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# A NOTE ON $R$-EQUITABLE $K$-COLORINGS OF TREES

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**Abstract:** A graph $G = (V, E)$ is $r$-equitably $k$-colorable if there exists a partition of $V$ into $k$ independent sets $V_1, V_2, \cdots, V_k$ such that $|\,|V_i| - |V_j|\,| \leq r$ for all $i, j \in \{1, 2, \cdots, k\}$. In this note, we show that if two trees $T_1$ and $T_2$ of order at least two are $r$-equitably $k$-colorable for $r \geq 1$ and $k \geq 3$, then all trees obtained by adding an arbitrary edge between $T_1$ and $T_2$ are also $r$-equitably $k$-colorable.

**Keywords:** Trees, equitable coloring, independent sets.

**MSC:** 05C15, 05C69.

## 1 INTRODUCTION

All graphs in this paper are finite, simple and loopless. Let $G = (V, E)$ be a graph. We denote by $|G|$ its order, i.e, the number of vertices in $G$. For a vertex $v \in V$, let $N(v)$ denote the set of vertices in $G$ that are adjacent to $v$. $N(v)$ is called the *neighborhood* of $v$ and its elements are *neighbors* of $v$. The *degree* of vertex $v$, denoted by $deg(v)$, is the number of neighbors of $v$, i.e., $deg(v) = |N(v)|$. $\Delta(G)$ denotes the *maximum degree* of $G$, i.e., $\Delta(G) = \max\{deg(v)|\ v \in V\}$. For a set $V' \subseteq V$, we denote by $G - V'$ the graph obtained from $G$ by deleting all vertices in $V'$ as well as all edges incident to at least one vertex of $V'$.

An *independent set* in a graph $G = (V, E)$ is a set $S \subseteq V$ of pairwise nonadjacent vertices. The maximum size of an independent set in a graph $G = (V, E)$ is called the *independence number* of $G$ and denoted by $\alpha(G)$.

A *k-coloring* $c$ of a graph $G = (V, E)$ is a partition of $V$ into $k$ independent sets which we will denote by $V_1(c), V_2(c), \cdots, V_k(c)$ and refer to as *color classes.*

The cardinality of a largest color class with respect to a coloring $c$ will be denoted by $Max_c$. A graph $G$ is *$r$-equitably $k$-colorable*, with $r \geq 1$ and $k \geq 2$, if there exists a $k$-coloring $c$ of $G$ such that $|\,|V_i(c)| - |V_j(c)|\,| \leq r$ for all $i, j \in \{1, 2, \cdots, k\}$. Such a coloring is called an *$r$-equitable $k$-coloring of $G$*. A graph which is 1-equitably $k$-colorable is simply said to be *equitably $k$-colorable.*

The notion of equitable colorability was introduced in [8] and has been studied since then by many authors (see [2, 3, 4, 5, 6, 7, 9]). In [3], the authors gave a complete characterization of trees which are equitably $k$-colorable. This result was then generalized to forests in [2]. More precisely, for a forest $F = (V, E)$, let $\alpha^*(F) = \min\{\alpha(F - N[v])| \, v \in V \text{ and } deg(v) = \Delta(F)\}$

**Theorem 1.1** *([2]) Suppose $F = (V, E)$ is a forest and $k \geq 3$ is an integer. Then $F$ is equitably $k$-colorable if and only if $k \geq \lceil \frac{|F|+1}{\alpha^*(F)+2} \rceil$.*

This result can easily be generalized to $r$-equitable $k$-colorings.

**Theorem 1.2** *([1]) Suppose $F = (V, E)$ is a forest and $r \geq 1, k \geq 3$ are two integers. Then $F$ is $r$-equitably $k$-colorable if and only if $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$.*

**Proof:** Suppose $F$ is $r$-equitably $k$-colorable for $r \geq 1$ and $k \geq 3$. Let $v$ be a vertex in $F$ such that $deg(v) = \Delta(F)$ and $\alpha(F - N[v]) = \alpha^*(F)$. Clearly, for such a coloring, there are at most $\alpha^*(F) + 1$ vertices in the color class that contains $v$. It follows that all other color classes contain at most $\alpha^*(F) + r + 1$ vertices. Thus $|F| \leq \alpha^*(F) + 1 + (k - 1)(\alpha^*(F) + r + 1) = k(\alpha^*(F) + r + 1) - r$, and we therefore have $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$.

