

# A FULL-NEWTON STEP INFEASIBLE-INTERIOR-POINT ALGORITHM FOR $P * (\kappa)$ -HORIZONTAL LINEAR COMPLEMENTARITY PROBLEMS

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**Abstract:** In this paper we generalize an infeasible interior-point method for linear optimization to horizontal linear complementarity problem (HLCP). This algorithm starts from strictly feasible iterates on the central path of a perturbed problem that is produced by suitable perturbation in HLCP problem. Then, we use so-called feasibility steps that serves to generate strictly feasible iterates for the next perturbed problem. After accomplishing a few centering steps for the new perturbed problem, we obtain strictly feasible iterates close enough to the central path of the new perturbed problem. The complexity of the algorithm coincides with the best known iteration complexity for infeasible interior-point methods.

**Keywords:** Horizontal Linear Complementarity Problem (HLCP), Infeasible-interior-point Method, Central Path.

**MSC:** 90C33, 90C51.

## 1. INTRODUCTION

This paper deals with the solution of the horizontal linear complementarity

problem (HLCP) that consists in finding a pair of vectors  $(x, s) \in \mathbf{R}^{2n}$  satisfying

$$Qx + Rs = b, \quad (x, s) \geq 0, \quad x^T s = 0, \quad (P)$$

where  $b$  is in  $\mathbf{R}^n$ ,  $Q$  and  $R$  are real  $n \times n$  matrices. The standard (monotone) linear complementarity problem (LCP) is obtained by taking  $R = -I$  and  $Q$  positive semidefinite. There are other formulations of the HLCP as well, but, as shown in [1], all popular formulations are equivalent, and the behavior of a large class of interior point methods is identical to those formulations, so it is sufficient to prove results for only one of the formulations. We have chosen HLCP because of its symmetry. The LP problem, and the QP problem can be formulated as HLCPs. Therefore, the HLCP provides a convenient general framework for studying interior point methods. Throughout this paper, we assume that the HLCP is  $P_*(\kappa)$ , in the sense that

$$Qu + Rv = 0 \Rightarrow (1 + 4\kappa) \sum_{i \in \mathcal{I}^+} u_i v_i + \sum_{i \in \mathcal{I}^-} u_i v_i \geq 0 \quad \forall u, v \in \mathbf{R}^n, \quad (1)$$

where  $\kappa$  is nonnegative constant and  $\mathcal{I}^+ = \{i : u_i v_i > 0\}$  and  $\mathcal{I}^- = \{i : u_i v_i < 0\}$ . If the above condition is satisfied, then we say that the pair  $(Q, R)$  is a  $P_*(\kappa)$ -pair and write  $(Q, R) \in P_*(\kappa)$ . For  $\kappa = 0$ ,  $P_*(0)$ -HLCP is called the monotone HLCP.

There is great variety of solution approaches for HLCP, which have been studied intensively. Among them, the interior-point methods (IPMs) gained more attention than other methods. After the seminal work of Karmarkar [2], many researchers have proposed IPMs for linear programming (LP), and linear complementarity problem (LCP), and HLCP. One may distinguish between feasible IPMs and infeasible IPMs (IIPMs). Feasible IPMs start with a strictly feasible interior-point and maintain feasibility during the solution process. IIPMs start with an arbitrary positive point and feasibility is reached as the optimality is approached. The choice of the starting point in IIPMs is crucial for the performance. In [3, 4, 5, 6, 7, 8] the authors proposed some feasible IPMs for solving HLCP. Note that there is not always a strictly feasible point to starting an interior-point algorithm. So it is worth while paying attention to infeasible interior-point algorithm. Zhang [9] presented a class of infeasible IPMs for HLCP and showed that the algorithm has  $O(n^2 \log \frac{1}{\varepsilon})$  under some mild assumptions. Stoer, Wechs, and Mizuno have described in [10] simple short-step infeasible-interior-point methods of predictor-corrector type of arbitrarily high local convergence order. In [10], only the local convergence was studied and the complexity of these high-order methods remained open. In [11] was showed that all these methods have the following complexity: Let  $\gamma_0 = (x^0)^T s^0$  and  $\varepsilon > 0$  be arbitrary. If the method is started with a "sufficiently large" infeasible point  $(x^0, s^0) > 0$ , then these methods need at most

$$N = O\left(\left(1 + \kappa\right)^2 n \left\lceil \log \left(\frac{\gamma_0}{\varepsilon}\right) \right\rceil\right)$$

predictor-corrector steps to find a strictly feasible  $\varepsilon$ -solution. This is the best iteration bound for infeasible-start methods. It should be noted that all of most known polynomial variants of IPMs used the so-called central path as a guideline to the

optimal set, and some variants of the Newton method to follow the central path approximately.

In this paper we generalize an infeasible interior-point method for linear optimization introduced by Roos [12], extended by Mansouri and Roos [13] to SDO, and by Mansouri et al. [14] to monotone  $(P_*(0)-)$ LCP, to  $P_*(\kappa)$ -HLCP. We prove that the complexity of the algorithm coincides with the best known iteration bound for infeasible IPMs.

The paper is organized as follows. In Section 2, we summarize a feasible interior point method from our previous work [7]. In Section 3, we introduce the perturbed problem and its central path. Also, we present the full-Newton step IIPM and we explain its main iteration. In Section 4, we analyze the feasibility step. The complexity of the algorithm is obtained in Section 5. Finally, some concluding remarks are presented in Section 6.

