

## THE LOGISTIC MODELING POPULATION; HAVING HARVESTING FACTOR

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**Abstract:** The present paper deals with the logistic equation having harvesting factor, which is studied in two cases constant and non-constant. In fact, the nature of equilibrium points and solutions behavior has been analyzed for both of the above cases by finding the first integral, solution curve and phase diagram. Finally, a theorem describing the stability of a real model of single species is proved.

**Keywords:** Equilibrium Points, Harvesting Factor, Logistic Equation, O.D.Es.

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### 1. INTRODUCTION

Main concern of population ecology is growth (or decay) and interaction rates of the entire population. Study of population change started with F.Fibonacci, the greatest European mathematician of the middle ages, who introduced a mathematical model for a growing rabbit population in his arithmetic book [2]. If population is defined as a collection of individuals of a particular species that live within a well-defined area, then the occurred changes are due to the number of its individuals, caused by reproduction, death or migration. This notion of population change is illustrated in the following form which is known as population balance equation or conservation equation for population:

Population Change = Births - Deaths – Migration

The phrase “balance equation” refers to the fact that any change in population abundance is a balance between processes that may decrease or increase it. Population in which migration can be neglected is usually referred to as closed population or closed system, opposed to open population or open system. We suppose that the population has changed only by the occurrence of birth and death, so there is no immigration or emigration. The number of births and the number of deaths are denoted by  $\alpha(t)$ ,  $\beta(t)$  respectively, and the population changes during time  $\Delta t$  is presented as follows:

$$\Delta x = (\alpha(t) - \beta(t))x(t)\Delta t$$

Now, assume that the variable  $x$  is extended (by interpolation) to a nonnegative real valued function of a real variable, then

$$\frac{dx}{dt} = (\alpha(t) - \beta(t))x(t) \quad (1)$$

This is known as general population equation. If  $\alpha(t)$  and  $\beta(t)$  are constant (as  $\alpha$  and  $\beta$ ) and by taking  $r = \alpha - \beta \neq 0$ , then the equation (1) may be reduced to the following exponential growth (or decay) equation:

$$\frac{dx}{dt} = rx \quad (2)$$

Where  $r$  is known as growth (or decay) rate. The equation (2) serves as a mathematical model for a wide range of natural phenomena involving the quantity whose rate of change is proportional to its current size. If the initial population is  $x_0 = x(0)$ , then the solution of the equation (2) is a familiar formula for unlimited growth,  $x(t) = x_0 e^{rt}$ , which is known as exponential growth rate. It can depend on many things; assume for a moment, that it depends only upon the capital food supply  $\delta$ , which is a nonnegative constant, then a minimum of  $\delta_0$  would be necessary to sustain the population. If  $\delta > \delta_0$ , the growth rate is positive; and if  $\delta < \delta_0$ , it is negative, while for  $\delta = \delta_0$ , it is zero. The simplest way to ensure this manner is to make the growth rate as a linear function of  $\delta - \delta_0$  i.e.  $r = \eta(\delta - \delta_0)$ , where  $\eta > 0$ . Here the parameters  $\delta_0$  and  $\eta$  are constant dependents only on the species, and  $\delta$  is a parameter which is depended on the particular environment but constant for a given ecology. Thus, the population increase remains constant or approaches zero, depending on whether  $\delta > \delta_0$ , or  $\delta = \delta_0$  and  $\delta < \delta_0$ .

## 2. THE LOGISTIC POPULATION

There are various reasons for checking exponential growth, and limited food supply is the main one. Furthermore, a larger population has fewer resources and food supply,

which leads to its smaller growth rate. It is also noticed that when a population is in a closed container, the birth rate decreases and hence, the population decreases itself.