Conversely, let $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$. Consider the forest $F' = (V', E')$ obtained from $F$ by adding $r-1$ new isolated vertices. Then $|F'| = |F|+r-1$ and $\alpha^*(F') = \alpha^*(F)+r-1$. Thus $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil = \lceil \frac{|F'|+1}{\alpha^*(F')+2} \rceil$. By Theorem 1.1, $F'$ is equitably $k$-colorable. Restricting the color classes to $V$ gives an $r$-equitable $k$-coloring of $F$.

In this note, we are interested in a different sufficient condition for a tree to be $r$-equitably $k$-colorable. More precisely, given a tree $T = (V, E)$ and an edge $e \in E$ such that its removal from $T$ creates two trees $T_1$ and $T_2$ of order at least two, we show that if both $T_1$ and $T_2$ are $r$-equitably $k$-colorable, for $r \geq 1$ and $k \geq 3$, then $T$ is also $r$-equitably $k$-colorable. We also explain why $|T_1|, |T_2| \geq 2$ and $k \geq 3$ are necessary conditions.

## 2 A SUFFICIENT CONDITION

Consider a tree $T$ and two integers $r \geq 1$ and $k \geq 3$. Let $c$ be an arbitrary $r$-equitable $k$-coloring of the vertex set of $T$ such that $|V_1(c)| \geq |V_2(c)| \geq \cdots \geq |V_k(c)|$. Then there may be vertices in $T$ which are forced to be colored with color $k$. Indeed, if for instance $T$ is a star on $(k-1)r + k$ vertices, then the vertex $v$ of degree $> 1$ necessarily belongs to $V_k(c)$ and actually $V_k(c) = \{v\}$. Furthermore, we have $|V_i(c)| = r + 1$ for $i \in \{1, 2, \cdots, k-1\}$. It turns out that this is no longer true for colors $1, 2, \cdots, k-1$, as shown in the following property.

**Lemma 2.1** *Consider an $r$-equitably $k$-colorable tree $T$ of order at least two, where $r \geq 1$ and $k \geq 3$. Also, let $\ell$ be any element in $\{1, 2, \cdots, k-1\}$. Then, for any vertex $u$ in $T$, there exists an $r$-equitable $k$-coloring $c$ of $T$ with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$ such that $u \notin V_\ell(c)$.*

**Proof:** Suppose the lemma is false. We then clearly have $|T| \geq 3$. Let $c$ be an $r$-equitable $k$-coloring of $T$ with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$. Among all such colorings we choose one such that, for each $t = 1, 2, \cdots, k$, there is no $r$-equitable $k$-coloring $c'$ of $T$ with $|V_i(c)| = |V_i(c')|$ for $i = 1, 2, \cdots, t-1$ and $\max_{i=t}^{k}\{|V_i(c')|\} < |V_t(c)|$. In other words, $Max_c = |V_1(c)|$ is minimum among all $r$-equitable $k$-colorings of $T$, $|V_2(c)|$ is mininum among all $r$-equitable $k$-colorings $c'$ of $T$ with $Max_{c'} = Max_c$, and so on.