The notations used throughout the paper are rather standard: capital letters denote matrices, lower case letters denote vectors, script capital letters denote sets, and Greek letters denote scalars. All vectors are considered to be column vectors. The components of a vector  $u \in \mathbf{R}^n$  will be denoted by  $u_i, i = 1, \dots, n$ . The relation  $u > 0$  is equivalent to  $u_i > 0, i = 1, \dots, n$ , while  $u \geq 0$  means  $u_i \geq 0, i = 1, \dots, n$ . We denote  $\mathbf{R}_+^n = \{u \in \mathbf{R}^n : u \geq 0\}$ ,  $\mathbf{R}_{++}^n = \{u \in \mathbf{R}^n : u > 0\}$ . For any vector  $x \in \mathbf{R}^n$ ,  $x_{\min} = \min(x_1; x_2; \dots; x_n)$  and  $x_{\max} = \max(x_1; x_2; \dots; x_n)$ . If  $u \in \mathbf{R}^n$ , then  $U := \text{diag}(u)$  denotes the diagonal matrix having the components of  $u$  as diagonal entries. If  $x, s \in \mathbf{R}^n$ , then  $xs$  denotes the componentwise (Hadamard) product of the vectors  $x$  and  $s$ . Furthermore,  $e$  denotes all-one vector of length  $n$ . The 2-norm and the infinity norm for vectors are denoted by  $\|\cdot\|$  and  $\|\cdot\|_\infty$ , respectively. We denote the set of feasible points of the HLCP by

$$\mathcal{F} = \{(x, s) \in \mathbf{R}_+^{2n} : Qx + Rs = b\}, \quad (2)$$

and the set of strictly feasible (or interior) points by

$$\mathcal{F}^0 = \{(x, s) \in \mathbf{R}_{++}^{2n} : Qx + Rs = b\}, \quad (3)$$

and the solution set of HLCP by

$$\mathcal{F}^* = \{(x^*, s^*) \in \mathcal{F} : x^* s^* = 0\}. \quad (4)$$

Throughout this paper, it will be assumed that  $\mathcal{F}^*$  is not empty, i.e.  $(P)$  has at least one solution.

## 2. FEASIBLE FULL NEWTON STEP IPM

Note that since in  $(P)$   $x$  and  $s$  are nonnegative,  $x^T s = 0$  holds if and only if  $xs = 0$ . Therefore, solving HLCP is equivalent to finding a solution of the following system of equations:

$$\begin{aligned} Qx + Rs &= b, & x &\geq 0, \\ xs &= 0, & s &\geq 0, \end{aligned} \quad (5)$$

where  $xs = 0$  is the so-called complementarity condition. IPMs replace the complementarity condition by the parameterized equation  $xs = \mu e$ , where  $\mu > 0$  and  $e$  denotes the all-one vector. This gives rise to the following system:

$$\begin{aligned} Qx + Rs &= b, & x &\geq 0, \\ xs &= \mu e, & s &\geq 0, \end{aligned} \quad (6)$$

It has been established (e.g., see[15]) that if HLCP satisfies the interior-point condition (IPC), i.e.,  $\mathcal{F}^0$  be nonempty, then the system (6) has a unique solution denoted by  $(x(\mu), s(\mu))$ , called the  $\mu$ -center of HLCP; the limit  $\lim_{\mu \rightarrow 0} (x(\mu), s(\mu))$  exists and is a solution of the system (5). The set of all  $\mu$ -centers forms a virtual path inside the feasibility region leading to the optimal solution. This path is called central path of HLCP.

A direct application of Newton's method to solve the system (6) with  $\mu$  fixed and assuming  $(x, s) > 0$ , produces the following system for the displacements  $\Delta x$  and  $\Delta s$ .

$$\begin{aligned} Q(x + \Delta x) + R(s + \Delta s) &= b, \\ (x + \Delta x)(s + \Delta s) &= \mu e. \end{aligned}$$

By omitting the quadratic term  $\Delta x \Delta s$  in the second equation, we have the following linear system of equations.

$$\begin{aligned} Q\Delta x + R\Delta s &= b - (Qx + Rs), \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned}$$

Note that if  $(x, s)$  is a feasible solution of HLCP, then  $Qx + Rs = b$ . Hence, the above system is reduce to

$$\begin{aligned} Q\Delta x + R\Delta s &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned} \quad (7)$$

We will refer to the assignment

$$(x^+, s^+) = (x + \Delta x, s + \Delta s), \quad (8)$$

as a full Newton step.

Below, we discuss the feasible IPM method presented in Figure 1. First, note that in this algorithm  $\delta(x, s; \mu)$  is a quantity that measures proximity of the feasible  $(x, s)$  to the  $\mu$ -center  $(x(\mu), s(\mu))$ . This quantity is defined as follows:

$$\delta(x, s; \mu) = \frac{1}{\sqrt{2}} \|v - v^{-1}\|, \quad (9)$$

where

$$v = \sqrt{\frac{xs}{\mu}}. \quad (10)$$

So if  $(x, s)$  lies on the central path, then  $v = e$  and hence,  $\delta(x, s; \mu) = 0$ . Otherwise  $\delta(x, s; \mu) > 0$ . Without loss of generality, we assume that a pair  $(x, s) \in \mathcal{F}$  is given, that is close to  $(x(\mu), s(\mu))$ , for some  $\mu$ . Then,  $\mu$  is decreased to  $\mu^+ = (1 - \theta)\mu$ , for some  $\theta \in (0, 1)$ . Next, we redefine  $\mu = \mu^+$ , and we solve the Newton system (7). The solution  $(\Delta x, \Delta s)$  is known as the Newton direction. By taking a step along this search direction, we construct a new iterate  $(x^+, s^+)$  from (8). We repeat this process until  $\mu$  is small enough, i.e.  $n\mu < \varepsilon$ , where  $\varepsilon$  is a small positive number. The following lemmas are crucial in the analysis of the algorithm. We recall them

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**Feasible IPM for  $P_*(\kappa)$ -HLCP**

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**Input:**

Accuracy parameter  $\varepsilon > 0$ ;  
 threshold parameter  $\tau < 1$ ;  
 barrier update parameter  $\theta, 0 < \theta < 1$ ;  
 feasible pair  $(x^0, s^0)$  with  $(x^0)^T s^0 = n\mu^0$  and  $\mu^0 > 0$  such that  
 $\delta(x^0, s^0; \mu^0) \leq \tau$ .

