

COMPUTATION OF GORDIAN DISTANCES AND H₂-GORDIAN DISTANCES OF KNOTS

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Abstract: In this paper we discuss the possibility of computing unknotting number from minimal knot diagrams, Bernhard-Jablan Conjecture, unknown knot distances between non-rational knots and of searching minimal distances by using a graph with weighted edges representing knot distances. Since topoisomerases are enzymes involved in changing crossing of DNA, knot distances can be used to study topoisomerases actions. We compute some undecided knot distances 1 known from the literature, and extend the computations by computing knots with smoothing number one with at most $n = 11$ crossings and smoothing knot distances of knots with at most $n = 9$ crossings. All computations are done in the program *LinKnot*, based on Conway notation and non-minimal representations of knots.

Keywords: Unknotting number, Gordian distances, Smoothing distances, Weighted graph search.

MSC: 57M25, 57M27.

1. INTRODUCTION

The question of unknotting numbers, or Gordian numbers is one of the most difficult in knot theory [1, 2]. In order to compute unknotting numbers, we need a link surgery. In every crossing of a knot, it is possible to make a crossing change (Fig. 1a): to transform an overcrossing to undercrossing or *vice versa*. Crossing change is unknotting operation.

Definition 1. *The unknotting number $u(D)$ of a knot diagram D is the minimal number of crossing changes on the diagram required to obtain a diagram representing an unknot;*

The $u_M(K)$ of a knot K in R^3 is the minimum of $u(D)$ over all minimal crossing number diagrams D representing knot K ;

The unknotting number $u(K)$ of a knot K in R^3 is the minimum of $u(D)$ over all diagrams D representing K .

Since every knot K has an infinite number of diagrams and a finite number of minimal diagrams, the main question is: Is the unknotting number computable from minimal diagrams of K ? There are two different approaches for obtaining the unknotting number of K :

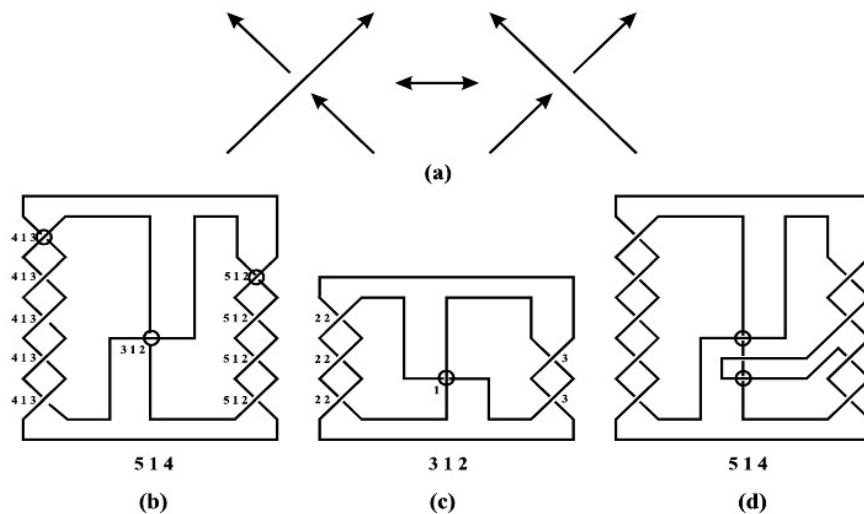


Figure 1: A crossing change (a) and the Nakanishi-Bleiler example: (b) the minimal projection of the knot 5 1 4 that requires at least three crossing changes to be unknotted; (c) the minimal projection of the knot 3 1 2 with the unknotting number 1; (d) non-minimal projection of the knot 5 1 4 from which we obtain the correct unknotting number $u(5 1 4) = 2$

- according to the classical definition, one is allowed to make an ambient isotopy after each crossing change and then continue the unknotting process with the newly obtained projection;
- the standard definition requires all crossing changes to be done simultaneously in a fixed projection.

Those two definitions are equivalent (see, e.g.,[3]).

If in the standard definition we take only minimal projections instead of working with all projections, we cannot always obtain the correct unknotting number. This is illustrated by the well known example of the knot 10_8 given by Nakanishi [4] and Bleiler [5]. The rational knot 10_8 (or 5 1 4 in Conway notation) has only one minimal projection

(Fig. 1b). According to the standard definition, unknotting takes at least three crossing changes (in the crossings denoted by circles). If we apply the classical definition: make a crossing change in the middle point of the diagram (Fig. 1b) followed by the reduction $5 - 1 \ 4 = 3 \ 1 \ 2$, we obtain the minimal projection of the knot $6_2 = 3 \ 1 \ 2$ (Fig. 1c) that can be unknotted by only one crossing change. As the result we obtain the correct unknotting number 2. The same unknotting number will be obtained if we take the non-minimal projection of the knot $5 \ 1 \ 4$ (Fig. 1d) and use the standard definition. Hence, by using the classical definition, we can obtain the correct unknotting number $u(10_8) = 2$ from the minimal projection of the knot $10_8 = 5 \ 1 \ 4$. Therefore, Bernhard [6] and Jablan [8] independently proposed the following conjecture:

Conjecture 1. *Bernhard-Jablan Conjecture:*

$u(K) = 0$ for any unknot K ;

$u(K) = \min(u(K_-)) + 1$, where the minimum is taken over all minimal projections of knots K_- , obtained from all minimal projections of K by one crossing change [8].

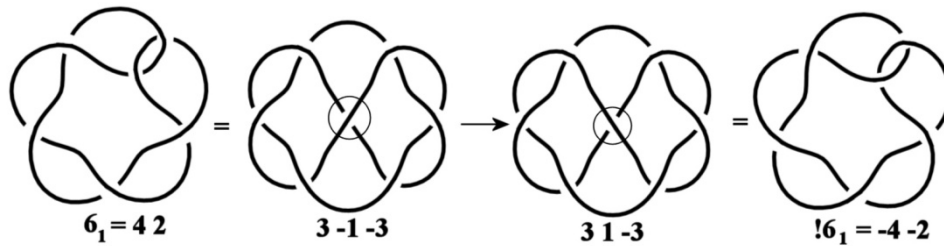
This means that we take all minimal projections of a knot K , make a crossing change in every crossing, and then minimize all the projections obtained. The same algorithm is applied to the first, second, . . . k^{th} generation of the knots obtained. The unknotting number is the minimal number of steps k in this recursive unknotting process. Many unknotting numbers computed according to Bernhard-Jablan Conjecture are confirmed by computing their signature (or Rasmussen signature), where for the both signatures holds the inequality $\frac{\sigma(K)}{2} \leq u(K)$.