Let $\ell \in \{1, 2, \cdots, k-1\}$ be an integer for which the lemma does not hold. We define $x = 1,\ y = 2,\ z = 3$ if $\ell = 1$, and $x = \ell - 1,\ y = \ell,\ z = \ell + 1$ if $\ell > 1$. Since we assume that the lemma is false, it follows that $u \in V_\ell(c)$, which means that $u \in V_x(c)$ if $\ell = 1$ and $u \in V_y(c)$ if $\ell > 1$. Then $|V_x(c)| > |V_y(c)|$, otherwise we could assign color $x$ to all vertices in $V_y(c)$ and color $y$ to all vertices in $V_x(c)$ to obtain an $r$-equitable $k$-coloring $c'$ with $u \notin V_\ell(c')$, a contradiction. Similarly, we must have $|V_y(c)| > |V_z(c)|$ when $\ell > 1$ since otherwise we could assign color $y$ to all vertices in $V_z(c)$ and color $z$ to all vertices in $V_y(c)$, and thus the lemma would hold.

We define $F$ as the subgraph of $T$ induced by $V_x(c) \cup V_y(c) \cup V_z(c)$. If $F$ is disconnected, we add some edges to make $F$ become a tree $T'$ such that no two adjacent vertices have the same color with respect to $c$; otherwise we set $T' = F$. Let $V'$ denote the vertex set of $T'$. Moreover, for $q = y$ or $z$, we denote $\overline{q} = y + z - q$. This implies that $\overline{q} = z$ if $q = y$ and $\overline{q} = y$ if $q = z$. We start by proving the following two claims.

*Claim 1: There exists no $r$-equitable 3-coloring $c'$ of $T'$ (using colors $x, y, z$) with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_q(c')| = |V_q(c)| + 1$ and $|V_{\overline{q}}(c')| = |V_{\overline{q}}(c)|$ for $q = y$ or $z$.*

Indeed, if such a coloring $c'$ exists, then the assumption on $c$ implies $|V_q(c')| = |V_x(c)| > |V_x(c')|$. Now we can obtain an $r$-equitable $k$-coloring $c^*$ of $T$ by letting $V_x(c^*) = V_q(c')$, $V_q(c^*) = V_x(c')$, and $V_i(c^*) = V_i(c')$ if $i \neq x, q$. We distinguish two cases:

- If $\ell = 1$, we have $|V_1(c^*)| > \max_{i=2}^{k}\{|V_i(c^*)|\}$ and $u \notin V_1(c^*)$.
- If $\ell > 1$, we have $q = y$ since otherwise $|V_z(c')| = |V_z(c)| + 1 = |V_x(c)|$ which contradicts $|V_x(c)| > |V_y(c)| > |V_z(c)|$. Then $|V_1(c^*)| \geq \cdots \geq |V_{\ell-1}(c^*)| > |V_\ell(c^*)| \geq |V_{\ell+1}(c^*)| \geq \cdots \geq |V_k(c^*)|$ and $u \in V_{\ell-1}(c^*)$.

Thus, in both cases, $c^*$ is an $r$-equitable $k$-coloring of $T$ such that $|V_i(c^*)| \geq |V_j(c^*)|$ for all $1 \leq i < j \leq k$ and $u \notin V_\ell(c^*)$, a contradiction.

*Claim 2: No leaf of $T'$, except possibly $u$, is in $V_x(c)$.*

Indeed, assume $T'$ has a leaf $v \neq u$ in $V_x(c)$ and let $w$ be its unique neighbor in $T'$. We can change the color of $v$ from $x$ to $\overline{c(w)}$ to obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(w)}}(c')| = |V_{\overline{c(w)}}(c)| + 1$ and $|V_{c(w)}(c')| = |V_{c(w)}(c)|$, contradicting Claim 1.

Let **vec**$T$ be the oriented rooted tree obtained from $T'$ by orienting the edges from root $u$ to the leaves. Let us partition the vertices in $V_x(c)$ into subsets $U_1, \cdots, U_p$ such that $U_q$ $(q = 1, 2, \cdots, p)$ contains all vertices in $V_x(c)$ having no successor in $V_x(c) - \bigcup_{j=1}^{q-1} U_j$. For a vertex $v \in U_1$, let $L(v)$ denote the set of leaves in **vec**$T$ having $v$ as predecessor.