**begin**

$x := x^0; s := s^0; \mu := \mu^0$ ;

**while**  $n\mu \geq \varepsilon$  **do**

**begin**

update of  $\mu$ :

$\mu := (1 - \theta)\mu$ ;

$(x, s) := (x, s) + (\Delta x, \Delta s)$ ;

**end****end**


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Figure 1: Feasible full-Newton-step algorithm for  $P_*(\kappa)$ -HLCP

without proof. They describe the effect of a  $\mu$ -update and of a full Newton step on  $\delta := \delta(x, s; \mu)$ .

**Lemma 2.1 (Lemma 1.5 in [7]).** *After a full Newton-step, one has*

$$(x^+)^T s^+ \leq (n + \delta^2) \mu.$$

**Lemma 2.2 (Corollary 1.7 in [7]).** *If  $\delta = \delta(x, s; \mu) \leq \frac{1}{\sqrt{2(1+2\sqrt{2}\kappa)}}$ , then we have*

$$\delta(x^+, s^+; \mu) \leq \left( \sqrt{1 + 2\sqrt{2}\kappa} \delta \right)^2,$$

*i.e. quadratic convergence to the  $\mu$ -center is obtained.*

**Lemma 2.3 (Lemma 2.1 in [7]).** *Let  $(x, s)$  be a positive pair and  $\mu > 0$  such that  $x^T s \leq (n + \delta^2)\mu$ . Moreover, let  $\mu^+ = (1 - \theta)\mu$ , then one has*

$$\delta(x, s; \mu^+)^2 \leq (1 - \theta)\delta^2 + \frac{n\theta^2}{2(1 - \theta)} + \frac{\delta^2}{2(1 - \theta)}.$$

From the above lemmas, we may derive the following result that establishes a polynomial iteration bound of the above described algorithm.

**Theorem 2.4 (Theorem 3.1 in [7]).** *If  $\theta = \frac{1}{(1+2\sqrt{2}\kappa)\sqrt{8n}}$ , the number of iterations of the feasible path following algorithm with full-Newton steps does not exceed*

$$\sqrt{8n} \left(1 + 2\sqrt{2}\kappa\right) \log \frac{n\mu^0}{\varepsilon}.$$

### 3. INFEASIBLE FULL-NEWTON STEP IPM

The infeasible-start algorithm is presented in Figure 2.

#### 3.1. The Perturbed Problem and Its Central Path

Let  $x^0 s^0 = \mu^0 e$ , for some arbitrarily positive pair  $(x^0, s^0)$  and some (positive) number  $\mu^0$ , and the initial value of the residual is as  $r^0$

$$r^0 = b - Qx^0 - Rs^0. \quad (11)$$

We consider the perturbed problem  $(P_\nu)$ , defined by

$$b - Qx - Rs = \nu r^0, \quad (x, s) \geq 0, \quad (P_\nu)$$

for any  $\nu$  with  $0 < \nu \leq 1$ . Note that if  $\nu = 1$  then,  $(x, s) = (x^0, s^0)$  yields a strictly feasible solution of  $(P_\nu)$ . We conclude that if  $\nu = 1$ , then  $(P_\nu)$  satisfies the IPC. The following lemma gives a sufficient condition for  $(P_\nu)$  that satisfies the IPC for every  $0 < \nu \leq 1$ .

**Lemma 3.1.** *Let the original problem  $(P)$  be feasible. Then, the perturbed problem  $(P_\nu)$  satisfies the IPC.*

*Proof.* *The proof is similar to the proof of Lemma 4.1 in [14].*  $\square$

Let the original problem  $(P)$  be feasible and  $0 < \nu \leq 1$ . Lemma 3.1 implies that the central path of the problem  $(P_\nu)$  exists. This means that the system

$$\begin{aligned} b - Qx - Rs &= \nu r^0, & x \geq 0, & s \geq 0, \\ xs &= \mu e, \end{aligned} \quad (12)$$

has a unique solution, for each  $\mu > 0$ . This unique solution is the  $\mu$ -center of the perturbed problem  $(P_\nu)$ , denoted by  $(x(\mu, \nu), s(\mu, \nu))$ . Note that since  $x^0 s^0 = \mu^0 e$ ,  $(x^0, s^0)$  is the  $\mu^0$ -center of the perturbed problem  $(P_1)$ . In other words,  $(x(\mu^0, 1), s(\mu^0, 1)) = (x^0, s^0)$ . In the sequel the parameters  $\mu$  and  $\nu$  always satisfy the relation  $\mu = \nu\mu^0$ .