Definition 2. *Knot distance between knots K_1 and K_2 is minimum number of crossing changes required to convert K_1 into K_2 .*

One of the applications of knot distances is the transformation of knots in DNA. Since topoisomerases are enzymes involved in crossing changes in DNA, knot distances can be used to study topoisomerase action. In this way, a knot distance is the minimum number of times needed for topoisomerase to mediate strand passage (crossing change) on DNA in order to convert K_1 to K_2 . Note that minimum is taken over all diagrams of K_1 . In particular, the unknotting number of a knot K is its distance from the unknot 0_1 . For unknotting number, according to Bernhard-Jablan Conjecture, there is a chance that it can be computed by a recursive algorithm from minimal knot diagrams, but knot distances are mostly realized only on non-minimal diagrams. For example, the distance of the knot $6_1 = 4 \ 2$ and its mirror-image $!6_1 = -4 \ -2$ is one. It cannot be obtained from minimal diagrams of 6_1 and $!6_1$, but can be obtained from non-minimal diagrams $3 \ -1 \ -3$ and $3 \ 1 \ -3$ of the knot 6_1 and its mirror-image $!6_1$, which are related by a single crossing change (Fig. 2).

Knot distances have the properties of metric:

1. $d(K_1, K_2) = 0$ iff $K_1 = K_2$;
2. $d(K_1, K_2) = d(K_2, K_1)$;
3. $d(K_1, K_2) \leq d(K_1, K) + d(K, K_2)$, where K is an arbitrary knot [7].

Figure 2: Knot 6_1 and its mirror-image $!6_1$ with distance 1

In particular, for unknotting numbers $u(K_1)$ and $u(K_2)$ the triangle inequality 3) reduces to $d(K_1, K_2) \leq u(K_1) + u(K_2)$.

Except crossing changes, another way to unknot some knot is to use smoothing. Even more difficult for computation, and hence less elaborated are smoothing unknotting numbers and smoothing distances of knots. In the second part of this paper we present computations of smoothing numbers and smoothing distances. As well as Gordian distances, H_2 -Gordian distances (or smoothing distances) mostly cannot be computed from minimal knot diagrams, so this represents the main obstacle for their computation.

The paper is organized in the following way: after the Introduction (Section 1) and a brief explanation of Conway notation (Section 2), in Section 3, we considered unknot recognition. Section 4 presents the overview of the algorithmic methods, and our recent computation results of knot distances for knots with at most 9 crossings and a generalization of these results to families of knots with distance 1, given in Conway notation. In Section 4, we present new results about Gordian distances and strand passage metrics. In Section 5, we analyze smoothing as unknotting operation, and equivalence between band unknotting number and smoothing number. In Section 6, we present computations of knots with smoothing number 1, based on knots with unknotting number 1, and Section 7, is dedicated to the computation of smoothing distances for knots up to $n = 9$ crossings. All computations are done in *Mathematica* based program *LinKnot* [8] by using Conway notation.

2. BASICS OF CONWAY NOTATION

Knots and links can be given in Conway notation [8–10]. For the readers non familiar with it, we explain Conway notation, introduced in Conway's seminal paper [10] published in 1967, and effectively used since (e.g., [8–10]). Conway symbols of knots with up to 10 crossings and links with at most 9 crossings are given in the Appendix of the book [9]. The explanation of the Conway notation is based on the book [8].

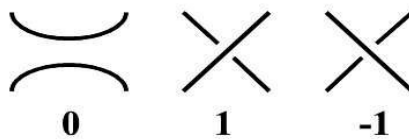


Figure 3: The elementary tangles

The main building blocks in the Conway notation are elementary tangles. We distinguish three elementary tangles, shown in Fig. 3 and denoted by 0, 1 and -1 . All other tangles can be obtained by combining elementary tangles, while 0 and 1 are sufficient for generating alternating knots and links (abbr. *KLs*). Elementary tangles can be combined by following operations: *sum*, *product*, and *ramification* (Figs. 4-5). For some given tangles a and b , image of a under reflection with mirror line NW-SE is denoted by $a0$, and sum is denoted by $a + b$. Product ab is defined as $ab = a0 + b$, and ramification by $(a, b) = a0 + b0$.

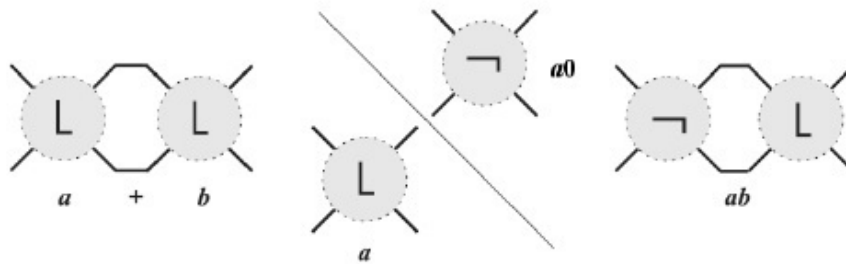


Figure 4: A sum and product of tangles

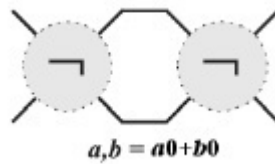


Figure 5: Ramification of tangles

Tangle can be closed in two ways (without introducing additional crossings): by joining in pairs NE and NW, and SE and SW ends of a tangle to obtain a *numerator closure*; or by joining in pairs NE and SE, and NW and SW ends we obtain a *denominator closure* (Fig. 6a,b).

Definition 3. A rational tangle is any finite product of elementary tangles. A rational *KL* is a numerator closure of a rational tangle.

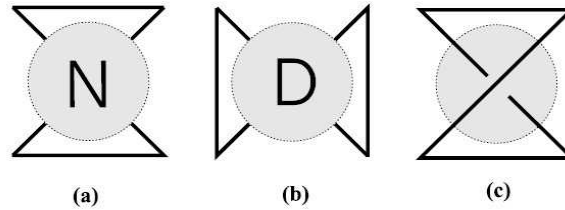


Figure 6: (a) Numerator closure; (b) denominator closure; (c) basic polyhedron 1^*

Definition 4. A tangle is algebraic if it can be obtained from elementary tangles using the operations of sum and product. KL is algebraic if it is a numerator closure of an algebraic tangle.