If $|L(v)| = 1$ for some $v \in U_1$, then let $P = v \to s_1 \to \cdots \to s_a$ denote the path from $v$ to the leaf $s_a$ in $L(v)$. If $v = u$ (and hence $\ell = 1$ since $u \in V_x(c)$) then $T'$ is a chain with only one vertex in $V_x(c)$, which means that $V_y(c) = V_z(c) = \emptyset$ since $|V_x(c)| > |V_y(c)| \geq |V_z(c)|$. Thus $T'$ has only one vertex, namely $u$, and since $u \in V_1(c)$ this implies that $T$ has only one vertex, a contradiction. Hence $v \neq u$. Let $w$ be the predecessor of $v$ in $\mathbf{vec}T$:

- if $c(w) = c(s_1)$, we change the color of $v$ to $\overline{c(w)}$ to obtain an r-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(w)}}(c')| = |V_{\overline{c(w)}}(c)| + 1$ and $|V_{c(w)}(c')| = |V_{c(w)}(c)|$, contradicting Claim 1;
- if $c(w) \neq c(s_1)$, we assign color $c(s_1)$ to $v$, color $c(s_{j+1})$ to $s_j$ ($j = 1, 2, ..., a-1$), and color $x$ to $s_a$; we obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $|V_i(c')| = |V_i(c)|$ ($i = x, y, z$), $c'(u) = c(u)$ and a leaf $s_a \in V_x(c')$. But this contradicts Claim 2.

We therefore conclude that $|L(v)| \geq 2$ for all $v \in U_1$. By denoting $W_1 = \bigcup_{v \in U_1} L(v)$, we get $|W_1| \geq 2|U_1|$. For each set $U_q$, with $q > 1$, we will now construct a set $W_q$ containing vertices in $V_y(c) \cup V_z(c)$ that are successors of vertices in $U_q$ but not successors of vertices in $U_{q-1}$. So let $v$ be any vertex in $U_q$ ($q > 1$). If $v$ has at least 2 immediate successors in $\mathbf{vec}T$, we add two of them to $W_q$. If $v$ has a unique immediate successor in $\mathbf{vec}T$, then let $P = v \to s_1 \to \cdots \to s_a \to v'$ denote a path from $v$ to a vertex $v' \in U_{q-1}$. If $a > 1$, we add $s_1$ and $s_2$ to $W_q$. If $a = 1$ and $s_1$ has an immediate successor $w \notin V_x(c)$, then we add $s_1$ and $w$ to $W_q$. Assume now that $a = 1$ and all the immediate successors of $s_1$ are in $V_x(c)$. We will prove that such a case is not possible.

- If $v \neq u$, then $v$ has a predecessor $w$ in $\mathbf{vec}T$. We must have $c(w) = \overline{c(s_1)}$, otherwise we could assign color $\overline{c(s_1)}$ to $v$ to obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$, contradicting Claim 1. But now we can assign color $c(s_1)$ to $v$ and assign color $\overline{c(s_1)}$ to $s_1$ to obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$, contradicting Claim 1.
- If $v = u$, then $\ell = 1$ since $u \in V_x(c)$. By assigning color $c(s_1)$ to $u$ and color $\overline{c(s_1)}$ to $s_1$, we obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$. It follows from the assumptions on $c$ that $|V_{\overline{c(s_1)}}(c')| = |V_x(c)| > |V_{c(s_1)}(c)| = |V_{c(s_1)}(c')|$. Thus the lemma would hold, a contradiction.

In summary, we have $|W_q| \geq 2|U_q|$. Since all sets $W_q$ are disjoint, we have

$$|V_y(c)| + |V_z(c)| \geq \sum_{q=1}^{p} |W_q| \geq \sum_{q=1}^{p} 2|U_q| = 2|V_x(c)|.$$

Hence $|V_y(c)|$ or $|V_z(c)|$ is larger than or equal to $|V_x(c)|$, a contradiction.

Lemma 2.1 allows us to show our main result.