### 3.2. An iteration of the algorithm

The algorithm starts with  $(x, s) = (x^0, s^0)$  that is the  $\mu$ -center of the perturbed problem  $(P_\nu)$  for  $\nu = 1$ . We measure proximity to the  $\mu$ -center of the perturbed problem  $(P_\nu)$  by the quantity  $\delta(x, s; \mu)$  as defined in (9). We assume that at the start of each iteration, just before the  $\mu$ -update,  $\delta(x, s; \mu)$  is smaller than or equal to a (small) threshold value  $\tau > 0$ . This is certainly true at the start of the first iteration, because initially we thus have  $\delta(x, s; \mu) = 0$ .

One (main) iteration of the algorithm works as follows: Suppose that for some  $\mu \in (0, \mu^0)$ , we have  $(x, s)$  satisfying the feasibility condition (12) for  $\nu = \frac{\mu}{\mu^0}$ , and  $\delta(x, s; \mu) \leq \tau$ . We reduce  $\mu$  to  $\mu^+ = (1 - \theta)\mu$ , with  $\theta \in (0, 1)$ , and find new iterates  $(x^+, s^+)$  that satisfy (12), with  $\mu$  replaced by  $\mu^+$  and  $\nu$  by  $\nu^+ = \frac{\mu^+}{\mu^0}$ , such that  $x^T s \leq (n + \delta^2)\mu^+$  and  $\delta(x^+, s^+; \mu^+) \leq \tau$ . Note that  $\nu^+ = (1 - \theta)\nu$ .

To be more precise, this is achieved as follows. Each main iteration consists of a feasibility step and a few centering steps. The feasibility step serves to get iterates  $(x^f, s^f)$  that are strictly feasible for  $(P_{\nu^+})$ , and close to its  $\mu$ -center  $(x(\mu^+, \nu^+), s(\mu^+, \nu^+))$ . According to the definition of  $(P_\nu)$ , the feasibility equation for  $(P_\nu)$  is given by

$$b - Qx - Rs = \nu r^0, \quad (x, s) \geq 0.$$

and that of  $(P_{\nu^+})$  by

$$b - Qx - Rs = \nu^+ r^0, \quad (x, s) \geq 0.$$

To get iterates that are feasible for  $(P_{\nu^+})$ , we need search directions  $\Delta^f x$  and  $\Delta^f s$  such that

$$b - Q(x + \Delta^f x) - R(s + \Delta^f s) = \nu^+ r^0, \quad (x + \Delta^f x, s + \Delta^f s) > 0.$$

Since  $(x, s)$  is feasible for  $(P_\nu)$ , it follows that  $\Delta^f x$  and  $\Delta^f s$  should satisfy

$$Q\Delta^f x + R\Delta^f s = \theta \nu r^0.$$

Therefore, the following system is used to define  $\Delta^f x$  and  $\Delta^f s$ :

$$\begin{aligned} Q\Delta^f x + R\Delta^f s &= \theta \nu r^0, \\ s\Delta^f x + x\Delta^f s &= \mu e - xs, \end{aligned} \tag{13}$$

and after the feasibility step, the iterates given by

$$x^f = x + \Delta^f x, \quad s^f = s + \Delta^f s. \tag{14}$$

We conclude that after the feasibility step, the iterates satisfy the affine equation (12) with  $\nu = \nu^+$ . After the feasibility step, we perform a few ordinary (centering) full Newton steps in order to get iterates  $(x^+, s^+)$  which satisfy  $\delta(x^+, s^+; \mu^+) \leq \tau$ . The hard part in analysis will be to guarantee that  $(x^f, s^f)$  are positive and satisfy

$$\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2(1 + 2\sqrt{2}\kappa)}}. \tag{15}$$

If this is satisfied, then by using Lemma 2.2, the required number of centering steps can easily be obtained. Indeed, assume that  $\delta(x^f, s^f; \mu^+) \leq \frac{1}{2(1+2\sqrt{2\kappa})}$ , which is in agreement with (15). Starting at  $(x^f, s^f)$ , we repeatedly apply full Newton steps until the  $k$ -iterate, denoted as  $(x^+, s^+) := (x^k, s^k)$ , satisfies  $\delta(x^k, s^k; \mu^+) \leq \tau$ . To simplify notations, we define for the moment  $\delta(v^k) = \delta(x^k, s^k; \mu^+)$ ,  $\delta(v^0) = \delta(x^f, s^f; \mu^+)$  and  $\gamma = \sqrt{1 + 2\sqrt{2\kappa}}$ . Note that  $\gamma \geq 1$ . It follows that

$$\delta(v^k) \leq (\gamma \delta(v^{k-1}))^2 \leq \left( \gamma (\gamma \delta(v^{k-2}))^2 \right)^2 \leq \dots \leq \gamma^{2+4+\dots+2^k} (\delta(v^0))^{2^k}.$$

This gives

$$\delta(v^k) \leq \gamma^{2^{k+1}-2} (\delta(v^0))^{2^k} = \gamma^{-2} (\gamma^2 \delta(v^0))^{2^k} \leq (\gamma^2 \delta(v^0))^{2^k}.$$

Using the definition of  $\gamma$  and  $\delta(v^0) \leq \frac{1}{2(1+2\sqrt{2\kappa})}$ , we obtain

$$\gamma^2 \delta(v^0) \leq \left( \sqrt{1 + 2\sqrt{2\kappa}} \right)^2 \frac{1}{2(1 + 2\sqrt{2\kappa})} = \frac{1}{2}.$$

Hence, we certainly have  $\delta(x^+, s^+; \mu^+) \leq \tau$  if  $(\frac{1}{2})^{2^k} \leq \tau$ . From this, we easily deduce that  $\delta(x^+, s^+; \mu^+) \leq \tau$  will hold after at most

$$\left\lceil \log_2 \left( \log_2 \frac{1}{\tau} \right) \right\rceil \tag{16}$$

centering steps.