A Montesinos tangle and the corresponding Montesinos link, consisting of n alternating rational tangles t_i , with at least three non-elementary tangles t_k for $k \in \{1, 2, \dots, n\}$, is denoted by t_1, t_2, \dots, t_n , $n \geq 3$, $i = 1, \dots, n$ (Fig. 7). The number of tangles n is called the length of the Montesinos tangle. In particular, if all tangles t_i , $n \geq 3$, $i = 1, \dots, n$ are integer tangles, we obtain pretzel knots and links.

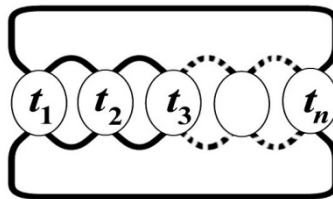


Figure 7: Montesinos link t_1, t_2, \dots, t_n

Definition 5. Basic polyhedron is a 4-regular, 4-edge-connected, at least 2-vertex connected plane graph.

Basic polyhedron [8–10] of a given KL can be identified by recursively collapsing all bigons in a KL diagram, until none of them remains. The basic polyhedron 6^* is illustrated in Fig. 8.

Definition 6. A link L is algebraic or 1^* -link if there exists at least one diagram of L which can be reduced to the basic polyhedron 1^* by a finite sequence of bigon collapses. Otherwise, it is a non-algebraic or polyhedral link.

Conway notation for polyhedral KL s contains additionally a symbol of a basic polyhedron that we are working with. The symbol $n^{*m} = n^{*m}1.1. \dots .1$, where $*m$ is a sequence of m stars, denotes the m -th basic polyhedron in the list of basic polyhedra with n vertices. A KL obtained from a basic polyhedron n^{*m} by substituting tangles t_1, \dots, t_k , $k \leq n$ instead of vertices, is denoted by $n^{*m}t_1 \dots t_k$, where the number of dots between two successive tangles shows the number of omitted substituents of value 1. For

example, $6^*2 : 2 : 2 0$ means $6^*2.1.2.1.2 0.1$, and $6^*2 1.2.3 2 : -2 2 0$ means $6^*2 1.2.3 2.1.-2 2 0.1$ (Fig. 8).

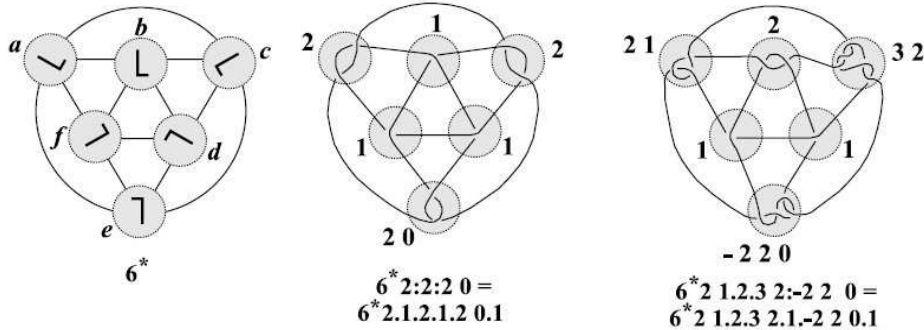


Figure 8: Basic polyhedron 6^* and knots $6^*2.1.2.1.2 0.1$, and $6^*2 1.2.3 2 : -2 2 0$

Definition 7. For a link, or a knot L given in an unreduced¹ Conway notation, $C(L)$, denote by S a set of numbers in the Conway symbol excluding numbers denoting basic polyhedron and zeros (determining the position of tangles in the vertices of polyhedron), and let $\tilde{S} = \{a_1, a_2, \dots, a_k\}$ be a non-empty subset of S . Family $F_{\tilde{S}}(L)$ of knots or links derived from L consists of all knots or links L' whose Conway symbol is obtained by substituting all $a_i \neq \pm 1$, by $\text{sgn}(a_i)|a_i + k_{a_i}|$, $|a_i + k_{a_i}| > 1$, $k_{a_i} \in \mathbb{Z}$.

An infinite subset of a family is called subfamily. If all k_{a_i} are even integers, the number of components is preserved within the corresponding subfamilies, i.e., adding full-twists preserves the number of components inside the subfamilies.

Definition 8. A link given by Conway symbol containing only tangles ± 1 and ± 2 is called a source link. A link given by Conway symbol containing only tangles ± 1 , ± 2 , or ± 3 is called a generating link.

For example, Hopf link 2 (link 2_1^2 in Rolfsen's notation) is the source link of the simplest link family p ($p = 2, 3, \dots$) (Fig. 9), and Hopf link and trefoil 3 (knot 3_1 in the classical notation) are generating links of this family. A family of KL s is usually derived from its source link by substituting $a_i \in \tilde{S}$, $a_i = \pm 2$, by $\text{sgn}(a_i)(2 + k)$, $k = 1, 2, 3, \dots$ (see Def. 7 and Def. 8).

Definition 9. Smoothing distance between knots K and K_1 is the minimal number of smoothings necessary to transform knot K to K_1

¹The Conway notation is called unreduced if in symbols of polyhedral links elementary tangles 1 in single vertices are not omitted.

For making computations of knot distances and smoothing distances the main advantage of the Conway notation is the use of twists (or chains of bigons): since crossings in a twist commute, every crossing change in a positive twist k ($k \geq 0$) gives the twist $k - 2$, and every crossing change in a negative twist $-k$ ($k \geq 0$) gives the twist $-k + 2$. Hence, crossing changes (or smoothings) in particular crossings of a twist are not necessary: it is sufficient to make a single crossing change (or smoothing) in the twist. Moreover, results obtained for particular pairs of knots can be extended to knot families.

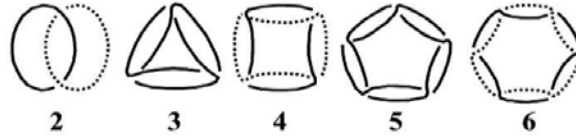


Figure 9: Hopf link $2_1^2 = 2$ and its family p ($p = 2, 3, \dots$)

As the only symbolic notation of knots, Conway notation is extremely powerful in all computations with rational knots (2-bridge knots or 4-plats), based on the connection of rational tangles and rational knots and links with continued fractions [10–12]. To a rational tangle $n_1 n_2 \dots n_k$ corresponds to the continued fraction

$$r = \frac{p}{q} = n_k + \frac{1}{n_{k-1} + \frac{1}{\dots + \frac{1}{n_2 + \frac{1}{n_1}}}}$$

where the rational number $r = \frac{p}{q}$ ($\text{GCD}(p, q) = 1$) is the slope of the rational tangle. A rational knot or link $L(\frac{p}{q})$ is a knot if p is odd, and link if p is even.