**Theorem 2.2** *Let $T_1$ and $T_2$ be two trees or order at least two. If both $T_1$ and $T_2$ are $r$-equitably $k$-colorable for $r \geq 1$ and $k \geq 3$, then a tree $T$ obtained by adding an arbitrary edge between $T_1$ and $T_2$ is also $r$-equitably $k$-colorable.*

**Proof:** Consider an $r$-equitable $k$-coloring $c$ of $T_1$ and an $r$-equitable $k$-coloring $c'$ of $T_2$ such that $|V_i(c)| \geq |V_j(c)|$ and $|V_i(c')| \geq |V_j(c')|$ for all $1 \leq i < j \leq k$. Let $u$ be a vertex in $T_1$ and $v$ a vertex in $T_2$, and let $T$ be the tree obtained by adding an edge which joins $u$ and $v$. According to Lemma 2.1, we may assume that $v \notin V_1(c')$. Hence $v \in V_{k-\ell+1}(c')$ for some $\ell \in \{1, 2, \cdots, k-1\}$ and it follows from Lemma 2.1 that we may assume that $u \notin V_\ell(c)$. We can therefore construct a $k$-coloring $c^*$ of $T$ such that $V_i(c^*) = V_i(c) \cup V_{k-i+1}(c')$, $i = 1, 2, \cdots, k$. For $i > j$, we have :

$$\begin{aligned} |V_i(c^*)| - |V_j(c^*)| &= |V_i(c)| + |V_{k-i+1}(c')| - (|V_j(c)| + |V_{k-j+1}(c')|) \\ &= (|V_i(c)| - |V_j(c)|) + (|V_{k-i+1}(c')| - |V_{k-j+1}(c')|). \end{aligned}$$

Since $V_j(c) \geq |V_i(c)|$ and $|V_{k-j+1}(c')| \leq |V_{k-i+1}(c')|$, we have :
- $|V_i(c^*)| - |V_j(c^*)| \geq |V_i(c)| - |V_j(c)| \geq -r$;
- $|V_i(c^*)| - |V_j(c^*)| \leq |V_{k-i+1}(c')| - |V_{k-j+1}(c')| \leq r$.

This proves that the considered $k$-coloring $c^*$ of $T$ is $r$-equitable.

Note that the condition $k \geq 3$ in Theorem 2.2 is necessary. Indeed, if both $T_1$ and $T_2$ are isomorphic to a star on 3 vertices (with $u$ being the vertex of degree two in $T_1$ and $v$ a leaf in $T_2$) then clearly $T_1$ and $T_2$ are 1-equitably 2-colorable. But by adding an edge which joins $u$ and $v$, we obtain a tree $T$ which is not 1-equitably 2-colorable.

Note also that the condition in Theorem 2.2 on the number of vertices in each tree is necessary. Indeed, if $T_1$ is an $r$-equitably $k$-colorable tree for some $k \geq 3$ and $r \geq 1$, and if $T_2$ contains a single vertex $v$, then the tree $T'$ obtained by adding an edge which joins $v$ and a vertex $u$ of $T_1$ is possibly not $r$-equitably $k$-colorable. For example, if $u$ is the vertex of degree four in the star $T_1$ on five vertices, and if we add a neighbor $v$ (the single vertex in $T_2$) to $u$, we obtain a star $T'$ on six vertices. While $T_1$ and $T_2$ are clearly 1-equitably 3-colorable, $T'$ is not 1-equitably 3-colorable. It is however not difficult to prove that if $T$ is an $r$-equitably $k$-colorable tree for some $k \geq 2$ and $r \geq 1$, then the tree $T'$ obtained by adding a new vertex $v$ and making it adjacent to some vertex $u$ of $T$ is $(r+1)$-equitably $k$-colorable. Indeed, given an $r$-equitable $k$-coloring $c$ of $T$, we can extend it to a $k$-coloring $c'$ of $T'$ by assigning any color $j \neq c(u)$ to $v$ with $j \in \{1, 2, \cdots, k\}$. If $|V_j(c)| \geq |V_i(c)|$ for all $i \neq j$, then $c'$ is $(r+1)$-equitable, otherwise $c'$ is $r$-equitable.

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