Now assume that the birth rate decreases with the population size  $x$  linearly, and so  $\alpha = \alpha_0 - \alpha_1 x$ , where  $\alpha_0$  and  $\alpha_1$  are real constants. Moreover, consider that the death rate  $\beta = \beta_0$  remains constant. Then, the equation (2) appears as the following differential equation which is known as the logistic equation <sup>2</sup>

$$\frac{dx}{dt} = ax - bx^2 \quad (3)$$

Where  $a = \alpha_0 - \beta_0$  and  $b = \alpha_1$ . It is useful to rewrite the equation (3) in the following form:

$$\frac{dx}{dt} = rx \left( 1 - \frac{x}{M} \right) \quad (4)$$

where  $r = a$ ,  $M = \frac{a}{b}$  [1]. Considering that the initial population  $x(0)$  is  $x_0$ , the solution of the above equation may have the following form:

$$x(t) = \frac{Mx_0}{x_0 + (M - x_0)e^{-rt}} \quad (5)$$

The equilibrium points of the equation (4) are  $x = 0$  and  $x = M$ . The change of  $\frac{dx}{dt}$  is zero if  $x = 0$  or  $x = M$ . If the positive initial population is smaller than the carrying capacity  $M$ , then the population density  $x(t)$  increases. While if the initial population is greater than the carrying capacity  $M$ , then it will decrease to  $M$  monotonically. Indeed, the behavior of the solution for the first case is logistic growth. For every small values of  $x$ , it is almost exponential, while  $x > \frac{M}{2}$ , it is asymptotically closed to the constant value  $M$ , describing the carrying capacity of the environment. Some of the related models are discussed in [3,4].

### 2.1. The Logistic Population; Having Constant Harvesting

Assume that the constant number  $h$  of a population is removed per each duration, and then the equation (4) can be extended as follows:

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<sup>2</sup> The logistic equation was published in 1838 by Pierre Franois Verhulst(1804 - 1849), the Belgian mathematician and demographer, as possible model for human population growth [5]

$$\frac{dx}{dt} = rx \left( 1 - \frac{x}{M} \right) - h \quad (6)$$

In fact, it acts like a limited population. It is obvious that there is no harvesting in the case of  $h = 0$ . Now let the harvesting factor be positive, i.e.  $h > 0$ . Then, the equilibrium points of the equation (6) are  $x_1, x_2 = \frac{M}{2} \left( 1 \pm \sqrt{1 - \frac{4h}{rM}} \right)$ , that are the roots of the following quadratic equation:

$$\frac{r}{M}x^2 - rx + h = 0$$

The term of  $\sqrt{1 - \frac{4h}{rM}}$  is a real number provided  $h < \frac{rM}{4}$ . Then the positive real numbers  $H_1$  and  $H_2$ , as the roots of right hand of the equation (6), may be obtained as follows:

$$H_1 = \frac{M}{2} \left( 1 - \sqrt{1 - \frac{4h}{rM}} \right), \quad H_2 = \frac{M}{2} \left( 1 + \sqrt{1 - \frac{4h}{rM}} \right) \quad (7)$$

As the function  $x(t)$ , density of population, can't be negative,  $H_1 < H_2$ , and

$$\sqrt{1 - \frac{4h}{rM}} < 1.$$

It implies that  $H_1 > 0$ ,  $H_2 < M$ .

And so, we can rewrite

$$rx \left( 1 - \frac{x}{M} \right) - h = a(x - H_1)(x - H_2) \quad (8)$$

By using equations (6) and (8), we find the following equation:

$$\frac{dx}{dt} = a(x - H_1)(x - H_2) \quad (9)$$

Considered that the initial population at zero time is  $x(0) = x_0$ , we get the solution to the last equation as follows:

$$x(t) = \frac{H_2(x_0 - H_1) - H_1(x_0 - H_2)e^{-\frac{r}{M}(H_2 - H_1)t}}{(x_0 - H_1) - (x_0 - H_2)e^{-\frac{r}{M}(H_2 - H_1)t}}, \quad -\frac{r}{M}(H_2 - H_1)t \quad (10)$$

Since the following inequality holds

$$-\frac{r}{M}(H_2 - H_1)t < 0 \text{ for all } t > 0$$

Since the value of  $e^{-\frac{r}{M}(H_2-H_1)t}$  approaches to zero whenever  $t \rightarrow \infty$ . So if  $x_0 > H_1$  and all of the coefficients in (10) are positive, we get  $x(t) = H_2$ . Therefore,  $x(t) = H_2$  is the solution of limiting equilibrium? Thus, the point of  $x(t) = H_2$  is a stable point.