Theorem 1. *The Unoriented rational links $L(\frac{p}{q})$ and $L(\frac{p'}{q'})$ are ambient isotopic iff:*

1. $p = p'$ and
2. either $q \equiv q' \pmod{p}$ or $qq' \equiv 1 \pmod{p}$ [13].

For rational tangles and rational knots and links the following theorem holds:

Theorem 2. *Two rational tangles are equivalent iff their continued fractions yield the same rational number [10–12].*

Following theorems above, we can conclude that some knot distance computations with rational knots and links in *Mathematica* can be reduced to the application of very fast *Mathematica* functions **ContinuedFraction** and **FromContinuedFraction**. On the basis of continued fractions, I. Darcy and her collaborators wrote computer programs for knot distances of rational knots and computed distances of rational knots with $n \leq 14$ crossings [14–17].

Since the basis of all computations of Gordian and smoothing distances of knots are non-minimal representations, in order to derive (as many as possible) non-minimal representations of knots up to $n = 9$ crossings, we used various methods. In the case of rational knots, to work with their non-minimal diagrams, it is possible to use the methods based on continued fractions, developed by Darcy and collaborators. In the case of non-rational knots, the simplest method for obtaining non-minimal knot diagrams is the replacement of complete twists (including twists of the length 1) in alternating knots by negative twists of the same or larger length, resulting in non-alternating minimal or nonminimal diagrams. The other method is the replacement of the complete positive rational tangles in certain knot by their equivalent non-alternating tangles. For example, by replacing in the 11-crossing Montesinos knot $3\ 1, 2\ 2, 2\ 1$ positive tangles $3\ 1, 2\ 2,$ and $2\ 1$ by their equivalent non-alternating tangles $-2\ -1\ 2, -3\ 1\ 1,$ and $-2\ 2,$ respectively, we obtain non-minimal 14-crossing representation $-2\ -1\ 2, -3\ 1\ 1, -2\ 2$ of the same knot. Hence, for this purpose we wrote special modules in the program *LinKnot* for the derivation of non-minimal knot representations.

3. UNKNOT RECOGNITION

The main problem in knot theory is the knot recognition: for every two knots we need to decide if they are ambient isotopic (i.e., equal) or not. Two knots, K and K_1 are ambient isotopic *iff* there exists a continuous movement (or deformation) of space S^3 that transforms K into K_1 . More precisely, K and K_1 are ambient isotopic if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself, where a diffeomorphism is a map between manifolds which is differentiable and has a differentiable inverse.

In other words, transformation of K to K_1 has to be smooth with smooth inverse, i.e., tearing a thread and regluing, or shrinking one part of a knot to a point is not allowed. If we imagine that the curve defining a knot is made of flexible and elastic thread, then the ambient isotopy is equivalent to allowing the threads to be twisted and moved continuously in space (cutting and gluing back together is not allowed). Instead of working with 3D knots and ambient isotopies, the problem of knot recognition can be reduced to the transformations of their projections (diagrams) by Reidemeister moves. Reidemeister moves $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ and \mathcal{Q}_3 are illustrated in Fig. 10. Reidemeister proved that they are sufficient to diagrammatically represent every ambient isotopy by a finite sequence of the moves. We represent Reidemeister moves as polygonal moves in order to avoid wild knots (the piecewise-linear and the tame knot theory give the same classification of knots).

Our next goal is the minimization of knot diagrams, i.e., their reduction to diagrams with the minimal number of crossings. As the final result, we obtain all knots, each represented by a minimal diagram. In particular, for unknot this means that we will obtain a single diagram: a circle.

From Fig. 10, it is clear that \mathcal{Q}_1 and \mathcal{Q}_2 decrease the number of crossings, the first by 1, and the second by 2. So, at the first glance it looks that reduction process fulfils by consecutive application of \mathcal{Q}_1 and \mathcal{Q}_2 . Unfortunately, things are not so straightforward: sometimes it is necessary to increase the number of crossings (by reverse move \mathcal{Q}_1 or \mathcal{Q}_2) in order to continue with reduction. Some (un)knots with this property, recognized for the

first time by Goeritz, are illustrated in Fig. 11. Hence, for the complete knot reduction we use – Reidemeister moves, but not their optimal finite reducing sequence.

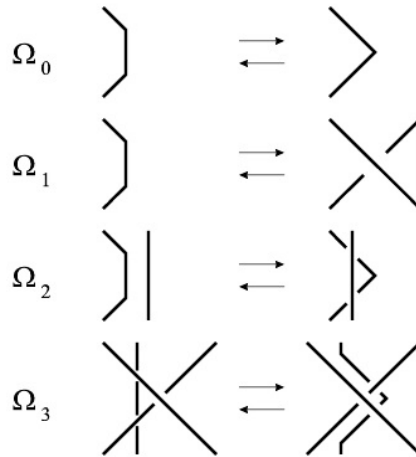


Figure 10: Reidemeister moves for polygonal *KLs*

There is a finite algorithm, based on Haken-Hemion method [19, 20], that guarantees a minimization and solves knot recognition problem. However, it is impossible to implement it because of its complexity. Even its special case, the unknot

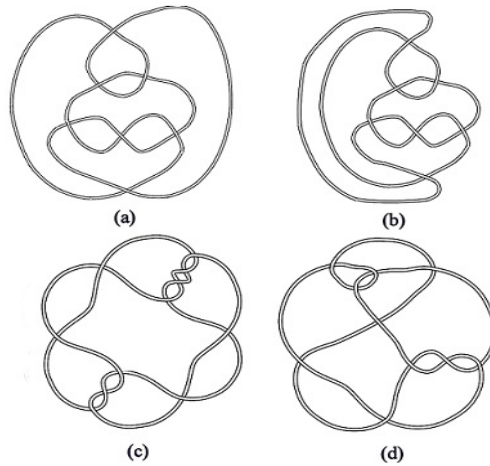


Figure 11: (a), (b) Nasty unknot; (c) Goeritz's unknot; (d) Monster unknot

recognition by Haken algorithm is NP-hard (where NP means "nondeterministic polynomial time"), and the upper bound for the number of Reidemeister moves needed to unknot an n -crossing unknot diagram is 2^{cn} , where $c = 2^{11}$.