The algorithm is presented in figure 2.

#### 4. ANALYSIS OF THE ALGORITHM

##### 4.1. The feasibility and the effect of the feasibility step

Let  $(x, s)$  denote the iterates at the start of an iteration, and assume that  $\delta(x, s; \mu) \leq \tau$ . As we established in Section 3.2, the feasibility step generates new iterates  $(x^f, s^f)$  that satisfy the feasibility condition for  $(P_{\nu^+})$ . A crucial element in the analysis is to show that these iterates are positive and lie within the region where the Newton process targeting at  $\mu^+$ -center of  $(P_{\nu^+})$  is quadratically convergent, i.e.,  $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2(1+2\sqrt{2\kappa})}}$ .

Define

$$d_x^f = \frac{v \Delta^f x}{x}, \quad d_s^f = \frac{v \Delta^f s}{s}, \tag{17}$$

where  $v$  is defined in (10). we have the following lemmas.

**Lemma 4.1 (Lemma 5.1 in [14]).** *The iterates  $(x^f, s^f)$  are strictly feasible if and only if  $e + d_x^f d_s^f > 0$ .*



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**Infeasible IPM for  $P_*(\kappa)$ -HLCP**


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**Input:**

Accuracy parameter  $\varepsilon > 0$ ;  
 threshold parameter  $\tau < 1$ ;  
 barrier update parameter  $\theta$ ,  $0 < \theta < 1$ ;  
 feasible pair  $(x^0, s^0)$  with  $(x^0)^T s^0 = n\mu^0$  and  $\mu^0 > 0$  such that  
 $\delta(x^0, s^0, \mu^0) \leq \tau$ .

**begin**

$x := x^0$ ;  $s := s^0$ ;  $\mu := \mu^0$ ;

**while**  $\max(n\mu, \|r\|) \geq \varepsilon$  **do**

**begin**

feasibility step:

$\mu := (1 - \theta)\mu$ ;

$(x, s) := (x, s) + (\Delta^f x, \Delta^f s)$ ;

$\mu$ -update:

$\mu := (1 - \theta)\mu$ ;

centering steps:

**while**  $\delta(v) \geq \tau$  **do**

**begin**

$(x, s) := (x, s) + (\Delta x, \Delta s)$ ;

**end****end****end**


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 Figure 2: Infeasible full-Newton-step algorithm for  $P_*(\kappa)$ -HLCP

**Corollary 4.2 (Corollary 5.2 in [14]).** *The iterates  $(x^f, s^f)$  are strictly feasible if  $\|d_x^f d_s^f\|_\infty < 1$ .*

**Lemma 4.3 (Lemma 5.3 in [14]).** *If  $\|d_x^f\|^2 + \|d_s^f\|^2 < 2$ , then the iterates  $(x^f, s^f)$  are strictly feasible.*

The following lemma gives an upper bound for  $\delta(x^f, s^f; \mu^+)$ . Let  $v^f = \sqrt{\frac{x^f s^f}{\mu^+}}$ , in the sequel, we denote  $\delta(x^f, s^f; \mu^+)$  shortly by  $\delta(v^f)$ .

**Lemma 4.4 (Lemma 5.4 in [14]).** *Let  $\|d_x^f\|^2 + \|d_s^f\|^2 < 2$ , which guarantees strict feasibility of the iterates  $(x^f, s^f)$ . Then, one has*

$$2\delta(v^f)^2 \leq \frac{n\theta^2}{1-\theta} + \frac{\|d_x^f\|^2 + \|d_s^f\|^2}{2(1-\theta)} + (1-\theta) \frac{\|d_x^f\|^2 + \|d_s^f\|^2}{2 - \left(\|d_x^f\|^2 + \|d_s^f\|^2\right)}.$$

Recall that for using the quadratically convergent of the Newton step, we need to have  $\delta(v^f) \leq \frac{1}{\sqrt{2(1+2\sqrt{2}\kappa)}}$ . But from Lemma 4.4, it is sufficient to have

$$\frac{n\theta^2}{1-\theta} + \frac{\|d_x^f\|^2 + \|d_s^f\|^2}{2(1-\theta)} + (1-\theta) \frac{\|d_x^f\|^2 + \|d_s^f\|^2}{2 - \left(\|d_x^f\|^2 + \|d_s^f\|^2\right)} \leq \frac{1}{1+2\sqrt{2}\kappa}. \quad (18)$$

Considering  $\|d_x^f\|^2 + \|d_s^f\|^2$  as a single term, by some elementary calculation, we obtain that if

$$\|d_x^f\|^2 + \|d_s^f\|^2 \leq \frac{1}{2(1+2\sqrt{2}\kappa)}, \quad (19)$$

and

$$0 < \theta \leq \frac{1}{5\sqrt{(n+1)(1+2\sqrt{2}\kappa)}}, \quad (20)$$

then, the inequality (18) is satisfied. In other words, the inequalities (19) and (20) imply that after the feasibility step,  $(x^f, s^f)$  is strictly feasible and lies in the quadratic convergence neighborhood with respect to the  $\mu^+$ -center of  $(P_{\nu^+})$ .

In the remainder of this section, we investigate some bounds for the statement  $\|d_x^f\|^2 + \|d_s^f\|^2$ .

#### 4.2. An upper bound for $\|d_x^f\|^2 + \|d_s^f\|^2$

We start by finding some bounds for the unique solution of the linear system (13).