Now, we are going to analyze the solution behavior of equation (6) in case of  $x_0 < H_1$ .

Put

$$x_0 - H_1 = (x_0 - H_2)e^{-\frac{r}{M}(H_2-H_1)t}$$

$$\Rightarrow \ln \frac{x_0 - H_1}{x_0 - H_2} = -\frac{r}{M}(H_2 - H_1)t$$

$$\Rightarrow t = \frac{1}{-\frac{r}{M}(H_2 - H_1)} \ln \frac{x_0 - H_1}{x_0 - H_2}$$

Now, let  $t_1$  be as following:

$$t_1 = \frac{1}{-\frac{r}{M}(H_2 - H_1)} \ln \frac{x_0 - H_1}{x_0 - H_2} \tag{11}$$

By considering the value of  $t_1$ , which is calculated in equation (11), at equation (10) we see that:

$$x(t_1) = \frac{H_2(x_0 - H_1) - H_1(x_0 - H_2)e^{\left[-\frac{r}{M}(H_2-H_1) \frac{1}{-\frac{r}{M}(H_2-H_1)} \ln \frac{x_0 - H_1}{x_0 - H_2}\right]}}{(x_0 - H_1) - (x_0 - H_2)e^{\left[-\frac{r}{M}(H_2-H_1) \frac{1}{-\frac{r}{M}(H_2-H_1)} \ln \frac{x_0 - H_1}{x_0 - H_2}\right]}}$$

Since the numerator of the above fraction is negative and it's denominator is zero, we get  $x(t_1) = -\infty$ . Therefore, the value of the population function  $x(t)$  at  $t_1 = H_1$  is a threshold and hence, the point of  $x_0 = H_1$  is a unstable equilibrium point.

## 2.2. The Logistic Population; Having Variable Harvesting Factor

Now we are going to study the extension of the logistic population harvesting in case of harvesting parameter  $h$  in system (6), which isn't constant. Consider the situation that the population has a logistic growth rate and harvesting coefficient isn't constant. Therefore, the general case of this model is as follows:

$$\frac{dx}{dt} = rx \left( 1 - \frac{x}{M} \right) - f(x) \quad (12)$$

There is no general way to solve the equation (12), so we should restrict our attention to a few special cases. In the following equation, we make assumption that the harvesting function is  $f(x) = h \frac{x}{1+x}$  and moreover, we have  $M = 1$ .

$$\frac{dx}{dt} = rx(1-x) - h \frac{x}{1+x} \quad (13)$$

The equation (13) exhibits that harvesting coefficient for each value depends on its population density. Since  $\frac{x}{1+x} < 1$ , the term of harvesting in the equation (13) has inverse relation respect to density of population. In other words, the term of harvesting decreases as the population increases. Put  $\frac{dx}{dt} = 0$ , then if  $h < r$ , there are three equilibrium points, and one of them is  $x = 0$ . For finding other equilibrium points, one should consider:

$$-rx^2 - r - h = 0 \quad (14)$$

Therefore, two other equilibrium points are  $x = \pm \sqrt{1 - \frac{h}{r}}$ . Considering that

$H = \sqrt{1 - \frac{h}{r}}$ , one may get

$$-rx^2 + r - h = x^2 - H^2 \quad (15)$$

$$\Rightarrow \frac{1+x}{x(x^2 - H^2)} dx = dt \quad (16)$$

One may find that parameters  $H_1$ ,  $H_2$ , and  $H_3$  are satisfying in the following equation:

$$\frac{1+x}{x(x^2-H^2)} = \frac{H_1}{x} + \frac{H_2}{x-H} + \frac{H_3}{x+H} \quad (17)$$