The best program for knot recognition is *KnotFind*, the part of the program *Knotscape* written by M. Thistlethwaite et al [21]. Instead of using only three Reidemeister moves, this heuristic program uses 13 different moves, and successfully reduces knots up to $n = 49$ crossings.

New progress in the field of unknot recognition present the works by C. Musick [22], based on 3D representation of knots by trails and risers. This algorithm, tested in his Ph.D. dissertation on hard examples of unknots created by Ochiai (with 45 crossings) and Haken, recognizes unknot in polynomial time.

An infinite number of unknots with an arbitrary number of crossings is easy to create: knowing the general form of rational knots with unknotting number 1, the program *LinKnot* produces the complete list of rational unknots with n crossings (Fig. 12). Moreover, replacing in any knot with unknotting number 1 realized in a minimal diagram the unknotting crossing by a rational equivalent of -1 tangle of arbitrary size, we obtain the unknot.



Figure 12: Rational unknot 3 2 1 1 2 3 3 -1 1 2 3 2 1 1 2

The standard method for knot recognition is a polynomial recognition. After Alexander polynomial, introduced in 1923, many different polynomial knot invariants appeared: Conway polynomial, Jones polynomial, HOMFLYPT, Kauffman two-variable polynomial, colored Jones polynomial, *etc.* All of them have the same property: two knots K and K_l are different if their corresponding polynomials are different. However, if two polynomials $P(K)$ and $P(K_l)$ are equal, we cannot make conclusion about K and K_l . So, for every knot polynomial P , we have an infinite collection of knots that P cannot distinguish. Typical examples are mutant knots, which cannot be distinguished by any of the mentioned polynomials.

Even with regard to recognition of unknot, polynomial invariants sometimes fail. E.g., Kinoshita-Terasaka knot, $11n42 = .-(3, 2).2$, and Conway knot, $11n34 = .-(2, 3).2$, have trivial Alexander polynomial. Both of them have non-minimal 12-crossing algebraic representations: $(3,-2), (2,-3), 2$ and $(3,-2), (-3, 2), 2$. They are members of the families of knots with $n = 4k + 2l + 1$ crossings, given by their non-minimal representations $((2k + 1), -2k), (2k, -(2k + 1)), 2$ 1 and $((2k + 1), -2k), (-2k + 1), 2k), 2$ 1, that have trivial Alexander and Conway polynomial.

For Jones polynomial, it is still unknown if "Jones unknot", meaning a knot with the trivial Jones polynomial, exists or not. However, for the categorization of Jones

polynomial–Khovanov polynomial, Kronheimer and Mrowka [23] proved that it recognizes unknot.

4. GORDIAN DISTANCES AND STRAND PASSAGE METRIC

A *knot distance* between knots K_1 and K_2 is defined as the minimum number of crossing changes required to convert K_1 into K_2 . Since topoisomerases are enzymes involved in crossing changes in DNA, knot distances can be used to study topoisomerase action. In this way, a knot distance is the minimum number of times needed for topoisomerase to mediate strand passage (crossing change) on DNA in order to convert K_1 to K_2 . Note that minimum is taken over all diagrams of K_1 . In particular, the *unknotting number* of a knot K is its distance from the unknot 0_1 . For unknotting number, according to Bernhard-Jablan Conjecture [8], there is a chance that it can be computed by a recursive algorithm from minimal knot diagrams although knot distances are usually realized on non-minimal diagrams. For example, the distance of the knot $6_1 = 4_2$ and its mirror-image is one, and it can be obtained from non-minimal diagrams $3_1 - 1 - 3$ and $3_1 - 1 - 3$ of the knot 6_1 and its mirror-image $!6_1$, which are related by a single crossing change (Fig. 13).

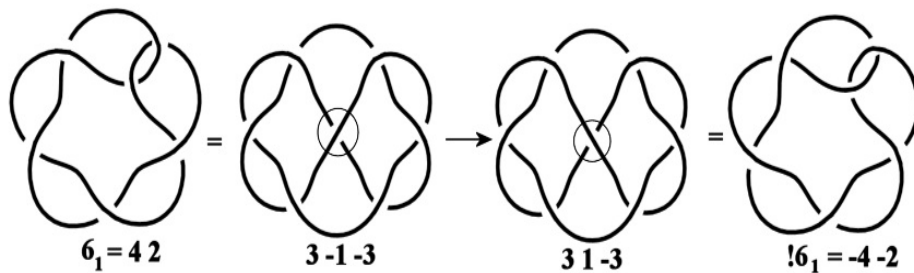


Figure 13: Knot 6_1 and its mirror-image $!6_1$ with distance 1

After the first strand passage metric table [14], containing only rational knots and composites of rational knots up to the knot 8_8 , Darcy computed knot distances and composites of rational knots up to $n = 14$ crossings by using computer programming based on algebra of rational tangles and continued fraction representations of rational knots. However, some undetermined values remained even in the knot distance tables of "small" rational knots. These results are extended to knots up to $n = 10$ crossings [24], and improved in the Ph.D. dissertation by H. Moon [24] (containing knots up to $n = 9$ crossings), where both works include non-rational knots. However, for many non-rational knots, results (e.g., the knots $9_{31} - 9_{49}$ and $10_{46} - 10_{165}$) are completely missing, and many values of knot distances are still undetermined.

The main goal of our computations is to determine some new distances one and to resolve cases of knot pairs with undetermined distances, for which, it is only known that $1 \leq d(K_1, K_2)$. Such improvements can be made even in cases where the known upper and lower bounds are $1 - 8$ [24]; e.g., for the knot 9_{22} and its distances from knots $6_2, 6_3, 7_7, 9_{19}, 9_{44}$, which are, in fact, equal to one. For all computations we used different methods based on non-minimal representations of knots, including the property that every pair of

knots such that $K_1 = N(T_+)$ and $K_2 = N(T_-)$ has distance one, where T_+ and T_- are tangles that differ by one crossing change, and $N(T)$ is its numerator closure of a tangle T [24]. Thanks to the Conway notation, that kind of results can be extended to knot families.

The basis for computation of all knot distances is distance 1 knots. In the case of alternating knots, since all minimal diagrams of an alternating knot are flype-equivalent, as the input is used only one diagram of the input knot, for non-alternating knots, all minimal non-alternating diagrams are used. By additional very large computations, based on non-minimal knot representations, as the result we obtained 429 pairs of knots with $n \leq 9$ crossings and with knot distance 1.

