	0.19	0.17	0.00303	(0.059, 0.281)
	50	0.19	0.18	0.00158	(0.102, 0.258)
(2,1.5,1.5)	15	0.25	0.26	0.00730	(0.085, 0.435)
	25	0.25	0.25	0.00442	(0.116, 0.384)
	50	0.25	0.25	0.00222	(0.158, 0.342)
(0.5,0.5,1)	15	0.82	0.76	0.00331	(0.642, 0.878)
	25	0.82	0.78	0.00222	(0.685, 0.875)
	50	0.82	0.79	0.00114	(0.724, 0.856)
(0.5,1.5,3)	15	0.82	0.73	0.00488	(0.587, 0.873)
	25	0.82	0.75	0.00229	(0.654, 0.846)
	50	0.82	0.83	0.00046	(0.788, 0.872)

From Table 2, we can see that even for small sample sizes, confidence intervals based on the MLE's work quite well in terms of interval lengths. It is observed that when the sample size is increased, then the lengths of the confidence intervals and variance of $\hat{R}$ decrease.

3.1. Numerical example

Here, we present a numerical example. Suppose that we have two data sets that represent the failure time of the air conditioning system of two different air planes (see [1, 5]). Let X be the failure time for the first plane (namely 7911 in [1]) and Y be the failure time for the second plane (namely 7912 in [1]). Let us suppose that we have 11 realizations of the random variable X: 33, 47, 55, 56, 104, 176, 182, 220, 239, 246, 320 and 11 realizations of the random variable Y: 1, 5, 7, 11, 12, 14, 42, 47, 52, 225, 261. We can model these X values with G (1, 152.5) or E(152.5) distribution (KS statistic is approximately 0.23 and p value is greater than 0.05). Also, we can model these Y values with E(61.5) distribution (KS statistic is approximately 0.34 and p value is greater than 0.05). Therefore, R=0.29.

We obtain the MLE of α by using the iterative procedure (8). We start with the initial value 1.00 and the iteration stops whenever two consecutive values are less than 10^{-3}. This iteration process provides $\hat{\alpha}$ approximately 0.999 (with arbitrarily constant C equals 1.25). Now, using (6) and (9), we obtain $\hat{\lambda}$=61.5454 and $\hat{\beta} = 152.5455$. Therefore $\hat{R} = 0.288$. Based on the sample values, we can conclude that there is 28.8% chances that air conditioning system of the second plane will work longer than air conditioning system of the first plane.

4. CONCLUSION

In this paper, we have considered the estimation of the probability $P\{X < Y\}$ when X and Y are two independent random variables from gamma and exponential distributions, respectively. We found maximum likelihood estimator and used its asymptotic distribution to construct confidence intervals. We performed a simulation study to show the consistency property of the MLE estimators of R.

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REFERENCES

[1] Bain, L.J. and Engelhardt, M., Statistical Analysis of Reliability and Life-Testing Models, 2nd. Edition, Marcel and Dekker, New York, 1991.
[2] Constantine, K. and Karson, M., "The Estimation of P(Y<X) in Gamma Case", *Communications in Statistics: Simulation and Computation*, 15(1986) 365-388.
[3] Constantine, K., Karson, M. and Tse, S.K., "Confidence Interval Estimation of P(Y<X) in the Gamma Case", *Communications in Statistics: Simulation and Computation*, 19 (1) (1990) 225-244.
[4] Downtown, F., "The Estimation of P(Y<X) in the Normal Case", Technometrics, 15 (1973), 551-558.
[5] Gupta, R. D. and Kundu, D., "Closeness of Gamma and Generalized Exponential Distribution", *Communications in statistics – Theory and Methods*, 32, (4) (2003) 705-721.
[6] Hogg, R. V., McKean, J. W. and Craig, A. T., Introduction to Mathematical Statistics, Sixth Edition, Pearson Prentice Hall, New Jersey, 2005, 348-349.
[7] Ismail, R, Jeyaratnam, S. S. and Panchapakesan, S., "Estimation of Pr[X>Y] for Gamma Distributions", *Journal of Statistical Computation and Simulation*, 26 (3-4) (1986) 253-267.
[8] Kundu, D. and Gupta, R. D., "Estimation of P[Y<X] for Generalized Exponential Distribution", *Metrika* 61 (2005) 291-308.
[9] Kundu, D. and Gupta, R. D., "Estimation of P[Y<X] for Weibull Distributions", IEEE Transactions on Reliability 55(2) (2006) 270-280.
[10] Nadarajah, S., "Reliability for Logistic Distributions", *Elektronnoe Modelirovanie* 26 (3) (2004) 65-82.
[11] Nadarajah, S., "Reliability for Laplace Distributions", *Mathematical Problems in Engineering* (2) (2004) 169-183.
[12] Nadarajah, S., "Reliability for Some Bivariate Beta Distributions", *Mathematical Problems in Engineering* (1) (2005) 101-111.
[13] Nadarajah, S., "Reliability for Some Bivariate Gamma Distributions", *Mathematical Problems in Engineering* (2) (2005) 151-163.
[14] Nadarajah, S. and Kotz, S., "Reliability for Pareto Models", *Metron* 61 (2003) 191-204.
[15] Rezaei, S., Tahmasbi, R. and Mahmoodi, M., "Estimation of P[Y<X] for Generalized Pareto Distribution", *Journal of Statistical Planning and Inference*, 140 (2010) 480-494.
[16] Tong, H., "A Note on the Estimation of P(Y<X) in the Exponential Case", *Technometrics,* 16 (1974) 625.