**Lemma 4.5 (Lemma 3.3 in [16]).** *If HLCP is  $P_*(\kappa)$ , then the linear system*

$$\begin{aligned} su + xv &= a, \\ Qu + Rv &= \tilde{b}, \end{aligned} \quad (21)$$

has a unique solution  $w = (u, v)$ , for any  $z = (x, s) \in R_{++}^{2n}$  and any  $a, \tilde{b} \in R^n$ , and the following inequality is satisfied:

$$\|w\|_z \leq \sqrt{1+2\kappa} \|\tilde{a}\| + (1 + \sqrt{2+4\kappa}) \xi(z, \tilde{b}),$$

where

$$\tilde{a} = (xs)^{-\frac{1}{2}} a, \quad \|w\|_z^2 = \|(u, v)\|_z^2 = \|Du\|^2 + \|D^{-1}v\|^2, \quad D = X^{-\frac{1}{2}} S^{\frac{1}{2}},$$

and

$$\xi(z, \tilde{b})^2 = \min\{\|(\tilde{u}, \tilde{v})\|_z^2 : Q\tilde{u} + R\tilde{v} = \tilde{b}\}.$$

Comparing system (21) with the system (13) and considering  $w = (u, v) = (\Delta^f x, \Delta^f s)$ ,  $a = \mu e - xs$ ,  $\tilde{b} = \theta \nu r^0$ ,  $z = (x, s)$  in the system (21), we have

$$\begin{aligned} & \|D\Delta^f x\|^2 + \|D^{-1}\Delta^f s\|^2 \\ & \leq \left( \sqrt{1+2\kappa} \left\| (xs)^{-\frac{1}{2}}(\mu e - xs) \right\| + (1 + \sqrt{2+4\kappa})\xi(z, \theta \nu r^0) \right)^2. \end{aligned} \quad (22)$$

Note that

$$\left\| (xs)^{-\frac{1}{2}}(\mu e - xs) \right\| = \left\| \frac{\mu e - xs}{\sqrt{\mu} v} \right\| = \sqrt{\mu} \|v^{-1} - v\| = \sqrt{2\mu} \delta,$$

and by definition of  $\xi(z, \tilde{b})$ , we have

$$\xi(z, \theta \nu r^0) = \theta \nu \xi(z, r^0).$$

Also by definitions of  $d_x^f$  and  $d_s^f$ , we obtain  $D\Delta^f x = \sqrt{\mu} d_x^f$  and  $D^{-1}\Delta^f s = \sqrt{\mu} d_s^f$ . By substituting the above equations in (22), we have

$$\|d^f x\|^2 + \|d^f s\|^2 \leq \frac{1}{\mu} \left( \sqrt{2\mu(1+2\kappa)} \delta + (1 + \sqrt{2+4\kappa})\theta \nu \xi(z, r^0) \right)^2. \quad (23)$$

To proceed, we have to specify our initial iterates. In creating the appropriate starting point, we choose

$$x^0 = \rho_p e, \quad s^0 = \rho_d e, \quad \mu^0 = \rho_p \rho_d, \quad (24)$$

where  $\rho_p$  and  $\rho_d$  are such that

$$\|x^*\| \leq \rho_p, \quad \|s^*\| \leq \rho_d, \quad (25)$$

for some optimal solution  $(x^*, s^*) \in \mathcal{F}^*$  and some  $\rho_p, \rho_d > 0$ . However, we also need to make sure that initial conditions for the application of the algorithm are satisfied, that is, we need to check if  $(x^0, s^0) > 0$  and  $\delta(x^0, s^0; \mu^0) < \tau$  are satisfied, but these are obviously satisfied. Note that for such starting points, we have clearly

$$x^* - x^0 \leq \rho_p e, \quad (26)$$

$$s^* - s^0 \leq \rho_d e. \quad (27)$$

Now we find an upper bound for  $\xi(z, r^0)$ .

**Lemma 4.6.** *Let  $\xi(\cdot, \cdot)$  be as defined in Lemma 4.5. Then, we have*

$$\xi(z, r^0) \leq \sqrt{\frac{\rho_p^2}{\mu v_{min}^2} \|s\|_1^2 + \frac{\rho_d^2}{\mu v_{min}^2} \|x\|_1^2}.$$

*Proof.* By definition of  $\xi(z, \tilde{b})$ , we have

$$\begin{aligned}\xi(z, r^0)^2 &= \min\{\|(\tilde{u}, \tilde{v})\|_z^2 : Q\tilde{u} + R\tilde{v} = r^0\} \\ &= \min\{\|D\tilde{u}\|^2 + \|(D)^{-1}\tilde{v}\|^2 : Q\tilde{u} + R\tilde{v} = r^0\}.\end{aligned}$$

We also have

$$r^0 = b - Qx^0 - Rs^0 = Qx^* + Rs^* - Qx^0 - Rs^0 = Q(x^* - x^0) + R(s^* - s^0),$$

thus by applying (26) and (27), the following inequalities are satisfied

$$\begin{aligned}\xi(z, r^0)^2 &\leq \|D(x^* - x^0)\|^2 + \|D^{-1}(s^* - s^0)\|^2 \leq \|\rho_p D e\|^2 + \|\rho_d D^{-1} e\|^2 \\ &= \rho_p^2 \left\| \sqrt{\frac{s}{x}} \right\|^2 + \rho_d^2 \left\| \sqrt{\frac{x}{s}} \right\|^2 \leq \rho_p^2 \left\| \sqrt{\frac{s}{x}} \right\|_1^2 + \rho_d^2 \left\| \sqrt{\frac{x}{s}} \right\|_1^2 \\ &= \frac{\rho_p^2}{\mu} \left\| \sqrt{\frac{\mu}{xs}} s \right\|_1^2 + \frac{\rho_d^2}{\mu} \left\| \sqrt{\frac{\mu}{xs}} x \right\|_1^2 = \frac{\rho_p^2}{\mu} \left\| \frac{s}{v} \right\|_1^2 + \frac{\rho_d^2}{\mu} \left\| \frac{x}{v} \right\|_1^2 \\ &\leq \frac{\rho_p^2}{\mu v_{\min}^2} \|s\|_1^2 + \frac{\rho_d^2}{\mu v_{\min}^2} \|x\|_1^2.\end{aligned}$$