The complete results are organized in knot distance table of knots with at most 9 crossings, which consists from pairs of knots. Among them, except for the mentioned 429 pairs of knots with distance 1, for 5015 pairs, we obtained exact knot distances. For the remaining pairs of knots, as the basic lower bounds are used, we obtain the results from polynomial criteria for lower bounds, based on HOMFLYPT, Jones, and Q-polynomials, signature and Arf invariant (Miyazawa, 2011 [29]). The upper bounds are given by triangle inequality based on unknotting numbers, or on the minimal distances obtained as shortest paths in knot distance table, treated as the graph with the knots as vertices, and with weighted edges representing knot distances (Fig. 14). A *star* algorithm is used for finding the shortest path, where heuristic function is defined as minimum of distance 2 and the lower bound [18].

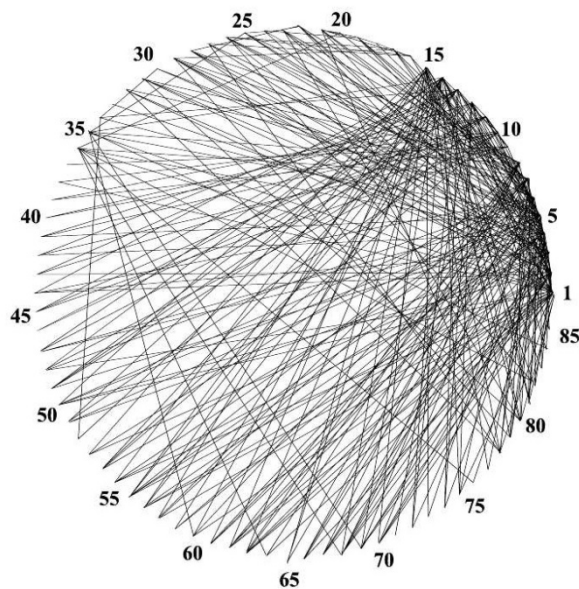


Figure 14: Graph G_1

After computing knot distances from knots given by any coding (Gauss codes, DT-codes, PD, etc.), from particular pairs of knots with distance 1, we cannot make any further conclusion about some other pairs of knots. However, for pairs of knots with distance 1 given in Conway notation, we can extend the particular results from knot pairs

to their corresponding families. E.g., knowing that knots 6_1 and its mirror-image $!6_1$, given by their non-minimal representations $3 \text{---} 1 \text{---} 3$ and $3 \text{---} 1 \text{---} 3$ have distance 1, we can generalize this result to the families of knots given by their non-minimal representations $(2p+1) \text{---} 1 \text{---} (2q+1)$ and $((2p+1) \text{---} 1 \text{---} (2q+1))$. By minimization, after $(6_1, !6_1)$, as the result we obtain pairs of knots $(8_1, 8_3) = (6 \text{---} 2, 4 \text{---} 4)$, $(10_1, 10_3) = (8 \text{---} 2, 6 \text{---} 4)$, $(12a803, 12a1166) = (10 \text{---} 2, 8 \text{---} 4)$, . . . , i.e., the pairs of knots with distance 1 that are the members of families $(2p)(2q)$ and $(2p-2)(2q+2)$ ($p \geq 2$), respectively (Fig. 15).

5. SMOOTHING AS UNKNOTTING OPERATION

In knot theory, smoothing plays a special role since all skein relations for computing polynomial knot invariants (Conway, Alexander, Jones, HOMFLYPT, or Kauffman polynomial) use smoothing.

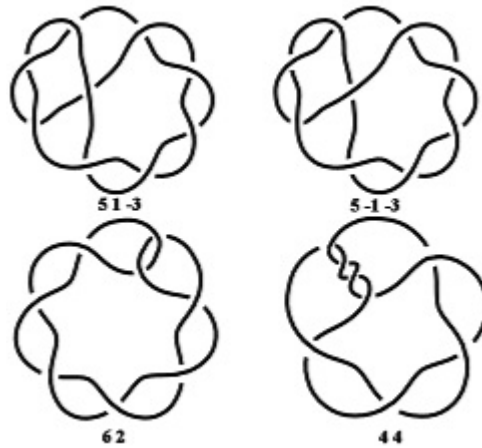


Figure 15: Knots $(8_1, 8_3)$ with distance 1 realized on non-minimal diagrams

J. Hoste, Y. Nakanishi, and K. Taniyama [25] introduced an $H(n)$ move, a deformation of a link diagram. In particular, they distinguished $H(2)$ move, a band surgery which requires to preserve the number of components, and proved that an $H(2)$ move is unknotting operation, i.e. that any knot can be deformed into the unknot by a sequence of $H(2)$ moves ([25], Theorem 1). The authors defined $H(2)$ -unknotting number $u_2(K)$ of a knot K as the minimal number of $H(2)$ moves necessary to unknot K . T. Abe and T. Kanenobu [26] generalized this concept by considering a band surgery of unoriented knots and permitting the change of the number of components in the unknotting process. They defined *band unknotting number*, $u_b(K)$, as the minimal number of band surgeries necessary to unknot K . The unknotting sequence that realizes $u_2(K)$ consists only of knots, whereas the unknotting sequence that realizes $u_b(K)$ may contain links as well. So, $u_2(K) \geq u_b(K)$ by definition. This inequality can be strong: $u_b(8_{18}) = 2$ and $u_2(8_{18}) = 3$. In fact, there exist infinitely many knots K with $u_b(K) = 2$ and $u_2(K) = 3$

([26], Theorem 7.1). Moreover, Abe and Kanenobu ([26], Corollary 3.2) proved the following (in)equalities:

Corollary 1. For a knot K ,

$$u_b(K) = u_2(K) - 1 \text{ or } u_2(K).$$

Furthermore, if $u_b(K)$ is odd, then $u_b(K) = u_2(K)$; equivalently, if $u_2(K)$ is either one or even, then $u_b(K) = u_2(K)$.

Using simple topological arguments we can prove that $H(2)$ operation is equivalent with the oriented smoothing in some crossing c . Namely, if in the crossing c (of an arbitrary sign) we perform an $H(2)$ -move (Fig. 16) followed by a Reidemeister move I, the result is an oriented (vertical) smoothing in c . In the same way, the unoriented band surgery in c is equivalent to a horizontal smoothing in c . Hence, the $H(2)$ -unknotting number $u_2(K)$ of a knot K and band unknotting number $u_b(K)$ of K can be simply called *component-preserving smoothing number* $u_2(L)$ (or u_∞ -unknotting number [8]), and *smoothing number* $u_b(L)$, where L is a link. In this case, the result of the both unlinking procedures can be not only a knot but a link, as well. In the first case, the number of components of links in every step of unlinking process remains unchanged; in the second case it can be changed in any step. In every step, both procedures produce a link with one fewer crossings, so both unlinking numbers are finite.