First, multiplying the equation (17), we have

$$\frac{1+x}{x^2-H^2} = H_1 + x \left( \frac{H_2}{x-H} + \frac{H_3}{x+H} \right)$$

After putting  $x = 0$  in the above equation, we get

$$H_1 = \frac{-1}{H_2} \quad (18)$$

Similarly, we obtain:

$$H_2 = \frac{1+H}{2H^2}, \quad H_3 = \frac{1-H}{2H^2} \quad (19)$$

$$\int \frac{1+x}{x(x^2-H^2)} dx = H_1 \ln x + H_2 \ln(x-H) + H_3 \ln(x+H) \quad (20)$$

$$\int \frac{1+x}{x(x^2-H^2)} dx = \ln \left[ x^{H_1} (x-H)^{H_2} (x+H)^{H_3} \right] \quad (21)$$

From the relations in equation (12) and (21), we obtain:

$$x^{H_1} (x-H)^{H_2} (x+H)^{H_3} = ce^{-t} \quad (22)$$

where  $c$  is an integral constant. It can be found by having an initial population  $x(0) = x_0$ , for the equation (12). Its solution is given by:

$$x^{H_1} (x-H)^{H_2} (x+H)^{H_3} = x_0^{H_1} (x_0-H)^{H_2} (x_0+H)^{H_3} e^{-t} \quad (23)$$

Function of  $x(t)$  is an implicit function, and we can't sketch the graph of solution  $x(t)$  with respect to  $t$ .

### 3. A REAL MODEL OF SINGLE SPECIES

In this section, we study a realistic model of single species which is governed by

$$\frac{dx}{dt} = xM(x) \quad (24)$$

In the last equation, we assume that variable growth rate  $M$  depends only on total population  $x(t)$ . It is conceivable to assume, as we did before, that there is a limiting population  $\delta$  in which  $M(\delta) = 0$  and  $M(x) < 0$  for  $x > \delta$ , and  $M(x) > 0$  for  $x < \delta$ . If a very small population behaves as an un-limiting growth rate model, we assume that  $M(0) > 0$ . The equilibrium points of the equation (22) are  $x = 0$ , and they are the roots of  $M(x)$ . Finally, we present the following theorem that describes the behavior of the equilibrium points of single species real model (22).

**Theorem 3.1.** *For the system (24), the following statements hold:*

- i) The equilibrium point  $x$  is unstable.
- ii) An equilibrium point  $x_1$  (The root of  $M(x)$  if exists) is asymptotically stable if and only if there is  $\varepsilon > 0$  such that  $M(x) > 0$  on  $[x_1 - \varepsilon, x_1)$  and  $M(x) < 0$  on  $(x_1, x_1 + \varepsilon]$
- iii) An equilibrium point  $x_2$  (The root of  $M(x)$ , if exists) is asymptotically unstable if and only if there is  $\varepsilon > 0$  such that  $M(x) < 0$  on  $[x_2 - \varepsilon, x_2)$  and  $M(x) > 0$  on  $(x_2, x_2 + \varepsilon]$ .

Proof. i )

$$\frac{d}{dx}(xM(x)) = M(x) + x \frac{dM(x)}{dx}$$

$$\left. \frac{d}{dx}(xM(x)) \right|_{x=0} = M(0)$$

Since  $\lambda = M(0)$  is positive, the point  $x = 0$  is a stable point.

ii ) Let  $x_1 > 0$ .

Now, consider that there is  $\varepsilon > 0$  such that  $M(x) > 0$  on the interval  $[x_1 - \varepsilon, x_1)$ , and  $M(x) < 0$  on the interval  $(x_1, x_1 + \varepsilon]$ .

As the growth rate  $M$  is positive on the interval  $[x_1 - \varepsilon, x_1)$  and negative on the interval  $(x_1, x_1 + \varepsilon]$ , it is easy to see that  $x(t) \rightarrow x_1$  as  $t \rightarrow \infty$ .

iii ) proof is similar to ii.

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