The proof is completed.  $\square$

We proceed aiming to obtain some bounds for  $\|x\|_1$  and  $\|s\|_1$  by presenting the following Lemmas:

**Lemma 4.7.** *Let  $(x, s)$  be feasible for the perturbed problem  $(P_\nu)$  and  $(x^0, s^0)$  as defined in (24). Then, for any optimal solution  $(x^*, s^*)$ , we have*

$$\nu(x^T s^0 + s^T x^0) \leq (1 + 4\kappa)(\nu^2 n \mu^0 + \nu(1 - \nu)((x^*)^T s^0 + (x^0)^T s^*) + x^T s).$$

*Proof.* Since  $r^0 = b - Qx^0 - Rs^0$  and  $b - Qx - Rs = \nu r^0$ , by definition of the perturbed problem, we have

$$\begin{aligned}Q(\nu x^0 + (1 - \nu)x^* - x) + R(\nu s^0 + (1 - \nu)s^* - s) \\ = \nu(Qx^0 + Rs^0) + (1 - \nu)(Qx^* + Rs^*) - (Qx + Rs) \\ = \nu(b - r^0) + (1 - \nu)b - (b - \nu r^0) = 0.\end{aligned}$$

Thus if  $\mathcal{I}^+ = \{i : (\nu x^0 + (1 - \nu)x^* - x)_i (\nu s^0 + (1 - \nu)s^* - s)_i > 0\}$  and  $\mathcal{I}^- = \{i : (\nu x^0 + (1 - \nu)x^* - x)_i (\nu s^0 + (1 - \nu)s^* - s)_i < 0\}$ , then the  $P_*(\kappa)$  property implies that

$$\begin{aligned}(1 + 4\kappa) \sum_{\mathcal{I}^+} (\nu x^0 + (1 - \nu)x^* - x)_i (\nu s^0 + (1 - \nu)s^* - s)_i \\ + \sum_{\mathcal{I}^-} (\nu x^0 + (1 - \nu)x^* - x)_i (\nu s^0 + (1 - \nu)s^* - s)_i \geq 0.\end{aligned}$$

So, we have

$$\begin{aligned} & \sum_{\mathcal{I}^+} (\nu x^0 + (1-\nu)x^* - x)_i (\nu s^0 + (1-\nu)s^* - s)_i \\ & + \sum_{\mathcal{I}^-} (\nu x^0 + (1-\nu)x^* - x)_i (\nu s^0 + (1-\nu)s^* - s)_i \\ & \geq -4\kappa \sum_{\mathcal{I}^+} (\nu x^0 + (1-\nu)x^* - x)_i (\nu s^0 + (1-\nu)s^* - s)_i. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & [\nu x^0 + (1-\nu)x^* - x]^T [\nu s^0 + (1-\nu)s^* - s] \\ & \geq -4\kappa \sum_{\mathcal{I}^+} (\nu x^0 + (1-\nu)x^* - x)_i (\nu s^0 + (1-\nu)s^* - s)_i \\ & \geq -4\kappa \sum_{\mathcal{I}^+} (\nu^2 x_i^0 s_i^0 + \nu(1-\nu)(x_i^* s_i^0 + x_i^0 s_i^*) + x_i s_i) \\ & \geq -4\kappa (\nu^2 (x^0)^T (s^0) + \nu(1-\nu)((x^*)^T s^0 + (x^0)^T s^*) + x^T s), \end{aligned}$$

Since  $(x^*)^T s^* = 0$ ,  $s^T x^* + x^T s^* \geq 0$  and  $s^T x^0 + x^T s^0 \geq 0$ , we deduce that

$$\begin{aligned} & -4\kappa (\nu^2 (x^0)^T (s^0) + \nu(1-\nu)((x^*)^T s^0 + (x^0)^T s^*) + x^T s) \\ & \leq [\nu x^0 + (1-\nu)x^* - x]^T [\nu s^0 + (1-\nu)s^* - s] \\ & = \nu^2 n\mu^0 + \nu(1-\nu)((x^*)^T s^0 + (x^0)^T s^*) - \nu(s^T x^0 + x^T s^0) \\ & + x^T s - (1-\nu)(s^T x^* + x^T s^*) + (1-\nu)(x^*)^T s^* \\ & \leq \nu^2 n\mu^0 + \nu(1-\nu)((x^*)^T s^0 + (x^0)^T s^*) - \nu(s^T x^0 + x^T s^0) + x^T s. \end{aligned}$$

Therefore, we have

$$\nu(x^T s^0 + s^T x^0) \leq (1+4\kappa) (\nu^2 n\mu^0 + \nu(1-\nu) ((x^*)^T s^0 + (x^0)^T s^*) + x^T s).$$