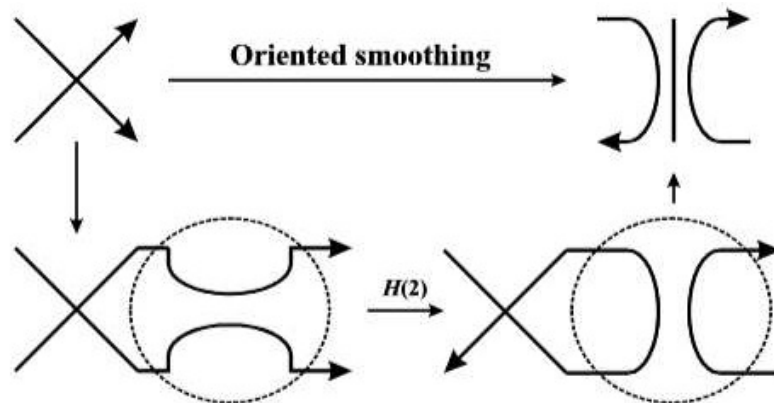


Figure 16: Equivalence of $H(2)$ -move and oriented smoothing

Abe and Kanenobu [26] defined another important concept: band-Gordian distance and $H(2)$ -Gordian distance of knots. For two knots, K_1 and K_2 , both smoothing distances are defined as the minimal number of smoothing necessary to obtain knot K_2 from K_1 . Gordian distances play an important role in DNA-knotting, where Topoisomerase I and Topoisomerase II can make crossing changes and smoothing in DNA. In the language of

Gordian numbers, smoothing numbers $u_2(L)$ and $u_b(L)$ are smoothing distances of a link L from an unlink.

6. KNOTS WITH SMOOTHING NUMBER ONE

Many efforts are made in order to make tables of smoothing numbers u_b and u_2 for knots with $n \leq 9$ crossings. Certain smoothing numbers are still undecided. Y. Bao [28] established the criterion for rational knots with smoothing number one. Remember that for a knot with $u_2(K) = 1$, the smoothing number $u_b(K)$ is one, as well. In the case of unknotting number of a knot $u(K)$, a simple tool facilitating the confirmation of certain unknotting numbers is the existence of the lower bound of unknotting number given by a half of signature. For smoothing numbers we have no simply computable lower bound, so every smoothing number different from one needs independent confirmation. We will consider the simplest case, knots with smoothing number one.

In our work, all knots with smoothing number one are obtained by the following algorithm:

Algorithm 1. *1. Take a diagram of a knot K' with unknotting number one, given by its Conway symbol;
2. find the crossing in K' which need to be changed from $+1$ to -1 in order to obtain the unknot;
3. substitute this crossing by (un)tangle $(1,-1)$ and $(1,-1)$ $(1,-1)$. By these substitutions, we obtain a knot diagram K_0 and link diagram L_0 . Choose the substitution which gives a knot;
4. minimize diagram K_0 of and recognize the obtained knot K . Knot K has the smoothing number one realized on the non-minimal diagram K_0 .*

Example 1. Rational knot $K' = 4\ 1\ 1\ 1\ 2$ has unknotting number one. By changing third crossing $+1$ to -1 , we obtain unknot $4\ 1\ 1\ (-1)\ 2$. The replacement $(-1) \rightarrow (1,-1)$ gives knot diagram $K_0 = 4\ 1\ 1\ (1,-1)\ 2$ which reduces to $K = 4\ 1\ 3$. Hence, the knot $K = 4\ 1\ 3$ has the smoothing number one realized on the non-minimal diagram $K_0 = 4\ 1\ 1\ (1,-1)\ 2$. Unknot will be obtained by smoothing crossing 1 in $(1,-1)$.

Because all minimal diagrams of an alternating knot are flype-equivalent, in the case of alternating knots, it is sufficient to take arbitrary minimal diagram of a knot K' with unknotting number one. In the case of non-alternating knots, we need to take all minimal diagrams of K' . In all cases, we implicitly suppose that unknotting number one is always realized on minimal diagrams, but we can start from an arbitrary diagram of K' which can be unknotted by a single crossing change.

Open question: Can we prove that the proposed algorithm is exhaustive, i.e., that all knots with smoothing number one can be obtained in this way? For knots with $n \leq 9$ crossings this seems to be true.

The results we obtained for smoothing number one knots with $n \leq 9$ crossings almost coincide with the results from tables of smoothing numbers given in Table 2 of the paper

[26]. The only disagreements are the following: in smoothing number one knots 5_2^2 (Fig. 17a), 9_{23} (shown in Fig. 15 in [26] as one of the smoothing number one knots), and 9_{25} (Fig. 17b) are omitted, and knot 9_{24} is included in two lists of knots, those with $u_b = u_2 = 1$, and those with $u_b = u_2 = 2$. We believe that its smoothing number is not one. Also, to the list of composite knots with smoothing number one, we would like to add the knots $!3_1\#4_1$, $3_1\#5_2$, $4_1\#!5_2$, $4_1\#5_2$, and $3_1\#!6_2$ (Fig. 18), where $!$ is used to denote the mirror-image of a knot.

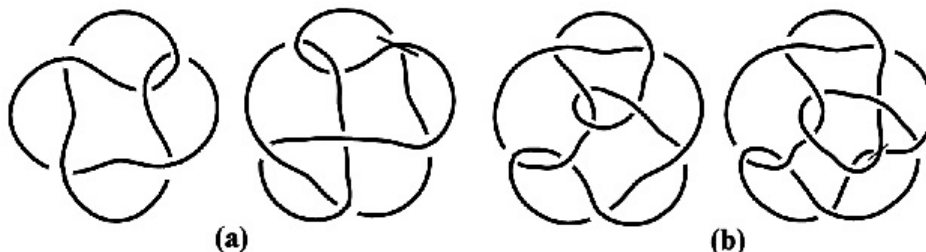


Figure 17: (a) Knot 5_2 and its non-minimal diagram $2(1,-1)1_2$ with smoothing number one; (b) knot 9_{25} and its non-minimal diagram $6*2_1 0.(1,-1) 0.2. : 2_0$ with smoothing number one

In the paper [26], Figure 15, are represented rational $H(2)$ -unknotting number one knots with $n \leq 9$ crossings, and we add the non-minimal diagrams of the rational knots 9_{21} , 9_{23} , 9_{26} , and 9_{31} (Fig. 19) with smoothing number one following from the results of Bao [28].

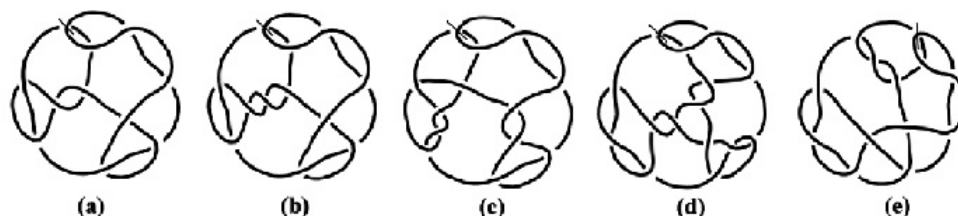


Figure 18: Composite knots (a) $!3_1\#4_1$; (b) $3_1\#5_2$; (c) $4_1\#!5_2$; (d) $4_1\#5_2$; (e) $3_1\#!6_2$ and their non-minimal diagrams with smoothing number one

By using Algorithm 1, we derived non-minimal diagrams of smoothing number one knots with $n \leq 11$ crossings. Their list can be downloaded from the address:

<http://www.mi.sanu.ac.rs/vismath/SmoothingNumber1Zekovic.pdf>.

² The correct result $u_2(5_2) = 1$ is given in [27], Table 2.

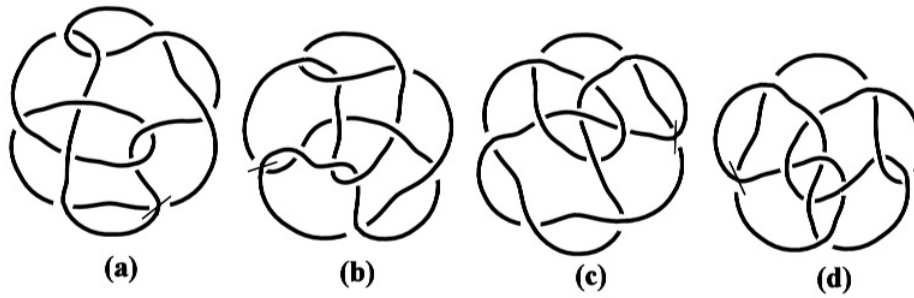


Figure 19: Knots (a) 9_{21} ; (b) 9_{23} ; (c) 9_{26} ; (d) 9_{31} and their non-minimal diagrams with smoothing number one

In this list, every knot is given in Alexander-Briggs notation (used for knots with $n \leq 10$ crossings) [9], or Knotscape notation (used for knots with $n = 11$ crossings) [21], its Conway symbol [9, 10], and the Conway symbol of its nonminimal diagram with smoothing number one.

7. SMOOTHING DISTANCES OF KNOTS

In this section, we consider H_2 -smoothing, i.e., smoothing of a knot that preserves the number of components and induced metrics of knot d_2 -distances (or H_2 -Gordian distances [27]) where $d_2(K, K_l)$ is the minimal number of H_2 -smoothings necessary to transform knot K to K_l . In the paper [27], H_2 -Gordian distances for knots with at most $n = 7$ crossings are computed. The final results of computations are given in Tables 3,4,5,6 (pages 826-827, [27]). As the partial result of our computations, we confirmed undecided (1 or 2) distances 1 for the pairs of knots $(5_1, !5_1)$, $(5_1, 7_2)$, $(5_2, !7_1)$, $(6_2, 7_2)$, $(6_3, 7_1)$, $(7_1, !7_5)$, and $(7_1, 7_7)$ (Fig. 20). Moreover, we computed knot d_2 -distances for knots with at most $n = 9$ crossings and obtained 702 knot pairs with d_2 -distance equal 1. These pairs served as the basis for the complete tables of d_2 -distances for all knots with at most $n = 9$ crossings. Among other results, together with 702 knot pairs with d_2 -distance equal 1, we confirmed distance 2 for 3192 pairs, and distance 3 for 88 pairs. In order to confirm these distances, we used the criteria I-VII from [2] (page 828), which we implemented in the Mathematica program with different obstructions proving that some knots K and K_l with undecided d_2 -distance 1 or 2 cannot have distance 1, and that certain knots have distance 3 (the complete results will be presented in the Ph.D. dissertation of the author).

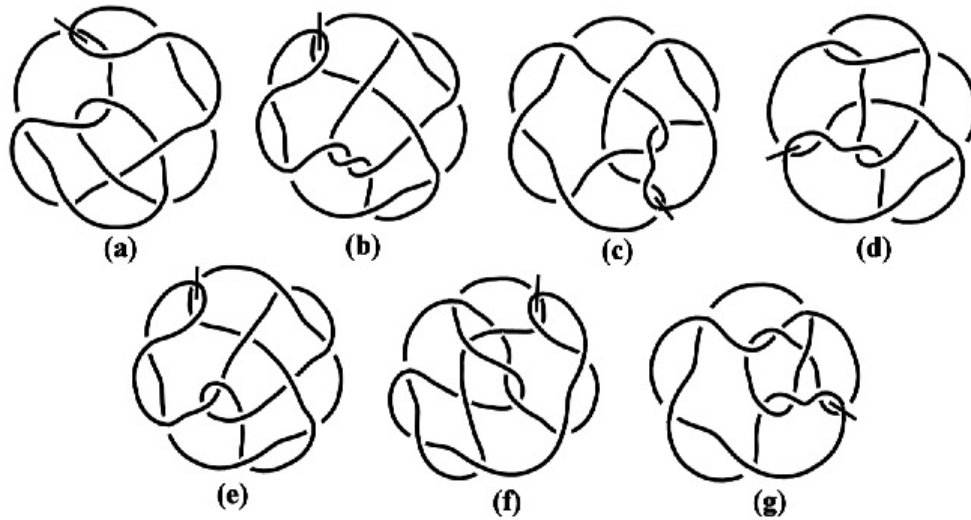


Figure 20: Knots with confirmed smoothing distance one: (a) $(5_1, !5_1)$; (b) $(5_1, 7_2)$; (c) $(5_2, !7_1)$; (d) $(6_2, 7_2)$; (e) $(6_3, 7_1)$; (f) $(7_1, !7_3)$, (g) $(7_1, 7_7)$

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