The proof is completed.  $\square$

**Lemma 4.8.** Let  $\delta = \delta(\nu)$  be given by (9). Then

$$\frac{1}{q(\delta)} \leq v_i \leq q(\delta),$$

where

$$q(\delta) := \frac{\sqrt{2}}{2}\delta + \sqrt{\frac{1}{2}\delta^2 + 1}.$$

*Proof.* The proof of this lemma is exactly the same as for Lemma II.60 in [17].  $\square$

**Lemma 4.9.** Let  $(x, s)$  be feasible for the perturbed problem  $(P_\nu)$  and  $(x^0, s^0)$  as defined in (24). Then we have

$$\|x\|_1 \leq (1+4\kappa) (q^2(\delta) + 2) n\rho_p, \quad (28)$$

$$\|s\|_1 \leq (1+4\kappa) (q^2(\delta) + 2) n\rho_d. \quad (29)$$

*Proof.* Using Lemma 4.7 and Lemma 4.8, this lemma may be proved in the same way as in the proof of Lemma 16 in [14].  $\square$

Substituting (28) and (29) in Lemma 4.6 and noting Lemma 4.8, it gives

$$\begin{aligned}\xi(z, r^0) &\leq \sqrt{\frac{2}{\mu} q^2(\delta) (1 + 4\kappa)^2 (q^2(\delta) + 2)^2 n^2 \rho_p^2 \rho_d^2} \\ &= \sqrt{\frac{2}{\mu} q(\delta)(1 + 4\kappa) (q^2(\delta) + 2) n \rho_p \rho_d}.\end{aligned}\quad (30)$$

Also by substituting (30) in (23), we deduce that

$$\begin{aligned}&\|d^f x\|^2 + \|d^f s\|^2 \\ &\leq \frac{1}{\mu} \left( \sqrt{2\mu(1 + 2\kappa)} \delta + \sqrt{2} (1 + \sqrt{2 + 4\kappa}) \theta \frac{\nu}{\sqrt{\mu}} q(\delta)(1 + 4\kappa) (q^2(\delta) + 2) n \rho_p \rho_d \right)^2 \\ &= 2 \left( \sqrt{1 + 2\kappa} \delta + (1 + \sqrt{2 + 4\kappa}) (1 + 4\kappa) q(\delta) (q^2(\delta) + 2) n \theta \right)^2.\end{aligned}\quad (31)$$

Now we are ready for fixing a value to barrier parameter  $\theta$ .

## 5. FIXING THE PARAMETERS AND COMPLEXITY ANALYSIS

We have found that  $\delta(v^f) \leq \frac{1}{\sqrt{2(1+2\sqrt{2\kappa})}}$  holds if the inequalities (19) and (20) are satisfied. Then, by (31), inequality (19) holds if

$$2 \left( \sqrt{1 + 2\kappa} \delta + (1 + \sqrt{2 + 4\kappa}) (1 + 4\kappa) q(\delta) (q^2(\delta) + 2) n \theta \right)^2 \leq \frac{1}{2(1 + 2\sqrt{2\kappa})}.$$

The left-hand side of the above inequality is increasing in  $\delta$ . Using this, one may easily verify that the above inequality is satisfied if

$$\tau = \frac{1}{8(1 + 4\kappa)}, \quad \theta = \frac{1}{25n(1 + 4\kappa)^2 (1 + \sqrt{2 + 4\kappa})},\quad (32)$$

which is in agreement with (20).

Note that in the previous section we have found that if at the start of an iteration, the iterates satisfy  $\delta(x, s; \mu) \leq \tau$ , with  $\tau$  as defined in (32), then after the feasibility step, with  $\theta$  as defined in (32), the iterates satisfy  $\delta(x^f, s^f; \mu) \leq \frac{1}{\sqrt{2(1+2\sqrt{2\kappa})}}$ .

According to (16), at most

$$\lceil \log_2(\log_2 8(1 + 4\kappa)) \rceil,$$

centering steps then suffice to get iterates  $(x^+, s^+)$  that satisfy  $\delta(x^+, s^+; \mu^+) \leq \tau$  again. It has become a custom to measure the complexity of an IPM by the required number of inner iterations. In each main iteration, both the value of  $x^T s$  and the

norm of the residual are reduced by the factor  $1 - \theta$ . Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max \left\{ (x^0)^T s^0, \|r^0\| \right\}}{\varepsilon}.$$

So, due to (32) the total number of inner iterations is bounded above by

$$25 n(1 + 4\kappa)^2 (1 + \sqrt{2 + 4\kappa}) \log_2 (\log_2 8 (1 + 4\kappa)) \log \frac{\max \left\{ (x^0)^T s^0, \|r^0\| \right\}}{\varepsilon}.$$

From the above statements, we have the following main result of the paper:

**Theorem 5.1.** *If (P) has optimal solution  $(x^*, s^*) \in \mathcal{F}^*$  such that  $\|x^*\|_\infty \leq \rho_p$  and  $\|s^*\|_\infty \leq \rho_d$ , for some  $\rho_p, \rho_d > 0$ , then after at most*

$$25 n(1 + 4\kappa)^2 (1 + \sqrt{2 + 4\kappa}) \log_2 (\log_2 8 (1 + 4\kappa)) \log \frac{\max \left\{ (x^0)^T s^0, \|r^0\| \right\}}{\varepsilon}.$$

*iterations the algorithm finds an  $\varepsilon$ -solution of HLCP.*

## 6. CONCLUDING REMARKS AND FURTHER RESEARCH

We presented an infeasible interior-point method for solving  $P_*(\kappa)$  horizontal linear complementarity problem. We proved that the complexity of this algorithm coincides with the currently best known iteration bounds of infeasible IPMs for HLCP. Our future work will focus on analyzing the algorithm with self-regular functions.

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